Fibre Bundles and Spin Structures

Week 8: Curvature and Chern-Weil Theory

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There are many different interpretations of the curvature of a connection. These range from formal algebraic properties to geometric ideas. At least some of the interpretations are:

1. Curvature as a measure of the failure of ∇ to satisfy certain natural-looking formulas,
2. Connections as a failure of the covariant de Rham complex,
3. Curvature as a measure of the failure of commuting partial derivatives, and
4. Curvature as a measure of the change of data once parallel transported along infintesimal loops.

One of the interesting features of curvature is that it allows us to obtain data relating to the underlying topology of the base manifold M. With an eye towards this, we will take a formal treatment of curvature as a failure of ∇ to act like the derivatives $\mathcal{L}$ and d. This algebraic interpretation allows us to compute the local and global properties of curvature, and importantly the transformation properties. We will then move on to some basics of Chern-Weil theory. This theory relates curvatures to elements of the de Rham cohomology of M.

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1 Curvature in Vector Bundles

In our algebraic approach to curvature, we start with a definition of curvature in terms of the connection's deviation from the Lie derivative. Throughout this section we fix ∇ to be a Koszul connection defined on some real vector bundle E.

1.1 Defining Curvature

Our first presentation of curvature will be as a measure of the failure of ∇ to satisfy certain nice formulas. For our first formula, recall that

$$\mathcal{L}_{[v,w]}f = \mathcal{L}_v\mathcal{L}_w f - \mathcal{L}_w\mathcal{L}_v f.$$

This is sometimes taken as a defining feature of the Lie derivative, though it can also be derived from our definition of $\mathcal{L}$ in terms of local flows along vector fields. Since the covariant derivative ∇ generalises directional derivatives like $\mathcal{L}_v f$ to expressions like $\nabla_v s$, it is natural to ask whether

$$\nabla_{[v,w]}s \overset{?}{=} \nabla_v\nabla_w s - \nabla_w\nabla_v s.$$

In general, the above equality does not hold. This motivates the following definition.

Definition 1.1. *The curvature F_∇ of the connection ∇ is a map $F_\nabla : \mathfrak{X}(M)\times \mathfrak{X}(M) \times \Gamma(E) \to \Gamma(E)$ given by*

$$F_\nabla(v, w, s) := \nabla_v\nabla_w s - \nabla_w\nabla_v s - \nabla_{[v,w]}s.$$

Recall that the connection ∇ on E is not a tensor – specifically it is not a section of the bundle $Hom(T^*M \otimes E, E)$. There are several reasons for this, the most relevant of which is that ∇ is not $C^\infty(M)$ linear in $\Gamma(E)$. Indeed, the Leibniz law for connections specifies that

$$\nabla_v(fs) = f\nabla_v s + \mathcal{L}_v fs \neq f\nabla_v s.$$

Interestingly, although the connection ∇ is not a tensor, its curvature is. We can see this as follows.

Proposition 1.2. *The curvature F_∇ is $C^\infty(M)$-multilinear.*

Proof. The preservation of addition follows immediately from the linearity of ∇ and the Lie bracket:

$$\begin{aligned}
F_\nabla(v+u,w,s) &= \nabla_{v+u}\nabla_w s - \nabla_w\nabla_{v+u}s - \nabla_{[v+u,w]}s \\
&= \nabla_v\nabla_w s + \nabla_u\nabla_w s - \nabla_w\nabla_v s - \nabla_w\nabla_u s - \nabla_{[v,w]+[u,w]}s \\
&= \nabla_v\nabla_w s + \nabla_u\nabla_w s - \nabla_w\nabla_v s - \nabla_w\nabla_u s - \nabla_{[v,w]}s - \nabla_{[u,w]}s \\
&= \left(\nabla_v\nabla_w - \nabla_w\nabla_v s - \nabla_{[v,w]}s\right) + \left(\nabla_u\nabla_w - \nabla_w\nabla_u s - \nabla_{[u,w]}s\right) \\
&= F_\nabla(v,w,s) + F_\nabla(u,w,s).
\end{aligned}$$

The other two cases are similar. As for the scaling of F by smooth functions, we have that:

$$\begin{aligned}
F_\nabla(v,w,fs) &= \nabla_v\nabla_w(fs) - \nabla_w\nabla_v(fs) - \nabla_{[v,w]}fs \\
&= (\nabla_v(f\nabla_w s + \mathcal{L}_w fs)) - (\nabla_w(f\nabla_v s + \mathcal{L}_v fs)) - \left(f\nabla_{[v,w]}s + \mathcal{L}_{[v,w]}fs\right) \\
&= (f\nabla_v\nabla_w s + \mathcal{L}_v f\nabla_w s + (\nabla_v s)\mathcal{L}_w fs + \mathcal{L}_v\mathcal{L}_w fs) - (v \leftrightarrow w) \\
&\quad - \left(f\nabla_{[v,w]}s + \mathcal{L}_{[v,w]}fs\right) \\
&= f\left(\nabla_v\nabla_w s - \nabla_w\nabla_v s - \nabla_{[v,w]}s\right) - \left(\mathcal{L}_v\mathcal{L}_w f - \mathcal{L}_w\mathcal{L}_v f - \mathcal{L}_{[v,w]}f\right)s \\
&= f\left(\nabla_v\nabla_w s - \nabla_w\nabla_v s - \nabla_{[v,w]}s\right) \\
&= fF_\nabla(v,w,s).
\end{aligned}$$

The other two cases are similar, except that they make use of the equalities $[v,fw] = f[v,w] + (\mathcal{L}_v f)w$ and $[fv,w] = f[v,w] - (\mathcal{L}_w f)v$. □

It follows that F_∇ is a section of the bundle $Hom(\bigwedge^2 T^*M \otimes E, E)$. We can use several bundle isomorphisms to redescribe F_∇. We have that

$$\begin{aligned} F_\nabla \in \Gamma\left(Hom(\bigwedge^2 T^*M \otimes E, E)\right) \\ \cong \Gamma\left((\bigwedge^2 T^*M \otimes E)^* \otimes E\right) \\ \cong \Gamma\left(\bigwedge^2 T^*M \otimes E \otimes E^*\right) \\ \cong \Gamma\left(\bigwedge^2 T^*M \otimes End(E)\right) = \Omega^2(M, End(E)). \end{aligned}$$

Thus we can equivalently interpret the curvature F_∇ as an $End(E)$-valued 2-form on M. This will become much more explicit when we determine the local expression of F_∇.

1.2 The de Rham non-Complex

Before computing the local expression of F_∇, we will first give an interesting interpretation of curvature. So far have interpreted curvature as a measure of the failure of ∇ to satisfy the same formulas as the Lie derivative. We will now interpret F_∇ as a measure of ∇ to satisfy the same formulas as the *exterior* derivative.

When discussing connections we shuffled the arguments of ∇ around to create an operator d_∇, which was locally equivalent to ∇. We also saw that this generalised this to the *exterior covariant derivative*, which acted on the E-valued k-forms on M by raising their degree by one. It is extremely tempting at this point to write down a de Rham-like complex as follows.

$$\Omega^0(M, E) \xrightarrow{d_\nabla} \cdots \xrightarrow{d_\nabla} \Omega^k(M, E) \xrightarrow{d_\nabla} \cdots \xrightarrow{d_\nabla} \Omega^n(M, E)$$

However, it is not clear whether or not we can create topological invariants from this information in a manner analogous to de Rham cohomology. The answer is in fact negative: although the maps d_∇ are $C^\infty(M)$-linear, in general they do not square to zero. This failure is captured by the following result.

Lemma 1.3. *For any η in $\Omega^k(M,E)$, the second exterior covariant derivative is given by $d_\nabla^2\eta = F_\nabla \wedge \eta$.*

We have thus arrived at an interesting interpretation of curvature: it is a measure of the failure of the existence of a covariant de Rham theory.

1.3 The Local Expression of F

We will now determine the local form of F. In order to do so, we will use the local properties of the connection ∇. Recall that locally, ∇ can be schematically expressed as "$\nabla = d + A$", where d is the standard exterior derivative and A is a matrix whose entries are local one-forms in $\Omega^1(U)$. Precisely, the components of ∇ in a local frame e_i are given by A^i_j.

We can compute the local expression of curvature by using the local expression of ∇ together with the fact that $F_\nabla = d_\nabla^2$. As a shorthand, we will denote the local expression of F_∇ in the local trivialisation U by F_A. The intention is clear – A is the local expression of ∇ in U. Using the same local frame e_i, we have that:

$$\begin{aligned}
F_A(e_i) &= d_\nabla(d_\nabla e_i) \\
&= d_\nabla(A^j_i e_j) \\
&= d_\nabla(A^j_i)e_j + A^j_i \wedge d_\nabla e_j \\
&= d_\nabla(A^j_i)e_j + A^j_i \wedge A^k_j e_k \\
&= \left(dA^j_i + A^k_i \wedge A^j_k\right) e_j.
\end{aligned}$$

Schematically, we can write $F_A = dA + A \wedge A$, where it is understood that this $\wedge$-operation is acting as a matrix multiplication in which the entries of A are wedged with each other.

1.4 Transformation Properties

Suppose that we have two overlapping trivialisations U_α and U_β of E, with corresponding frames e_i and $\tilde{e}_i$ respectively. According to our discussion thus far, we can represent the curvature locally by using the local expression of ∇. Moreover, we know the transition properties of ∇ under the transition map g that relates the two local frames by the equality $e_i = g_i^j \tilde{e}_j$. If we denote by A and B the local (matrix) representations of ∇, we may write $A = g^{-1}dg + g^{-1}Bg$. We will now use all of this information to derive the local transformation properties of F. We have:

$$
\begin{aligned}
F_A &= dA + A \wedge A \\
&= d(g^{-1}dg + g^{-1}Bg) + (g^{-1}dg + g^{-1}Bg) \wedge (g^{-1}dg + g^{-1}Bg) \\
&= (dg^{-1} \wedge dg + g^{-1}d^2 g + dg^{-1}Bg + g^{-1}dBg - g^{-1} \wedge Bdg) \\
&\quad + (g^{-1}dg \wedge g^{-1}dg + g^{-1}dg \wedge g^{-1}Bg + g^{-1}Bg \wedge g^{-1}dg + g^{-1}Bg \wedge g^{-1}Bg) \\
&= (dg^{-1} \wedge dg + g^{-1}d^2 g + dg^{-1} \wedge Bg + g^{-1}dBg - g^{-1}B \wedge dg) \\
&\quad + (g^{-1}dg \wedge g^{-1}dg + g^{-1}dg \wedge g^{-1}Bg + g^{-1}B \wedge dg + g^{-1}B \wedge Bg) \\
&= g^{-1}\left(dB + B \wedge B\right) g \\
&\quad + dg^{-1} \wedge dg + dg^{-1} \wedge Bg + g^{-1}dBg + g^{-1}dg \wedge g^{-1}dg + g^{-1}dg \wedge g^{-1}Bg.
\end{aligned}
$$

Since F is a section of the bundle $Hom(\bigwedge^2 T^*M \otimes E, E)$, we expect that the above expression should reduce to something simpler. This happens according to the following observation:

$$0 = d(\mathbb{I}_n) = d(gg^{-1}) = dgg^{-1} + gdg^{-1},$$

which implies that $dg = -gdg^{-1}g$ and $dg^{-1} = -g^{-1}dgg^{-1}$. It follows that both

$$
\begin{aligned}
dg^{-1} \wedge dg &= (-g^{-1}dgg^{-1}) \wedge dg = -g^{-1}dg \wedge g^{-1}dg, \text{ and} \\
dg^{-1} \wedge Bg &= (-g^{-1}dgg^{-1}) \wedge Bg = -g^{-1}dg \wedge g^{-1}Bg.
\end{aligned}
$$

As such, the extra terms in the above expression for F_A annihilate, and we are left with

$$F_A = g^{-1}F_B g \iff F_B = gF_A g^{-1}.$$

This is precisely the transformation law that we would expect from a section of the bundle $Hom\,(\wedge^2 T^*M \otimes End(E))$.

2 Characteristic Classes

In this section we will explore the prospect of extracting topological invariants from connections and curvature. Our starting point is the transformation properties of the curvature form:

$$F_A = g^{-1} F_B g.$$

In the derivation of this expression we assumed a globally-defined curvature form F_∇, and then computed its local expression. However, we can also proceed in the other direction: if we start with a collection of locally-defined curvature operators F_A satisfying the above transformation property on overlaps of open charts, then we can glue them all together to yield a globally defined curvature form F_∇. The gluing is well-defined due to the above transformation property.

We would like to relate F_∇ to elements of de Rham cohomology. In order to do this, we will employ the following strategy:

1. Locally define some operation P which acts on F_A and gives some local k-form $\Omega^k(U)$.

2. Do this in such a way that the transformation property of connection forms is preserved, that is, so that $P(F_A) = P(g^{-1} F_A g)$,

3. Conclude that these locally-defined forms on M then glue together to yield some ω in some $\Omega^k(M)$. We may naively write $P(F_\nabla) = \omega$.

4. Show that ω is actually a closed k-form.

If we follow the above strategy, we may use the closedness of the k-form ω to pass into the de Rham cohomology by considering $[\omega]$. Recall that this is an equivalence class of closed k-forms that differ from ω by an exact form. Under certain circumstances we may then argue that the equivalence class $[P(F_\nabla)]$ is independent of the particular choice of connection. Therefore, we may obtain some cohomological data that depends only on the bundle E on which ∇ is defined.

The above procedure captures the approach of Chern-Weil theory. The result is something known as a *characteristic class*. Here "class" refers to the equivalence class $[P(F_\nabla)]$ in the de Rham Cohomology $H^k_{dR}(M)$, and the adjective "characteristic" describes the independence of $[P(F_\nabla)]$ from ∇. We will now introduce the simplest characteristic class.

2.1 The First Chern Class

According to our strategy, we would like to create some operation P that takes local connection forms as input and yields local 2-forms on M. One way to do this is to use the trace operator. Recall that the trace of a square matrix is the sum over the diagonal entries. This is usually phrased within the context of pure linear algebra, however we can pass the operator into bundle language. Before defining the first Chern class, we will explore some important details about the trace operator on vector bundles.

2.1.1 Properties of the Trace Operator

An important property of the trace is that $Tr(AB) = Tr(BA)$, which implies that

$$Tr([A, B]) = Tr(AB - BA) = Tr(AB) - Tr(BA) = 0.$$

Importantly, the same holds when A and B are matrices with forms as entries, provided we adjust the commutator to keep track of relative antisymmetry. Let $A \in \mathcal{M}_{n\times n}(\Omega^k(M))$ and $B \in \mathcal{M}_{n\times n}(\Omega^l(M))$, we define the *graded commutator* as

$$[A, B] = A \wedge B + (-1)^{kl} B \wedge A.$$

We then have the following result, which can be proved by a direct check.

Proposition 2.1. *Let* $A \in \mathcal{M}_{n\times n}(\Omega^k(M))$ *and* $B \in \mathcal{M}_{n\times n}(\Omega^l(M))$. *Then* $Tr([A,B]) = 0$.

According to the above result, we may conclude that $Tr(gF_Ag^{-1}) = Tr(F_A)$ for all invertible matrices g. This means that the local trace operators can be used in conjunction to define a global trace operator on bundles.

We can interpret this in another way: the trace operator kills off the endomorphism part of the local curvature form F_A. In a purely linear algebraic context, the trace of a real matrix can be interpreted as a map $Tr : End(V) \to \mathbb{R}$. Similarly, in a global bundle context, we consider endomorphisms with forms as entries, we get a map

$$Tr : \Omega^2(M, End(E)) \to \Omega^2(M).$$

We finish this section with one final observation. Let $A \in \mathcal{M}_{n\times n}(\Omega^k(M))$. The exterior derivative dA is the n-by-n matrix consisting of the exterior derivative of each entry of A. In components, $(dA)^i_j = d(A^i_j)$. Since d is linear, by direct computation we have that

$$Tr(dA^i_j) = (dA)^i_i = d(A^i_i) = d(Tr(A)).$$

2.1.2 Defining the First Chern Class

We may now use the global trace operator to define the first Chern class. Our starting point is to show that the object $Tr(F_\nabla)$ is actually an element of the de Rham cohomology of M.

Lemma 2.2. *The element* $Tr(F_\nabla)$ *is a closed* 2*-form on* M.

Proof. First of all, observe that

$$dF_A = d(dA + A \wedge A) = d^2A + d(A \wedge A) = dA \wedge A - A \wedge dA.$$

Using the fact that $dA = dF_A - A \wedge A$, we have that

$$\begin{aligned} dF_A &= dA \wedge A - A \wedge dA \\ &= (F_A - A \wedge A) \wedge A - A \wedge (F_A - A \wedge A) \\ &= F_A \wedge A - A \wedge A \wedge A - A \wedge F_A + A \wedge A \wedge A \\ &= F_A \wedge A - A \wedge F_A \\ &= [F_A, A]. \end{aligned}$$

We can therefore conclude that

$$d(TrF_A) = Tr(dF_A) = Tr([F_A, A]) = 0,$$

as required. □

By the above result, we may conclude that the equivalence class $[Tr(F_\triangledown)]$ lies in $H^2_{dR}(M)$. We will now argue that this equivalence class is characteristic.

Lemma 2.3. *The equivalence class $[Tr(F_\triangledown)]$ is independent of the choice of connection.*

Proof. Suppose that $\tilde{\nabla}$ is another connection on E. We would like to show that $[Tr(F_{\tilde{\triangledown}})]$ is the same equivalence class in de Rham cohomology. By construction, this is truly if and only if $Tr(F_\triangledown)$ and $Tr(F_{\tilde{\triangledown}})$ differ by an exact form. Specifically, we seek some ω in $\Omega^1(M)$ such that

$$Tr(F_\triangledown) - Tr(F_{\tilde{\triangledown}}) = d\eta.$$

We will argue the existence of η locally. Recall that despite ∇ and $\tilde{\nabla}$ not being tensors, their difference $\nabla - \tilde{\nabla}$ *is*. This means that locally, we may consider the $End(E)$-valued one-form $\omega := A - B$, where A and B are the local connection matrices of ∇ and $\tilde{\nabla}$, respectively. Using that $A = B + \omega$, we have that

$$\begin{aligned} F_A &= dA + A \wedge A \\ &= d(B+\omega) + (B+\omega) \wedge (B+\omega) \\ &= dB + d\omega + B \wedge B + B \wedge \omega + \omega \wedge B + \omega \wedge \omega \\ &= F_B + d\omega + [B,\omega] + \frac{1}{2}[\omega,\omega]. \end{aligned}$$

Applying the trace operator, we have that

$$\begin{aligned} Tr(F_A) &= Tr\left(F_B + d\omega + [B,\omega] + \frac{1}{2}[\omega,\omega]\right) \\ &= Tr(F_B) + Tr(d\omega) + \frac{1}{2}Tr([\omega,\omega]) + Tr([B,\omega]) \\ &= Tr(F_B) + Tr(d\omega). \end{aligned}$$

The proof is complete by setting $\eta = Tr(\omega)$. Indeed, from the above expression it follows that

$$Tr(F_A) - Tr(F_B) = Tr(d\omega) = dTr(\omega),$$

as required. □

Using the above results, we may conclude that the class $[Tr(F_\nabla)]$ only depends on E. Therefore, we make the following definition.

Definition 2.4. *The first Chern class of a vector bundle E is equal to $c_1(E) := [Tr(F_\nabla)]$.*

There are several other interesting properties of the first Chern class that increase in sophistication.

Theorem 2.5. *1. If E is a real vector bundle then $c_1(E) = 0$.*

2. If E is a complex vector bundle then then the normalised Chern class $c_1(E) = \left[\frac{1}{2\pi i}Tr(F_\nabla)\right]$ is real.

3. *Let E and E' be two complex vector bundles of rank k and l respectively. Then $c_1(E \otimes E') = kc_1(E') + c_1(E)l$ and $c_1(E \oplus E') = c_1(E) + c_1(E')$.*

The proofs of these results are beyond the scope of this course, so we will omit them.

2.2 Invariant Polynomials

To appear

2.3 Other Characteristic Classes

To appear

2.3.1 The Chern Characteristic

2.3.2 The Pontryagin Class

2.3.3 The Euler Class