Fibre Bundles and Spin Structures

Part 6: Principal Bundles

David O'Connell

Okinawa Institute of Science and Technology
1919-1 Tancha, Onna-son, Okinawa 904-0495, Japan

Last week we saw vector bundles. We introduced two (equivalent) ways to interpret vector bundles: as a collection of vector spaces smoothly parameterised by M, and as a generalisation of the Cartesian product $M \times \mathbb{R}^k$. Principal bundles are a similar beast.

The fibres of a principal bundle are no longer vector spaces, but instead are something called a G-set, which is a set that carries a particularly nice action of a Lie group G. This might seem unusual, but principal bundles turn out to be simpler that their vector bundle cousins. In particular, we will see that isomorphisms and local trivialisations of a principal are much easier to describe.

There is also an interesting interplay between vector bundles and principal bundles. Every vector bundle can be used to construct a principal bundle formed from its structure group, and every principal bundle can define a family of vector bundles, provided that you have a representation.

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1 More on Group Actions

Before getting to the definition of a principal G-bundle, we first recall a few details about group actions. Recall that a smooth (right) group action is an action $\bullet : M \times G \to M$, where M is a manifold, G is a Lie group, and the map $\bullet$ is smooth. Throughout this document, whenever a group element acts on a manifold point, we will omit the "$\bullet$".

1.1 A Trivial Group Action

Using only the data of M and G, we can create a group action of G on the product manifold $M \times G$.

Example 1.1 (A Trivial Example)**.** *By starting with a smooth manifold M and a Lie Group G, we can create a smooth right action $\bullet$ by taking $M \times G$ as our starting manifold, and defining $\bullet : (M \times G) \times G \to M \times G$ by allowing the action to multiply the second component of a point using the multiplication of G:*

$$(x, g) \bullet h := (x, gh).$$

This is indeed a smooth right action, and moreover, the action $\bullet$ is free. This follows from the basic group-theoretic fact that

$$gh = gk \text{ implies } h = k,$$

which follows by multiplying both terms by g^{-1} on the left.

1.2 Quotients of Manifolds

We have seen previously that the quotient of a topological space by a continuous group action yields a topological space. A group action can induces the quotient $X/_G$ by identifying a pair of points in X if they lie in the same orbit space. We will now take this one step further, by considering the details of a smooth group action of a Lie group G on a smooth manifold M. In general, quotients will spoil a manifold's structure, for instance:

- we can quotient S^1 by an equivalence relation that identifies two antipodal points and leaves the rest alone. The resulting space will look like "∞" or "8", and will not be locally-Euclidean, or
- we can quotient two copies of $\mathbb{R}$ along some open interval. The resulting space will not be Hausdorff, since the boundary of the open intervals are not identified.

Even if we restrict our attention to those equivalence relations that arise via group actions, we can still find quotients which do not yield a manifold. This observation motivates the question: what properties of a smooth group action are required so that the quotient space ${}^{M}\!/_{G}$ is again a manifold? There a positive answer in the case that the group action is free and proper.

Normally, a continuous map between topological spaces is called *proper* if the preimage of any compact subset is again compact. We can mimic this terminology for group actions.

Definition 1.2. *A right action of G on M is called proper if the map $\theta : M \times G \to M \times M$ given by $(x, g) \mapsto (xg, x)$ is proper.*

The main benefit of considering a proper group action is that it guarantees the quotient space is Hausdorff. For our purposes, we have the following useful criteria.

Proposition 1.3. *Every smooth right action by a compact Lie group is proper.*

Moreover, in combination with freeness, we obtain the following useful theorem. This is commonly called the "Quotient Manifold Theorem".

Theorem 1.4. *If the smooth action of G on M is free and proper, then the quotient space $N = {}^{M}\!/_{G}$ has a unique smooth structure such that the natural projection $\pi : M \to N$ is smooth map.*

The full proof of the above result is beyond the scope of this course. The argument can be found in Chapter 21 of Lee.

2 Principal Bundles

Suppose that we have a Lie group G and a smooth manifold M. A principal G-bundle is a type of fibre bundle over M in which each fibre P_p carries a smooth group action of G. The fibres themselves are not Lie groups themselves, but they are at least diffeomorphic to G.

2.1 Basic Properties

A principal G-bundle will be locally isomorphic to the trivial example above. The definition is as follows.

Definition 2.1. *A principal bundle consists of the data (P, π, M, G), where*

1. *P and M are smooth manifolds,*
2. *G is a Lie group that acts on P from the right,*
3. *$\pi : P \to M$ is a smooth surjective map such that $\pi(pg) = \pi(p)$ for all $p \in P$ and $g \in G$, and*
4. *For each x in M there is some open neighbourhood U of x and a diffeomorphism $\Psi : \pi^{-1}(U) \to U \times G$ that sends each fibre $\pi^{-1}(y)$ to $\{y\} \times G$ and is G-equivariant, where $U \times G$ carries the trivial right-action by G.*

Observe the similarity between principal G-bundles and vector bundles. We won't really use the notation (P, π, M, G), instead we will just denote the entire object by P, where the base manifold M is understood from the context. In distinction to previous notation, in the setting of principal bundles we will now denote arbitrary elements of P by lowercase p, and we denote points in the base manifold by x. Thus we will denote fibres by P_x.[1] More-

[1] Presenting the material in this course involves passing through a minefield of inconsistent notation. Of the recommended texts I included in Week 1, I think that every single author uses slightly different notation. I've included a table of symbols in Week 1's notes, so please check it occasionally.

over, we will write P_U to mean the restriction of P to the open subset U of M, that is, $P_U = \pi^{-1}(U)$.

Lemma 2.2. *Let P be a principal G-bundle. The action of G on P is free, and is transitive on each fibre.*

Proof. Let p be a point in P. Then p lies in some fibre P_x. Suppose that g and h are elements of G such that $pg = ph$. Pick any local trivialisation Ψ that contains p. Then by G-equivariance of Ψ, we have that

$$\Psi(pg) = \Psi(p)g = (x,k)g = (x,kg), \Psi(ph) \quad = \Psi(p)h = (x,k)h = (x,kh).$$

Both of these lines are equal, so it follows that $kg = kh$, which implies that $g = h$ as required. As for the transitivity of P_x, we can use the same trivialisation Ψ. Suppose that q is another element of the fibre P_x. This means that $\Psi(q) = (x,\tilde{k})$. Since G is a group, there is some element l in G such that $k = \tilde{k}l$, i.e. $l = \tilde{k}^{-1}k$. It follows from the G-equivariance of Ψ that $p = q\tilde{k}^{-1}k$. □

Using the above result, we can now explain the terminology of "principal". In group theory, we refer to a set carrying a free and transitive action as a "G-torsor", or a *principal homogenous space*. This is where the terminology of principal bundles comes from – the fibres of a principal G-bundle are principal homogenous spaces for G. The following result reiterates a key point regarding the fibre structure of a principal G-bundle.

Proposition 2.3. *Any element p in P_x induces a G-equivariant diffeomorphism from P_x to G.*

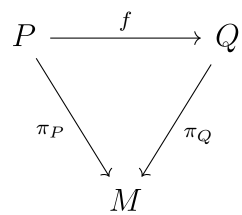
Proof. We map p to e in G. The free and transitive group action of G on P_x then allows us to extend this to a diffeomorphism. Indeed – suppose that we have another element $q \in P_x$. By transitivity of the action, there is a unique element g in G such that $q = pg$. Then we can map $q \mapsto g$. We do this for every element in P_x, and thus we define a bijective function from P_x to G. This map and its inverse are smooth since the group action is smooth. □

The diffeomorphism is given by restricting any local trivialisation that contains the fibre P_x. Note that a fibre P_x is not Lie group isomorphic to G, since P_x is not a group. Most obviously, P_x does not contain an identity element.

2.2 Morphisms

A morphism of principal bundles should preserve the smooth structure of P and the group action of G on P. As such, we obtain the following definition of a morphism of principal G-bundles over the same base manifold M.

Definition 2.4. *Let P and Q be principal G-bundles over the same base manifold M. A map $f : P \to Q$ is called a morphism of principal bundles if f is smooth, G-equivariant, and the diagram*



commutes.

An isomorphism of principal G-bundles is a morphism that is bijective and whose inverse is also a morphism. According to this notion of isomorphism, the fourth item of Definition 2.1 is stating that every principal bundle is locally isomorphic to a trivial bundle $U \times G$.

Interestingly, the following result shows that two non-isomorphic principal bundles are *really* different.

Theorem 2.5. *Any morphism between principal G-bundles is necessarily an isomorphism.*

Proof. To be released after the homework. ☐

2.3 Sections

Since the map π is defined to be a projection map, we can consider its right-inverses as we did for vector bundles.

Definition 2.6. *A global section of P is a smooth right-inverse of the map $\pi : P \to M$.*

We can again extend this idea to include local sections, which replace M in the above definition by a proper open subset U of M.

The sections of vector bundles allowed us to distinguish whether or not the bundle was trivial. Specifically, we required the existence of a global frame – k-many smooth sections that were linearly independent and non-vanishing. The fibres of a principal bundle don't have the structure of a vector space, so we do not have a notion of linear independence. Moreover, principal bundles do not have a preferred identity element in each fibre, so we cannot talk about the vanishing of sections. Instead we have a free and transitive group action on each fibre. This turns out to drastically simplify the relationship between sections and the trivial bundle.

Theorem 2.7. *A principal G-bundle P over M is trivial iff it admits a global section.*

Proof. Suppose first that $s : M \to P$ is a global section of P. We would like to use the section s to define an isomorphism $\chi : P \to M \times G$. In order to do so, we must first make a few observations. Let p be some point in P, say in the fibre above the point x in M. Since s is global, there exists some $s(x)$ that is also in the same fibre P_x. Since the group action G is transitive on each fibre, there exists some g in G such that $s(x)g = p$. Since we chose p to be arbitrary, every point p in P is of this form. We may thus define

$$\chi(p) = \chi(s(x)g) = (x, g).$$

This map is indeed an isomorphism.

For the converse, if P is trivial then we can use the global section $s(x) = (x, e)$. □

2.4 Pullbacks

Since principal bundles do not carry a linear-algebraic structure on their fibres, lots of the constructions detailed for vector bundles do not have analogues for principal bundles. However, we can still pull back principal bundles along a smooth map.

Definition 2.8. *Let $f : M \to N$ be a smooth map and (P, π, N, G) a principal bundle. The pullback bundle has as elements:*

$$f^*P = M \times_N P := \{(m, p) \in M \times P \mid f(m) = \pi(p)\},$$

with the projection map of f^*P is given by the projection onto the first factor, and the right group action of G on f^*P given by $(m, p)g = (m, pg)$.

The above definition is indeed well-defined, and forces the diagram

$$\begin{array}{ccc} f^*P & \overset{p_2}{\dashrightarrow} & P \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow[f]{} & N \end{array}$$

to commute.

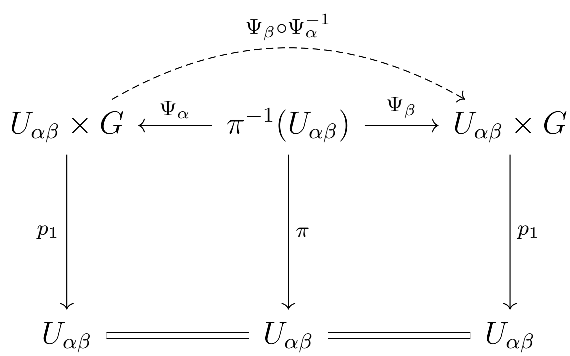
3 Other Expressions of a Principal Bundle

In the case of vector bundles, we have seen that there is a nice result that allows us to construct a vector bundle from Cartesian products $U_\alpha \times \mathbb{R}^k$ and $U_\beta \times \mathbb{R}^k$ and well-behaved transition maps $g_{\alpha\beta}$ defined along the common overlap $U_{\alpha\beta}$. In this section we will explore a similar feature of principal bundles. The ideas involved are analogous to the vector bundle case, but will change due to the different fibre structure that a principal bundle possesses.

Before getting to the reconstruction theorem for principal bundles, we first make an observation. By Theorem 2.3, the existence of a single global section forces a principal bundle to be trivial. As such, we can use local sections to characterise local trivialisations of P. Consider the following line of reasoning: let $s : U \to P$ be a local section and consider x in U. The point $s(x)$ lies somewhere in the fibre P_x. We define a local trivialisation $\Psi : \pi^{-1}(U) \to U \times G$ by mapping each $s(x)$ to (x, e). We can repeat the reasoning of 2.3 to conclude that Ψ is G-equivariant. Moreover, if we were to start with a local trivialisation $\Psi : P_U \to U \times G$, then we can always create a local section with these properties by simply taking $s(x) = \Psi^{-1}(x, e)$.

3.1 Transition Functions

Suppose that we have two local trivialisations U_α and U_β of M, with corresponding trivialisation maps Ψ_α and Ψ_β, respectively. According to the previous observation, let us consider the two local sections s_α and s_β that characterise Ψ_α and Ψ_β. We thus have the following diagram.



On the overlap $U_{\alpha\beta}$ we obtain a function $\Psi_\beta \circ \Psi_\alpha^{-1} : U_{\alpha\beta} \times G \to U_{\alpha\beta} \times G$. This function maps each (x, g) to (x, h), where h is an element of G that has smooth dependence on x. In fact, we can explicitly describe h using the sections s_α and s_β. Consider $s_\alpha(x)$. Since the action of G on P_x is free and transitive, there is some unique k in G such that $s_\alpha(x) = s_\beta(x)k$. Therefore

$$\Psi_\beta \circ \Psi_\alpha^{-1}(x,e) = \Psi_\beta(s_\alpha(x)) = \Psi_\beta(s_\beta(x)k) = \Psi_\beta(s_\beta(x))k = (x,e)k = (x,k).$$

It follows that $h = gk$. We can therefore define a smooth map $g_{\alpha\beta} : U_{\alpha\beta} \to G$ by

$$g_{\alpha\beta}(x) = k, \text{ where } s_\alpha(x) = s_\beta(x)k.$$

We will refer to $g_{\alpha\beta}$ as a *transition map*.

3.2 The Reconstruction Theorem

Suppose that we have a cover U_α of charts of M, together with a collection of local sections $s_\alpha : U_\alpha \to M$. We will now observe a few interesting features of the transition maps $g_{\alpha\beta}$.

- We trivially have that $s_\alpha(x) = s_\alpha(x)e$ for all x in U_α. Therefore the map $g_{\alpha\alpha} : U_\alpha \to G$ sends each x in U_α to e in G.

- On an overlap $U_{\alpha\beta}$, if we have that $s_\alpha(x) = s_\beta(x)k$, then it is also the case that $s_\beta(x) = s_\alpha(x)k^{-1}$. Therefore the transition map $g_{\beta\alpha}$ sends each x in $U_{\alpha\beta}$ to the inverse element $(g_{\alpha\beta}(x))^{-1}$ in G.

- Suppose that we have a triple overlap $U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cap U_\gamma$. Any element here has three transition maps that can be composed, namely $g_{\alpha\beta}$, $g_{\beta\gamma}$ and $g_{\gamma\alpha}$. Suppose that $s_\alpha(x) = s_\beta(x)k$, and that $s_\beta(x) = s_\gamma(x)l$. This means that $s_\alpha(x) = s_\gamma(x)kl$, or equivalently $s_\gamma(x) = s_\alpha(x)l^{-1}k^1$ It follows that

$$(x,e) \mapsto (x,k) \mapsto (x,l) \mapsto (x,kl) \mapsto (x, kll^{-1}k^{-1}) = (x,e),$$

and thus the triple $g_{\alpha\beta}(x)g_{\beta\gamma}(x)g_{\gamma\alpha}(x)$ equals e.

These three items are not independent. In fact, from the single observation that

$$g_{\gamma\alpha}(x) = g_{\alpha\beta}(x)g_{\beta\gamma}(x)$$

we can derive all three of the points above. This is known as a *cocycle condition* for reasons that we will see in a few weeks. We can now state the reconstruction theorem for principal bundles.

Theorem 3.1. *Let M be a smooth manifold and G a Lie group. Suppose that we have an open cover U_α of M and a collection of smooth maps $g_{\alpha\beta} : U_{\alpha\beta} \to G$ satisfying the relation $g_{\gamma\alpha}(x) = g_{\beta\gamma}(x)g_{\alpha\beta}(x)$ for all x in $U_{\alpha\beta\gamma}$. Then the set*

$$P := \bigsqcup U_\alpha \times G \big/ \sim,$$

where $(x, h) \sim (x, g)$ iff $g = hg_{\alpha\beta}(x)$ possesses a unique smooth structure and equivariant action by G that turns P into a principal G-bundle.

3.3 Equivalent Definition of a Principal Bundles

It is possible to define a principal bundle in terms of the behaviour of the group action G on P. In fact, the bundle P can be defined without ever appealing to the base manifold M.

Theorem 3.2. *Let P be a smooth manifold and G a Lie group acting on P from the right. If G acts freely and properly, then there is a unique smooth structure on P/G that makes P into a principal G-bundle.*

This theorem is similar in spirit to 1.4.

4 Examples of Principal Bundles

We will now run through some interesting examples of principal bundles. Specifically, we will classify the $\mathbb{Z}_2$-bundles over the circle (spoiler: there are two), and then we define the Hopf fibration, which can be seen as a principal

$SO(2)$-bundle over the 2-sphere. Finally we will introduce an important idea: we can define a principal G-bundle from a vector bundle E by considering the bundle of fibrewise frames. This is usually denoted $Fr(E)$, and is called the frame bundle of E.

4.1 Principal $\mathbb{Z}_2$-bundles over S^1

We will start with possibly the simplest non-trivial case of a principal bundle: the $\mathbb{Z}_2$ bundles over S^1. In order to do this, we will use 3.1. First, let us recall that the group $\mathbb{Z}_2$ consists of two points, namely $\{1, -1\}$. We can interpret this as a zero-dimensional manifold with two connected components. Thus $\mathbb{Z}_2$ is a (discrete) Lie group.

Let us describe S^1 as the gluing of two charts U_1 and U_2, heuristically pictured in Figure 1. The intersection U_{12} of these charts is then the disjoint union of two open intervals. Denote these intervals by V_1 and V_2.

Since there are only two charts, in order to use Theorem 3.1, we need to specify the transition function $g_{12} : U_{12} \to \mathbb{Z}_2$. Specifying different transition functions may yield different bundles. With this in mind, we would like to understand the possible values for g_{12}. In order to do so, we will make use of the following result from general topology.

Proposition 4.1. *A topological space is continuous iff every continuous function to a discrete topology is constant.*

The intersection U_{12} consists of two connected components, namely V_1 and V_2. According to the above result, the transition function g_{12} will be constant on both V_1 and V_2. Therefore, there are four possible choices of transition functions:

1. $g_{12}(V_1) = 1$ and $g_{12}(V_2) = 1$,

2. $g_{12}(V_1) = 1$ and $g_{12}(V_2) = -1$,

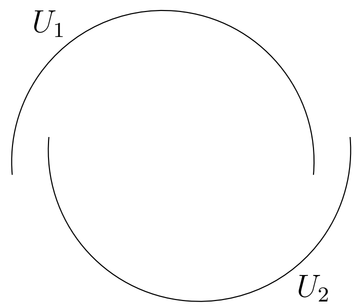


Figure 1: A construction of S^1

3. $g_{12}(V_1) = -1$ and $g_{12}(V_2) = 1$, and
4. $g_{12}(V_1) = -1$ and $g_{12}(V_2) = -1$.

As a matter of fact, up to isomorphism there are only two choices, namely whether or not $g_{12}(V_1)$ and $g_{12}(V_2)$ have the same sign. Let us consider these two cases seperately.

Case 1: $g_{12}(V_1) = 1$ and $g_{12}(V_2) = 1$. In this case, we construct a principal bundle by gluing $U_1 \times \mathbb{Z}_2$ to $U_2 \times \mathbb{Z}_2$ to yield two disjoint copies of S^1. Put differently, in this case we obtain the trivial principal bundle $S^1 \times \mathbb{Z}_2$. There are two global sections $s : S^1 \to S^1 \times \mathbb{Z}_2$, which send all of S^1 into either copy that sits in $S^1 \times \mathbb{Z}_2$.

Case 2: $g_{12}(V_1) = 1$ and $g_{12}(V_2) = -1$. In this case the V_1 and V_2 components of $U_1 \times \mathbb{Z}_2$ and $U_2 \times \mathbb{Z}_2$ are glued oppositely. This means that there is a "twist" in the gluing and we obtain a connected principal $\mathbb{Z}_2$-bundle that is isomorphic to a copy of S^1 that is wrapped around itself. We can concretely describe this bundle in the complex plane using the projection map $z \mapsto z^2$, which is clearly 2-to-1. Note that there is no global section $s : S^1 \to S^1$ since any map which sends z to $\sqrt{z}$ will be have a branch cut somewhere on the circle. It follows that this principal bundle is non-trivial, and is thus not isomorphic to the bundle of Case 1.

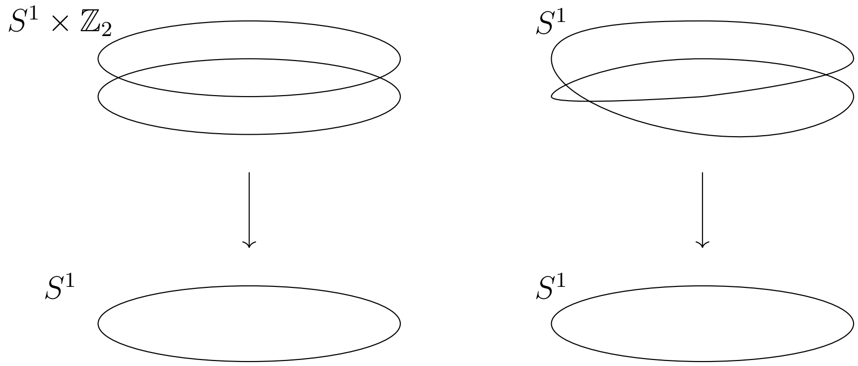


Figure 2: The two principal $\mathbb{Z}_2$-bundles of S^1.

If we were to consider the other two possibilities for the transition function g_{12} we would obtain the same two bundles described above. Figure 2 depicts the two principal $\mathbb{Z}_2$ bundles over S^1.

4.2 The Frame Bundle

We will now introduce an important example of a principal bundle. This is the so-called *frame bundle*. The idea is fairly simple – we know that there is a Lie group lurking in the definition of a vector bundle, the so-called structure group. This is the space of symmetries of the fibres of the vector bundle. We can take the set of frames, and endow this with a principal bundle structure by allowing the structure group to act on frames from the right.

Suppose that we have a rank-k vector bundle E. Recall that the transtion maps $g_{\alpha\beta}$ of E map from the overlap of charts $U_{\alpha\beta}$ into $GL(k)$. We would like to define a principal $GL(k)$ bundle using the frames of E. We will denote the frame bundle of E by $Fr(E)$. In order to properly describe $Fr(E)$, we will specify

- the fibres,
- the free (right) group action that is transitive on fibres,

- the local trivialisations, and
- the transition functions.

To begin with, we set each fibre $Fr(E)_x$ to be equal to the set of all frames of the fibre E_x. As a set, we take $Fr(E)$ to be the disjoint union of all $Fr(E_x)$.

We would like to define a smooth right action of $GL(k)$ on each fibre $Fr(E_x)$. In order to do this, we will need to make a few observations. Technically, the space $GL(k)$ is the automorphism group of $\mathbb{R}^k$, so an element A in $GL(k)$ is an isomorphism $A : \mathbb{R}^k \to \mathbb{R}^k$. We know from elementary linear algebra that A can either be seen as an invertible linear map, or as a $k \times k$ matrix. Since E is a rank-k vector bundle, each fibre E_x has dimension k and thus is isomorphic to $\mathbb{R}^k$. An isomorphism between two dimension-k vector spaces is not canonical – we need to fix a basis of one and map it to a basis of the other. In our context, this means that each frame $e_1, ..., e_k$ of E_x induces a vector space isomorphism $\chi_e : \mathbb{R}^k \to E_x$ which maps the i^{th} element of the standard basis of $\mathbb{R}^k$ to e_i in E_x.

We can now define a right action of $GL(k)$ via precomposition with the map χ_e. Let A be some matrix in $GL(k)$. Then the map $\chi \circ A : \mathbb{R}^k \to E_x$ is another isomorphism. However, the image of the standard basis of $\mathbb{R}^k$ under this map will not map to the frame e_i, instead it will map to some other frame f_i. The new frame can be explicitly described as $f_i = A_i^j e_j$, and thus the precomposition $\chi_e \circ A$ is simply a change-of-basis formula.[2] It is a basic fact of linear algebra that every basis of E_x can be obtain in this way. This means that the orbit space of a single frame e_i under the right action of $GL(k)$ covers all of $Fr(E_x)$. Moreover, the precomposition by A is clearly injective. This means the group $GL(k)$ acts freely and transitively on E_x from the right.

[2] In fact, using that A has a representation as a matrix A_j^i, we can explicitly describe the frame f_i as $f_i = A_i^j e_j$.

We will now define the local trivialisations of $Fr(E)$. Fix some U open in M. We would like a map $\Psi : Fr(E)_U \to U \times GL(k)$ that is smooth and $GL(k)$-equivariant. So, let e_i be a frame of E_x, for some x in U. Using the data that we have at our disposal, we can make two observations:

- without loss of generality there is a local trivialisation $\Phi : E_U \to U \times \mathbb{R}^k$. This map restricts to a linear isomorphism on each fibre, that is $\Phi|_x : E_x \to \mathbb{R}^k$ is an isomorphism,
- the frame e_i of E_x induces an isomorphism $\chi_e : \mathbb{R}^k \to E_x$.

The trivialisation map Ψ can then be defined as the composition of the above two isomorphisms. We map

$$\Psi(e_i) = (x, \chi_e \circ \Phi_x).$$

We emphasise that the function in the second argument is indeed an isomorphism from $\mathbb{R}^k$ to itself, thus is an element of $GL(k)$.

The local trivialisations of $Fr(E)$ and E can also be related using sections of both bundles. Suppose that $s^i : U \to E_U$ is a local frame of E that characterises the local trivialisation Φ of E. Then we can obtain a local trivialisation of $Fr(E)$ by using the section $s : U \to Fr(E)_U$ that acts by mapping each x in U to the frame $s^i(x)$ of E_x. Under this reading, the composition of the χ-map related to the frame $s^i(x)$ and the restricted map $\Phi|_{E_x}$ equals the identity in $GL(k)$.

Finally, we would like to describe the transition maps of $Fr(E)$. Suppose that we have two local trivialisations Ψ_α and Ψ_β. The transition map $\tilde{g}_{\alpha\beta}$ will be the second component of the map $\Psi_\beta \circ \Psi_\alpha^{-1}$. This should be a map which tells us which group element is used to relate two local expressions of the same frame. We know exactly what this is: it is the change-of-basis action induced from E. Put differently, the transition functions $\tilde{g}_{\alpha\beta}$ are precisely the transition functions $G_{\alpha\beta}$ of E.

4.3 The Frame Bundle LM

Recall that the tangent bundle TM is a rank-m vector bundle over M with

- local trivialisations $\Phi_\alpha : TM|_{U_\alpha} \to U_\alpha \times \mathbb{R}^m$ given by $v_p \mapsto (p, v^i)$, where v^i are the components of v_p expressed in the coordinate basis induced from the local chart U_α of M, and
- transition maps $g_{\alpha\beta} : U_{\alpha\beta} \to GL(m)$ given by the Jacobians $\frac{\partial y^j}{\partial x^i}$ where x^i are coordinates of U_α and y^j are coordinates of U_β.

We will now describe the frame bundle of TM. In the special case of the tangent bundle, we will denote the frame bundle by LM.

Suppose that $e_1, ..., e_m$ is a frame for the fibre T_xM. Then in a local chart U_α, each component e_μ of the frame can be represented as a linear combination

$$e_\mu = v^i_\mu \frac{\partial}{\partial x^i}.$$

If we do this for all $\mu = 1, .., m$, we can obtain a single $m \times m$ matrix v^i_μ which transforms the coordinate basis $\frac{\partial}{\partial x^i}$ into the basis e_μ. The matrix v^i_μ is invertible, so we can interpret it as an element of $GL(m)$. As such, we can define a local trivialisation $\Psi_\alpha : \tilde{\pi}^{-1}(LM) \to U_\alpha \times GL(m)$ by $e_\mu \mapsto (x, v^i_\mu)$. The right action of $GL(m)$ on the fibre L_xM is given by a change of basis formula. Suppose now we have another chart U_β of M, with coordinate functions y^i. We now have two representations of a frame e_μ:

$$e_\mu = v^i_\mu \frac{\partial}{\partial x^i}, \text{ and } e_\mu = w^i_\mu \frac{\partial}{\partial y^i},$$

where again we suppress the dependence on the point x for simplicity. We can now use the transformation properties of vector components to conclude that the transition map between local trivialisations Ψ_α and Ψ_β is given by the Jacobian, that is,

$$g_{\alpha\beta}(x) = \left.\frac{\partial y^i}{\partial x^j}\right|_x.$$

4.4 Reduction of the Structure Group

In various contexts it is useful to endow a vector bundle E with some additional structure. In the canonical case of TM, we can use the higher tensorial powers of TM to define fields of tensors on M. Specifically, we can define

- An orientation on M to be a non-vanishing section of the top exterior power $\bigwedge^m TM$, and
- a metric on M to be a section of the rank $(0,2)$-tensorial bundle $TM \otimes TM$.

In either of these cases, we can restrict our attention to those charts in M that consistently preserve the structure in question. We have already seen this in the case of an orientation – we defined M to be orientable whenever there exist some sub-atlas whose transition maps are orientation preserving. Similarly, for a metric on M we can consider those charts that preserve the metric – that is, the charts whose Jacobians satisfy $A^T = A^{-1}$.

This amounts to reducing the structure group $GL(m)$ to the smaller collection of symmetries that preserve the additional data. Indeed, for an orientation we may replace $GL(m)$ with $SL(m)$, and for a metric we may replace $GL(m)$ with $O(m)$. For an orientable manifold with a metric, we may use $SO(m)$. In terms of frames, we consider principal sub-bundles of LM, whose fibres carry an action of a subgroup of $GL(m)$. The frame bundle of an orientable manifold with a metric is then a principal $SO(m)$-bundle comprised of all the orthonormal frames of TM.

5 Associated Bundles

In the previous section we saw that we can obtain a principal bundle from a vector bundle by means of the frame bundle construction. We will now go in the opposite direction, and use a principal bundle to construct a vector bundle. In contrast to the frame bundle, the construction of a vector bundle from a principal bundle requires an additional piece of structure – we need a representation of the Lie group G on some vector space V. This vector space will then become the fibres of the resulting vector bundle, and the representation of G will induce appropriate transition maps. We detail this construction now.

5.1 Construction of an Associated Bundle

Let P be a principal G-bundle, and suppose that we have a representation $\rho : G \to GL(V)$. In order to simplify notation slightly, we will write gv instead of $\rho(g)v$.

We will define the associated bundle $E(P, V)$ using a quotient of the trivial bundle $P \times V$ by a particular group action. Consider the following right action of G on $P \times V$:

$$(p, v)g = (pg, g^{-1}v).$$

The above expression does indeed define a smooth right action of G on $P \times V$. We can thus consider the quotient space

$$E(P, V) = {}^{P \times V}\!\diagup_{G} = \{[p, v] \mid (p, v) \in P \times V\}.$$

The equivalence classes $[p, v]$ are defined by the equivalence relation that identifies every pair (p, v) with its orbit space under G. Put differently, we have that $(p, v) \sim (q, w)$ iff there exists some g in G such that $(p, v)g = (pg, g^{-1}v) = (q, w)$. In particular, we have that

$$(pg, v) = (pg, g^{-1}gv) = (p, gv)g,$$

and thus $[pg, v] = [p, gv]$ for all $p \in P$, $v \in V$ and $g \in G$.[3]

[3]According to the discussion of Section 1, there is no guarantee that this quotient yields a smooth manifold. One can verify that the above action is free and proper, but instead we will describe the smooth manifold structure explicitly.

There is a natural projection map $\tilde{\pi} : E(P, V) \to M$ that uses the projection map π of P:

$$\tilde{\pi}([p, v]) = \pi(p).$$

In order to show that $E(P, V)$ is a vector bundle we need to specify the vector space structure of $\tilde{\pi}^{-1}(x)$ and the local trivialisations to some Cartesian product. We start with the fibrewise structure.

Observe that each fibre of $E(P, V)$ naturally inherits the vector space structure of V:

$$\begin{aligned} [p, v] + [p, u] &= [p, u + v], \\ \lambda[p, v] &= [p, \lambda v], \text{ and} \\ 0 \in E(P, V)_x &\text{ is given by } [p, 0]. \end{aligned}$$

Technically we should show that these relations are independent on the choice of representative of each equivalence class. This is a routine check.

Regarding the local transitions of $E(P, V)$, we would like to construct some maps

$$\Phi_\alpha : \tilde{\pi}^{-1}(U_\alpha) \to U_\alpha \times V$$

using the local trivialisations $\Psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times G$ of the principal bundle P. This is quite straightforward – we define

$$\Phi_\alpha([p, v]) = (\pi(p), p_2 \circ \Psi_\alpha(p)(v)),$$

where p_2 projects onto the second factor of $U_\alpha \times G$. The function on the second coordinate of Φ_α is the vector that is obtained by mapping v under the representation of the local description of p. It is not hard to see that the map Φ_α actually acts as a linear isomorphism once restricted to fibres. As such, we may conclude that Φ_α is a vector bundle isomorphism from $E(P, V)|_{U_\alpha}$ to $U_\alpha \times V$.

The transition functions of $E(P, V)$ can be described using the transition functions of P. Indeed, suppose that we have two local trivialisations Ψ_α and Ψ_β of P. The transition functions $g_{\alpha\beta}$ will assign each point in $U_{\alpha\beta}$ a group element G. This transition map $g_{\alpha\beta}$ is obtained as the second component of the map $\Psi_\beta \circ \Psi_\alpha^{-1} : U_{\alpha\beta} \times G \to U_{\alpha\beta} \times G$.

5.2 Relationship to Frame Bundles

Suppose that we have a rank-k vector bundle E. Using the frame bundle construction, we can create a principal $GL(k)$-bundle from E. Moreover, we can use the trivial representation $id : GL(k) \to GL(k)$ to create an associated bundle $E(Fr(E), \mathbb{R}^k)$. This vector bundle will be isomorphic to E. This observation motivates the following interesting result.

Theorem 5.1. *Up to isomorphism, there is a one-to-one correspondence between rank-k vector bundles and principal $GL(k)$-bundles over M.*