SPECIAL RELATIVITY FINAL EXAM

Question 1. *A wall of unspeakable evil with mass M is approaching Earth, at a frighteningly relativistic initial velocity v. The planet's greatest scientific minds have decided to strike at it with a laser beam. The laser's power output is P, and the beam hits the wall perpendicularly. The wall is perfectly reflective.*

A) Find the force exerted by the laser beam on the wall when it first hits, both in the wall's reference frame and in the Earth's reference frame.

B) How much time, in the Earth's reference frame, is required to stop the wall and hurl it back into the blackness of space?

Question 2. *Consider the Maxwell equations:*

$$\partial_\nu F^{\mu\nu} = j^\mu \ ; \quad \partial_{[\mu} F_{\nu\rho]} = 0 \ . \tag{1}$$

To gain a deeper understanding of them, let us define the so-called dual electromagnetic field strength:

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \ , \tag{2}$$

where $\epsilon^{\mu\nu\rho\sigma}$ is the totally antisymmetric Levi-Civita tensor with $\epsilon^{txyz} = 1$.

A) Recall the decomposition $F^{ti} \equiv E^i$, $\frac{1}{2}\epsilon^{ijk} F_{jk} \equiv B^i$ of $F_{\mu\nu}$ into electric and magnetic parts. Perform a similar decomposition on $\tilde{F}_{\mu\nu}$, and express $(\tilde{\mathbf{E}}, \tilde{\mathbf{B}})$ in terms of $(\mathbf{E}, \mathbf{B})$.

B) Rewrite the second equation in (1) using $\tilde{F}_{\mu\nu}$ instead of $F_{\mu\nu}$.

C) Generalize the Maxwell equations to a world with not only a 4-current j^μ of electric charge, but also a 4-current J^μ of magnetic charge (such charges are known as magnetic monopoles). Note that the potential A_μ no longer exists, in defiance of some "axioms" from our course. Nevertheless, such a system can emerge from an underlying Yang-Mills theory.

Question 3. *Let us convert vector indices into spinor indices, via:*

$$v^\mu = (v^t, v^x, v^y, v^z) \quad \longrightarrow \quad v^{\alpha\dot{\alpha}} = \begin{pmatrix} v^t + v^z & v^x - iv^y \\ v^x + iv^y & v^t - v^z \end{pmatrix} \ . \tag{3}$$

Also, let us raise and lower spinor indices via:

$$\psi_\alpha = \psi^\beta \epsilon_{\beta\alpha} \ ; \quad \psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta \ ; \quad \psi_{\dot{\alpha}} = \psi^{\dot{\beta}}\epsilon_{\dot{\beta}\dot{\alpha}} \ ; \quad \psi^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\psi_{\dot{\beta}} \ , \tag{4}$$

where the left-handed and right-handed spinor metrics can be chosen as:

$$\epsilon_{\alpha\beta} = \epsilon^{\alpha\beta} = \epsilon_{\dot{\alpha}\dot{\beta}} = \epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ . \tag{5}$$

A) Express the scalar product $u_\mu v^\mu$ of two vectors using spinor indices, i.e. in terms of $u^{\alpha\dot{\alpha}}$ and $v^{\alpha\dot{\alpha}}$.

As discussed in class, the conversion into spinor indices of an electromagnetic field strength $F^{\mu\nu}$ takes the form:

$$F^{\mu\nu} \ \rightarrow \ F^{\alpha\dot{\alpha}\beta\dot{\beta}} = f^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}} + \bar{f}^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta} \ , \tag{6}$$

where $\bar{f}^{\dot{\alpha}\dot{\beta}}$ is the complex conjugate of $f^{\alpha\beta}$.

B) Write the corresponding decomposition of the dual field strength $\tilde{F}^{\mu\nu}$ from (2), i.e. express $\tilde{f}^{\alpha\beta}$ in terms of $f^{\alpha\beta}$.

Question 4. *Here, we pull together results from the previous two questions.*

A) Convert the Maxwell equations as formulated in Question 2B into spinor form. Show that they become a single (complex) equation!

B) Do the same for Question 2C, i.e. in the presence of magnetic charges.