SPECIAL RELATIVITY FINAL EXAM – SOLUTION OF QUESTION 1

The 4-current of the 4-momentum of the laser's photons is given by the stress-energy tensor $T^{\mu\nu} = j^\mu p^\nu$, where j^μ is the 4-current for the number of photons, and p^μ is the 4-momentum of each photon. j^μ and p^μ are pointing in the same lightlike direction, e.g. $(1, 1, 0, 0)$. In other words, for photons, momentum is the same as energy, and current is the same as density (in $c = 1$ units, of course). The momentum current (= momentum density = energy density = energy current) of the incoming photons is thus $j^x p^x = j^t p^x = j^t p^t = j^x p^t$.

Now, the energy per unit time emitted by the laser, i.e. $j^x p^x$ integrated over yz area, is P. If the wall was at rest, then the momentum per unit time of the photons hitting it would be P. Since the photons are reflected in the opposite direction, the total momentum transfer per unit time (i.e. the force) would be $2P$. However, the wall is not at rest!

Let's consider first the wall's reference frame. As we switch from the earth's frame to the lab's frame, we must boost both j^μ and p^μ by the boost angle:

$$\eta = \operatorname{arctanh} v = \frac{1}{2} \ln \frac{1+v}{1-v} \ . \tag{1}$$

As a result, both j^μ (the photon 4-current) and p^μ (the 4-momentum of each photon) are multiplied by e^η. Thus, in the wall's reference frame, the momentum per unit time of the incoming photons is not P, but $e^{2\eta} P$. The overall force from the reflection is thus:

$$F_{\text{wall frame}} = 2e^{2\eta} P = \frac{1+v}{1-v} \cdot 2P \ . \tag{2}$$

Now, a fun fact I wasn't aware of is that the force does not depend on reference frame (this is only true for motion in 1+1 dimensions). Indeed, it can be expressed as:

$$F = \epsilon_{\mu\nu} \frac{dp^\mu}{d\tau} \frac{dx^\nu}{d\tau} \ , \tag{3}$$

where τ is proper time, an $\epsilon_{\mu\nu}$ is the Levi-Civita tensor in 1+1d.

Alternatively, we can find the force in the Earth's reference frame directly. Here, the momentum per unit time of photons hitting a <u>stationary target</u> would be P. However, due to the wall's motion, the number of photons hitting it per unit time is greater by a factor of $1 + v$, where 1 is the photons' velocity (note that this factor has nothing to do with relativity – the same would happen with non-relativistic motion). Now, what about the reflected photons? In the wall's frame, their momentum is equal and opposite to that of the incoming ones, both being e^η times the photons' original momentum in the Earth's frame. However, when we boost back into Earth's frame, this has different effects on photons traveling in different directions: the momentum of the incoming ones is decreased back to its original value, but the momentum of the reflected ones is increased by another factor of e^η! Thus, overall, the reflected photons in the Earth's frame have $e^{2\eta}$ times the momentum of the incoming ones. Putting everything together, we obtain the force as:

$$F_{\text{Earth frame}} = (1+v)(1+e^{2\eta})P = \frac{1+v}{1-v}\cdot 2P \ . \tag{4}$$

Let us now stay in the Earth's frame, since it is at inertial one. The force is related to the change in the wall's boost parameter via:

$$F = -\frac{dp^x}{dt} = -\frac{d}{dt}(M\sinh\eta) = -M\cosh\eta\,\frac{d\eta}{dt} \ . \tag{5}$$

Plugging in our expression (2) or (4) for the force, we get:

$$dt = -\frac{M}{2P}e^{-2\eta}\cosh\eta\,d\eta = -\frac{M}{4P}(e^{-\eta}+e^{-3\eta})d\eta \ . \tag{6}$$

Integrating from η down to 0, we get the total time as:

$$t = \frac{M}{12P}\left(4-3e^{-\eta}-e^{-3\eta}\right) = \frac{M}{3P}\left(1-\frac{(1+v/2)\sqrt{1-v}}{(1+v)^{3/2}}\right) \ . \tag{7}$$