Fibre Bundles and Spin Structures
Part 9: Čech Cohomology

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So far we have seen that by introducing the structure of connections on vector bundles, we can define their curvatures and glean topological information via Chern-Weil theory. This topological data was related by extracting differential forms from the curvature forms, and then by mapping it into the de Rham cohomology of M.

When we eventually discuss the (non)-existence of spin structures, we would like to repeat this philosophy of relating bundle data to topological data of M. However, in the case of spin, we require a different cohomology theory: the Čech cohomology. Roughly speaking, Čech cohomology is a series of topological invariants that arises from the combinatorial data obtained from the intersection properties of local charts on M.

There are several flavours of Čech cohomology, some more sophisticated than others. We will only present a basic description of Čech cohomology with abelian coefficients. This version of the theory will give us what we wanted: Čech cohomology over $\mathbb{Z}_2$ allows us to determine whether or not M is orientable.

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1 Defining Čech Cohomology

Čech cohomology tracks the combinatorics of functions defined on intersecting open sets of a topological space. For our purposes we will assume that the topological space in question is a smooth manifold M.

A general algorithm for cohomology can be summarised as follows.

1. Create a cochain complex $C^k(M)$, which is some series of algebraic objects generated from M. The elements of $C^k(M)$ are called k-cochains.

2. Define a map δ which homomorphically raises the degree of $C^k(M)$. The chain complex can then be organised as:

$$C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} \cdots \xrightarrow{\delta^{k-1}} C^k \xrightarrow{\delta^k} \cdots \xrightarrow{\delta^{n-1}} C^n$$

   where here we have indexed the δ map for simplicity.

3. Confirm that $\delta^{k+1} \circ \delta^k = 0$. Schematically this is written as $\delta^2 = 0$.

4. k-cocycles are defined as $Z^k(M) = Ker(\delta^k)$ and k-coboundaries are defined as $B^k(M) = Im(\delta^{k-1})$.

5. The k^{th} cohomology group is then defined to be the k-cocycles modulo the k-coboundaries of something in $C^{k-1}(M)$. Formally

$$H^k(M) := \frac{Ker(\delta^k)}{Im(\delta^{k-1})}.$$

Recall that de Rham cohomology follows this procedure – we take k-cochains to be differential k-forms on M, and the δ map is the exterior derivative. We will now do a similar thing for overlapping charts of M.

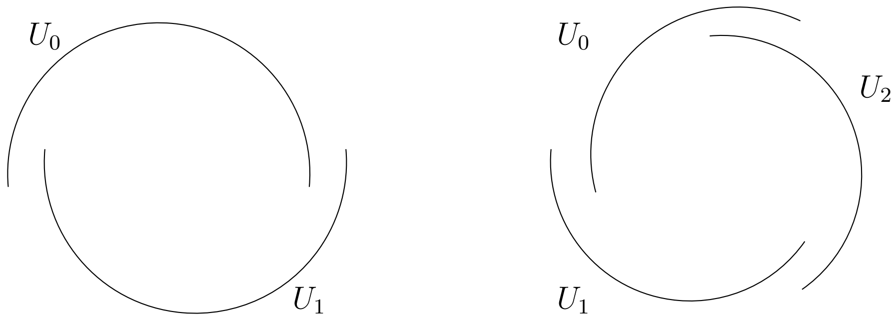


Figure 1: Two open covers of S^1. The left-hand cover is not good because the intersection U_{12} is not connected, let alone simply connected. As the right-hand-side demonstrates, this can be fixed by including an extra open set into the cover.

1.1 Good Covers

Let M be a manifold and consider some open cover $\mathcal{U}$ of M. Formally we can index $\mathcal{U}$ by:

$$\mathcal{U} = \{U_\alpha \mid \alpha \in A\}.$$

We can introduce a multi-index notation that keeps track of multiple intersections of open set:

$$U_{\alpha_0,\dots,\alpha_k} := U_{\alpha_0} \cap \dots \cap U_{\alpha_k}.$$

As a technical detail, we will require that the open cover $\mathcal{U}$ is *good*, meaning that every non-empty intersection $U_{\alpha_0,\dots,\alpha_k}$ is simply-connected. For our purposes we will gloss over this technical detail, but the quick explanation for this requirement is that it guarantees that the eventual Čech cohomology is actually independent of the choice of good cover.

We should also note that any manifold admits a good cover. The proof is technical and cumbersome, and is certainly beyond the scope of this course. The simplest case of a good cover is depicted in Figure 1. Without loss of generality we may assume that any open covers we are considering are good.

1.2 Čech Cohomology with Abelian Coefficients

Čech cohomology is defined in terms of an abelian group G and the functions f that map from elements of $\mathcal{U}$ into G. Put differently, Čech cohomology studies the structure of the G-valued functions defined on $\mathcal{U}$. We should note that there are many flavours of Čech cohomology, depending on what level of sophistication one wants to achieve, and what sort of restrictions one wants to impose on the functions f. We will only present the version that is required for our eventual purpose of describing spin structures.

Our presentation of Čech cohomology will follow the algorithm outlined previously. We will follow these steps in order. To simplify our notation, at this stage we will assume that G is an additive abelian group, that is, we denote the group operation of G by $+$.

A degree-k cochain is a collection of functions

$$\check{f} = \{f_{\alpha_0,...,\alpha_k} : U_{\alpha_0,...,\alpha_k} \to G \mid U_{\alpha_0,...,\alpha_k} \neq \emptyset\},$$

where we assume that $\alpha_0, ..., \alpha_k$ form an ordered subset of A. In addition, we suppose that each $f_{\alpha_0,...,\alpha_k}$ is a constant map. We denote the space of all k-cochains by $\check{C}^k(\mathcal{U}, G)$. Observe that $\check{C}^k(\mathcal{U}, G)$ inherits the group structure of G by defining the operation pointwise:

$$(f + f')_{\alpha_0,...,\alpha_k}(x) = f_{\alpha_0,...,\alpha_k}(x) + f'_{\alpha_0,...,\alpha_k}(x).$$

Therefore the identity element of $\check{C}^k(\mathcal{U}, G)$ is the collection f in which each $f_{\alpha_0,...,\alpha_k}$ is the constant function mapping everything in $U_{\alpha_0,...,\alpha_k}$ to the identity element of G.

There is a natural function that raises the degree of k-cochains by one. We define $\delta : \check{C}^k(\mathcal{U}, G) \to \check{C}^{k+1}(\mathcal{U}, G)$ function-wise as

$$(\delta f)_{\alpha_0,...,\alpha_{k+1}} := \sum_i (-1)^i f_{\alpha_0,...,\hat{\alpha_i},...,\alpha_{k+1}},$$

where here the $\hat{\alpha}_i$ notation denotes the exclusion of this entry.

Example 1.1. *For low degree cochains, the δ map acts as:*

$$\begin{aligned}(\delta f)_{\alpha_0\alpha_1} &= f_{\alpha_1} - f_{\alpha_0}\\ (\delta f)_{\alpha_0\alpha_1\alpha_2} &= f_{\alpha_0\alpha_1} - f_{\alpha_0\alpha_2} + f_{\alpha_1\alpha_2},\\ (\delta f)_{\alpha_0\alpha_1\alpha_2\alpha_3} &= f_{\alpha_0\alpha_1\alpha_3} - f_{\alpha_0\alpha_1\alpha_2} + f_{\alpha_1\alpha_2\alpha_3} - f_{\alpha_0\alpha_2\alpha_3}.\end{aligned}$$

Our next step is to confirm that δ is a group homomorphism. This follows easily from the definition of the group structure on the cochains.

Lemma 1.2. *The map $\delta : \check{C}^k(\mathcal{U}, G) \to \check{C}^{k+1}(\mathcal{U}, G)$ is a group homomorphism.*

Proof. This can be computed directly. We have that

$$\begin{aligned}\delta(f + f')_{\alpha_0,\ldots,\alpha_{k+1}} &= \sum_i (-1)^i (f + f')_{\alpha_0,\ldots,\hat{\alpha}_i,\ldots,\alpha_{k+1}}\\ &= \sum_i (-1)^i \left(f_{\alpha_0,\ldots,\hat{\alpha}_i,\ldots,\alpha_{k+1}} + f'_{\alpha_0,\ldots,\hat{\alpha}_i,\ldots,\alpha_{k+1}}\right)\\ &= \left(\sum_i (-1)^i f_{\alpha_0,\ldots,\hat{\alpha}_i,\ldots,\alpha_{k+1}}\right) + \left(\sum_i (-1)^i f'_{\alpha_0,\ldots,\hat{\alpha}_i,\ldots,\alpha_{k+1}}\right)\\ &= (\delta f)_{\alpha_0,\ldots,\alpha_{k+1}} + (\delta f')_{\alpha_0,\ldots,\alpha_{k+1}}.\end{aligned}$$

If f is the identity in $\check{C}^k(\mathcal{U}, G)$, this means that f sends everything to 0, the identity of G. Therefore we have that

$$\begin{aligned}(\delta f)_{\alpha_0,\ldots,\alpha_{k+1}} &= \sum_i (-1)^i (f)_{\alpha_0,\ldots,\hat{\alpha}_i,\ldots,\alpha_{k+1}}\\ &= \sum_i 0 + 0 + \ldots + 0\\ &= 0,\end{aligned}$$

so δf is the zero element in $\check{C}^{k+1}(\mathcal{U}, G)$. □

We would now like to confirm that the δ operator squares to zero. In order to see this, we start with a few low-k examples.

Example 1.3. *Suppose that f is a 1-cochain. We can compute the Čech differential twice:*

$$\begin{aligned}(\delta^2 f)_{\alpha_0\alpha_1\alpha_2} &= (\delta f)_{\alpha_0\alpha_1} - (\delta f)_{\alpha_0\alpha_2} + (\delta f)_{\alpha_1\alpha_2} \\ &= (f_{\alpha_1} - f_{\alpha_0}) - (f_{\alpha_2} - f_{\alpha_0}) + (f_{\alpha_2} - f_{\alpha_1}) \\ &= 0.\end{aligned}$$

Similarly, for 2-cochains we have that

$$\begin{aligned}(\delta^2 f)_{\alpha_0\alpha_1\alpha_2\alpha_3} &= (\delta f)_{\alpha_0\alpha_1\alpha_3} - (\delta f)_{\alpha_0\alpha_1\alpha_2} + (\delta f)_{\alpha_1\alpha_2\alpha_3} - (\delta f)_{\alpha_0\alpha_2\alpha_3} \\ &= (f_{\alpha_0\alpha_1} - f_{\alpha_0\alpha_3} + f_{\alpha_1\alpha_3}) - (f_{\alpha_0\alpha_1} - f_{\alpha_0\alpha_2} + f_{\alpha_1\alpha_2}) \\ &\quad + (f_{\alpha_0\alpha_1} - f_{\alpha_1\alpha_3} + f_{\alpha_2\alpha_3}) - (f_{\alpha_0\alpha_2} - f_{\alpha_0\alpha_3} + f_{\alpha_2\alpha_3}) \\ &= 0.\end{aligned}$$

The general idea is that we will always introduce two omissions of indices which carry the opposite sign. This can be used to prove the following general result.

Lemma 1.4. *The map $\delta^2 : \check{C}^k(\mathcal{U}, G) \to \check{C}^{k+2}(\mathcal{U}, G)$ sends everything to 0.*

Proof. By direct computation we have that

$$\begin{aligned}(\partial^2 f)_{\alpha_0,\ldots,\alpha_{k+2}} &= \sum_i (-1)^i (\partial f)_{\alpha_0,\ldots,\hat{\alpha_i},\ldots,\alpha_{k+1}} \\ &= \sum_{i,j} (-1)^i (-1)^j (\partial f)_{\alpha_0,\ldots,\hat{\alpha_i},\ldots,\hat{\alpha_j}\ldots,\alpha_{k+1}} \\ &= \left(\sum_{i,j} (-1)^i (-1)^j f_{\alpha_0,\ldots,\hat{\alpha_i},\ldots,\hat{\alpha_j}\ldots,\alpha_{k+2}} \right) \\ &\quad + \left(\sum_{i,j} (-1)^i (-1)^{j+1} f_{\alpha_0,\ldots,\hat{\alpha_j},\ldots,\hat{\alpha_i}\ldots,\alpha_{k+2}} \right) \\ &= 0.\end{aligned}$$

□

Since δ is a group homomorphism that squares to 0, we can use it to define cocycles and coboundaries:

- The k-cocycles are those k-cochains that are mapped to zero under δ. We denote the space of all Čech k-cocycles by $\check{Z}^k(\mathcal{U}, G)$. By definition, $\check{Z}^k(\mathcal{U}, G) = Ker(\delta^k)$.

- The k-coboundaries are those k-cochains that are equal to the image of some $(k-1)$-cochain under δ. We denote the space of all Čech k-coboundaries by $\check{B}^k(\mathcal{U}, G)$. By definition $\check{B}^k(\mathcal{U}, G) = Im(\delta^{k-1})$.

Finally, we define the k^{th} Čech cohomology group to be the quotient of k-cocycles by k-coboundaries. We will denote the k^{th} Čech cohomology group by $\check{H}^k(\mathcal{U}, G)$. We have that

$$\check{H}^k(\mathcal{U}, G) := \frac{\{k - \text{cocycles}\}}{\{k - \text{coboundaries}\}} = \frac{\check{Z}^k(\mathcal{U}, G)}{\check{B}^k(\mathcal{U}, G)} = \frac{Ker(\delta^k)}{Im(\delta^{k-1})}.$$

Since we are working with a good open cover, we may conclude that the resulting Čech cohomology groups are independent of the choice of $\mathcal{U}$, and thus may be written $\check{H}^k(M, G)$.

Remark 1.5. *In the case that we take M to be compact and G to be the additive real group $\mathbb{R}$, we can obtain an isomorphism $H^k(M, \mathbb{R}) \cong H^k_{dR}(M)$. The proof is very interesting and is certainly worth a look. See Chapter 8 of Bott/Tu.*

2 The First Stiefel-Whitney Class

We will now study the Čech cohomology with $\mathbb{Z}_2$ coefficients. Although $\mathbb{Z}_2$ is the simplest non-trivial abelian group, it is interesting enough to yield some useful characteristic classes of M. In this section we will introduce the first of these characteristic classes, namely the *first Stiefel-Whitney* class. We will see that this is intimately related to the notion of orientability of M.

Note that we will now switch to the multiplicative notation for abelian groups. This means that the identity element is now denoted 1, the group operation is multiplication, and the group inverse is now $^{-1}$. In particular, all summations presented in Section 1 are replaced with products.

2.1 Defining $w_1(E)$

Suppose now that we have a Riemannian manifold M. Pick some good open cover $\mathcal{U}$ that is also a trivialisation of TM. Since M is Riemannian we may regard all of the transition maps $g_{\alpha\beta}$ of U as mapping into the orthogonal group $O(n)$. This amounts to considering only orthonormal frames induced from $\mathcal{U}$.

As with our discussion of characteristic classes from last week, we will now use this bundle data to define some element of the cohomology of M. This time we will use the Čech cohomology $\check{H}^1(M, \mathbb{Z}_2)$. Observe that by assumption we are working with a good cover, so the cohomologies are related by

$$\check{H}^1(\mathcal{U}, \mathbb{Z}_2) = \check{H}^1(M, \mathbb{Z}_2).$$

To begin with we will define some element $\check{f}$ in $\check{C}^1(\mathcal{U}, \mathbb{Z}_2)$. This amounts to specifying a constant function $f_{\alpha\beta} : U_{\alpha\beta} \to \mathbb{Z}_2$. Since we are working with $O(n)$ transition maps, we can use their determinants. Indeed, we define

$$f_{\alpha\beta}(x) = \det(g_{\alpha\beta}) = \pm 1$$

for all x in $U_{\alpha\beta}$. Since this map is continuous and $U_{\alpha\beta}$ is connected, it follows that $f_{\alpha\beta}$ is actually constant on all of $U_{\alpha\beta}$. Therefore $\check{f}$ is a 1-cochain.

Lemma 2.1. *The cochain $\check{f}$ defined above is a cocyle.*

Proof. Fix some $f_{\alpha\beta}$. We can use basic properties about the determinant together with the cocycle condition $g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = 1$. To argue this directly. We have:

$$\begin{aligned}
(\delta f)_{\alpha\beta\gamma} &= f_{\alpha\beta}\,(f_{\alpha\gamma})^{-1}\,f_{\beta\gamma} \\
&= \det(g_{\alpha\beta})\det(g_{\alpha\gamma})^{-1}\det(g_{\beta\gamma}) \\
&= \det(g_{\alpha\beta})\det(g_{\beta\gamma})\det(g_{\alpha\gamma})^{-1} \\
&= \det(g_{\alpha\beta})\det(g_{\beta\gamma})\det(g_{\gamma\alpha}) \\
&= \det(g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha}) \\
&= \det(1) \\
&= 1,
\end{aligned}$$

as required. □

It follows that $\check{f}$ can be passed into the cohomology using its equivalence class:

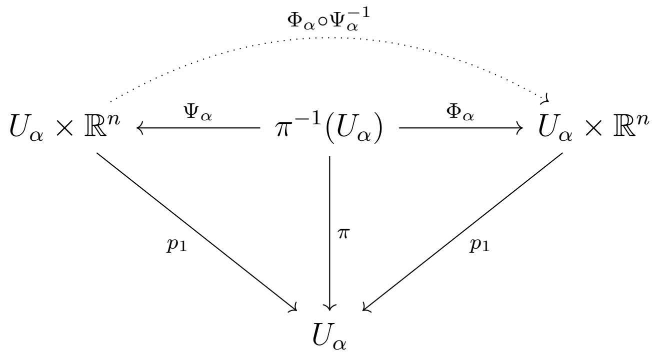
$$[\check{f}] \in \check{H}^1(M, \mathbb{Z}_2).$$

2.2 New Trivialisations from Old

We will now take a brief aside and discuss how to create new trivialisations from old ones. Suppose that we have a rank-n vector bundle E and an open trivialising cover $\mathcal{U}$, with local trivialisations denoted by Ψ_α and transition maps denoted by

$$g_{\alpha\beta} : U_{\alpha\beta} \to GL(n).$$

Suppose we have another collection of trivialisations also defined on $\mathcal{U}$, say Φ_α with transition maps $\tilde{g}_{\alpha\beta}$. For each U_α, we then have the following commutative diagram.



Observe the similarity between this diagram and that which we used to derive the transition maps between overlapping local trivialisations. Since the diagram commutes, the map $\Phi_\alpha \circ \Psi_\alpha^{-1}$ is really encoded in its behaviour in the second component. We have that $\Phi_\alpha \circ \Psi_\alpha^{-1}(x, v) = (x, h_\alpha(x)v)$, where $h_\alpha(x)$ is some invertible linear map. This means that $h_\alpha : U_\alpha \to GL(n)$. The family of these h_α maps can be used to describe the relationship between $g_{\alpha\beta}$ and $\tilde{g}_{\alpha\beta}$. We have that:

$$\begin{aligned}
(x, \tilde{g}_{\alpha\beta}(x)v) &= \Phi_\beta \circ \Phi_\alpha^{-1}(x, v) \\
&= \Phi_\beta \circ (\Psi_\beta^{-1} \circ \Psi_\beta) \circ (\Psi_\alpha^{-1} \circ \Psi_\alpha) \circ \Phi_\alpha^{-1}(x, v) \\
&= (\Phi_\beta \circ \Psi_\beta^{-1}) \circ (\Psi_\beta \circ \Psi_\alpha^{-1}) \circ (\Psi_\alpha \circ \Phi_\alpha^{-1})(x, v) \\
&= (\Phi_\beta \circ \Psi_\beta^{-1}) \circ (\Psi_\beta \circ \Psi_\alpha^{-1})(x, h_\alpha(x)v) \\
&= (\Phi_\beta \circ \Psi_\beta^{-1})(x, g_{\alpha\beta} \circ h_\alpha(x)v) \\
&= (x, h_\beta^{-1} \circ g_{\alpha\beta} \circ h_\alpha(x)v).
\end{aligned}$$

It follows that $\tilde{g}_{\alpha\beta} = h_\beta^{-1} \circ g_{\alpha\beta} \circ h_\alpha$. As a matter of fact, we can always reverse-engineer a new collection of local trivialisations using $g_{\alpha\beta}$ and the maps h_α. This is done by defining $\Phi_\alpha := (id, h_\alpha) \circ \Psi_\alpha$. The resulting collection of Φ_α will be local trivialisations of E, and moreover their transition maps will be of the form $h_\beta^{-1} \circ g_{\alpha\beta} \circ h_\alpha$. We can use this idea to prove the following.

Theorem 2.2. *The class $w_1(M)$ is independent of the choice of local trivialisation of TM.*

Proof. Suppose that Φ_α is some collection of local trivialisations of TM that are also defined on $\mathcal{U}$. Denote by $\tilde{g}_{\alpha\beta}$ the transition maps of the Ψ_α. Consider the class $[\check{f}']$ defined from the determinants of the transition maps $\tilde{g}_{\alpha\beta}$. In order to show that this class coincides with $[\check{f}]$, we need to show that $\check{f}$ and $\check{f}'$ differ by the coboundary of some 0-cochain. In multiplicative notation, this means we need to argue the existence of some $\check{h}$ in $\check{C}^0(\mathcal{U}, G)$ such that

$$f'_{\alpha\beta} = f_{\alpha\beta}(\delta h)_{\alpha\beta}.$$

According to the discussion preceeding this theorem, the desired $\check{h}$ is readily available. Since $\tilde{g}_{\alpha\beta}$ and $g_{\alpha\beta}$ are two trivialisations over the same open cover, there exists a family of functions $\tilde{h}_\alpha : U_\alpha \to O(n)$ such that $\tilde{g}_{\alpha\beta} = \tilde{h}_\beta^{-1} \circ g_{\alpha\beta} \circ \tilde{h}_\alpha$. We can define $h_\alpha : U_\alpha \to \mathbb{Z}_2$ as $h_\alpha = \det(\tilde{h}_\alpha)$. It then follows that

$$\begin{aligned} f'_{\alpha\beta} &= \det(\tilde{g}_{\alpha\beta}) \\ &= \det(\tilde{h}_\beta^{-1} \circ g_{\alpha\beta} \circ \tilde{h}_\alpha) \\ &= \det(\tilde{h}_\beta^{-1}) \det(g_{\alpha\beta}) \det(\tilde{h}_\alpha) \\ &= h_\beta^{-1} \det(g_{\alpha\beta}) h_\alpha \\ &= (h_\beta^{-1} h_\alpha) \det(g_{\alpha\beta}) \\ &= (\delta h)_{\alpha\beta} f_{\alpha\beta} \end{aligned}$$

as required. □

2.3 Orientations on M

Recall that an orientation on a manifold M is a choice of globally non-vanishing section of the top exterior bundle $\bigwedge^n(TM)$. This is a line bundle, so up to sign there are two possible orientations (if any). According to our discussion of vector bundles, an orientation of M amounts to reducing the structure group of TM from $GL(n)$ to $SL(n)$. If we also assume that M is Riemannian then we can further reduce the structure group of TM to $SO(n)$. The following result makes this precise.

Lemma 2.3. *A Riemannian manifold M is orientable iff there exists a cover of M inducing local trivialisations Ψ_α of TM whose transition maps $g_{\alpha\beta}$ take image in $SO(n)$.*

This result allows us to characterise the orientability of M in terms of the first Stiefel-Whitney class $w_1(M)$.

Theorem 2.4. *A Riemannian manifold M is orientable if and only if $w_1(M)$ vanishes.*

Proof. Suppose first that M is oriented. This amounts to reducing the structure group of TM to $SO(n)$. Therefore, M admits a cover $\mathcal{U}$ consisting of local trivialisations whose transition functions take image in $SO(n)$. Without loss of generality, we may assume $\mathcal{U}$ is also a good cover. Consider $w_1(M)$ computed according to $\mathcal{U}$. Since all the transition maps $g_{\alpha\beta}$ take image in $SO(n)$, in particular they have positive determinant. Computing $\check{f}$, we have:

$$f_{\alpha\beta} = \det(g_{\alpha\beta}) = +1.$$

Therefore $\check{f}$ consists of all the constant functions which map to the identity in $\mathbb{Z}_2$. Put differently, $\check{f} = 1$ and thus $w_1(M) = [\check{f}] = [1]$.

For the converse, suppose that $w_1(M)$ is trivial. Suppose that $\mathcal{U}$ is any good cover of M that trivialises TM. Let us denote the local trivialisations by $\Phi_\alpha : \pi^{-1}(U_\alpha) \to U \times \mathbb{R}^n$. With Lemma 2.3 in mind, we would like to use $w_1(M)$ to create a new set of local trivialisations Ψ of TM whose transition maps are all of positive determinant.

Consider the transition maps $g_{\alpha\beta} : U_{\alpha\beta} \to O(n)$. Since $w_1(M)$ is trivial, the class $[f]$ defined from $\mathcal{U}$ and $g_{\alpha\beta}$ is also trivial. Therefore $\check{f}$ is exact, that is, there exists some h in $\check{C}^0(\mathcal{U}, \mathbb{Z}_2)$ such that $\check{f} = \delta\check{h}$. By definition this means that

$$f_{\alpha\beta} = \det(g_{\alpha\beta}) = h_\beta^{-1} h_\alpha.$$

We can use these maps $h_\alpha : U_\alpha \to \mathbb{Z}_2$ to create a new trivialising cover in which trivialisations are now given by $\tilde{g}_{\alpha\beta} = \tilde{h}_\beta^{-1} \circ g_{\alpha\beta} \circ \tilde{h}_\alpha$, where the $\tilde{h}$ are a family of functions $\tilde{h}_\alpha : U_\alpha \to O(n)$ such that $\det(\tilde{h}_\alpha) = h_\alpha$ for all α. We then have that

$$\begin{aligned}
\det(\tilde{g}_{\alpha\beta}) &= \det(\tilde{h}_\beta^{-1} \circ g_{\alpha\beta} \circ \tilde{h}_\alpha) \\
&= \det(\tilde{h}_\beta^{-1}) \det(g_{\alpha\beta}) \det(\tilde{h}_\alpha) \\
&= h_\beta^{-1} \det(g_{\alpha\beta}) h_\alpha \\
&= (h_\beta^{-1} h_\alpha) \det(g_{\alpha\beta}) \\
&= f_{\alpha\beta} \det(g_{\alpha\beta}) \\
&= \det(g_{\alpha\beta}) \det(g_{\alpha\beta}).
\end{aligned}$$

Since $\det(g_{\alpha\beta}) = \pm 1$, the determinant of $\tilde{g}_{\alpha\beta}$ is guaranteed to be $+1$. We may thus use the trivialisations defined according to $\tilde{g}_{\alpha\beta}$ to reduce the structure group of TM to $SO(n)$. □