A Differential geometry refresher

This is meant to be a concise summary of those parts of differential geometry which are useful to know for this course. It is probably not the best reference for learning these concepts for the first time; there are many other references which would be more suitable for that. For example:

- *Geometry, topology and physics*, Nakahara.
- *Topology and geometry for physics*, Eschrig.

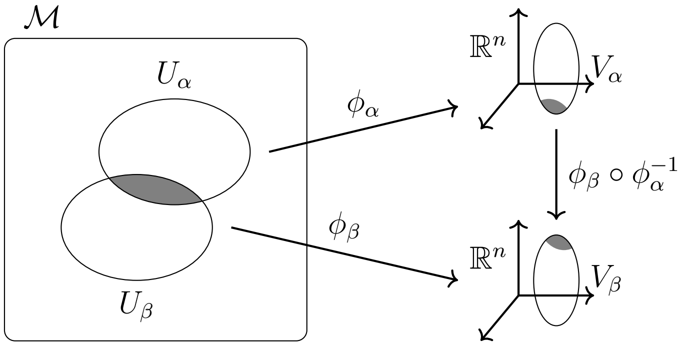
Or for a more mathematical perspective:

- *Foundations of differential geometry*, Kobayashi and Nomizu.

A.1 Manifolds

An n-manifold is a space which locally looks like $\mathbb{R}^n$. To be more precise:

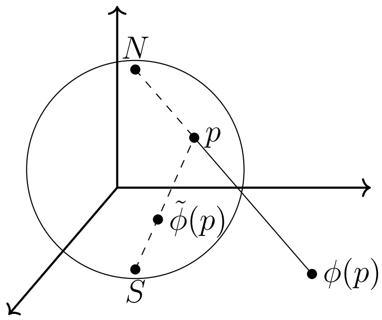
Definition. An n-dimensional (smooth) *manifold* is a set $\mathcal{M}$ together with a collection of open sets U_α, $\alpha = 1, 2, \ldots$, such that the U_α cover $\mathcal{M}$, and there exist bijections $\phi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^n$ (called *charts*) such that $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is smooth.



Informally, we can say that M is a topological space with some extra structure that allows us to introduce a differential calculus. The set of all charts is sometimes called an *atlas* or *differential structure.*

Example. The trivial manifold is $\mathcal{M} = \mathbb{R}^n$ with a single chart, for example the identity.

Example. The n-dimensional sphere $S^n = \{\mathbf{r} \in \mathbb{R}^{n+1} \text{ s.t. } |\mathbf{r}| = 1\} \subset \mathbb{R}^{n+1}$ is an n-manifold. We choose open sets $U = S^n \setminus N, \tilde{U} = S^n \setminus S$ where $N = \{0, \ldots, 0, 1\}$, $S = \{0, \ldots, 0, -1\}$, and define charts as follows:



$$\phi(\mathbf{r}) = \left(\frac{r_1}{1 - r_{n+1}}, \ldots, \frac{r_n}{1 - r_{n+1}}\right) = (x_1, \ldots, x_n) \text{ on } U$$

$$\tilde{\phi}(\mathbf{r}) = \left(\frac{r_1}{1 + r_{n+1}}, \ldots, \frac{r_n}{1 + r_{n+1}}\right) = (\tilde{x}_1, \ldots, \tilde{x}_n) \text{ on } \tilde{U}$$

Note that:

$$x_1^2 + \cdots + x_n^2 = \frac{r_1^2 + \cdots + r_n^2}{(1 - r_{n+1})^2} = \frac{1 - r_{n+1}^2}{(1 - r_{n+1})^2} = \frac{1 + r_{n+1}}{1 - r_{n+1}} \tag{A.1}$$

So:

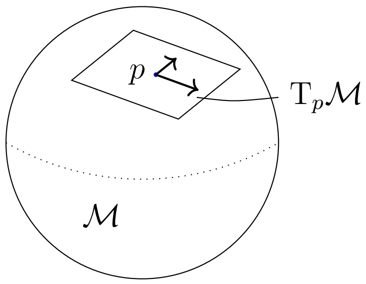
$$\tilde{x}_k = \frac{1 - r_{n+1}}{1 + r_{n+1}} x_k = \frac{x_k}{x_1^2 + \cdots + x_n^2} \tag{A.2}$$

is smooth on $U \cap \tilde{U}$.

A.2 Tangent vectors

Let $\mathcal{M}$ be a smooth manifold of dimension n, and let p be a point in $\mathcal{M}$. Since $\mathcal{M}$ is locally like $\mathbb{R}^n$, we can introduce a set of coordinates $\{x^i\}, i = 1, \ldots, n$ into an open subset of the manifold, with origin at p.

Definition. The tangent space to $\mathcal{M}$ at p, denoted $\mathrm{T}_p\mathcal{M}$, is the n-dimensional vector space spanned by the differential operators $\{\frac{\partial}{\partial x^j}\}, j = 1, \ldots, n$.



Since $\mathrm{T}_p\mathcal{M}$ is an n-dimensional real vector space, it is homeomorphic to $\mathbb{R}^n$.

Suppose $f : \mathcal{M} \to \mathbb{R}$ is a function on $\mathcal{M}$, and let $V = V^i \frac{\partial}{\partial x^i} \in \mathrm{T}_p(\mathcal{M})$. The action of V on f is defined as

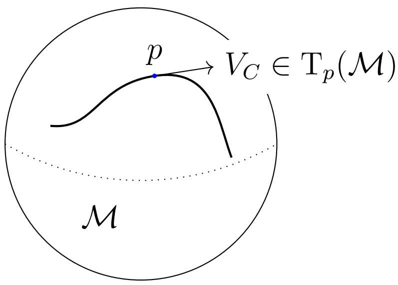
$$V(f) = V^i \frac{\partial f}{\partial x^i}\bigg|_{x=0}. \tag{A.3}$$

Consider a smooth curve on $\mathcal{M}$,

$$C : \mathbb{R} \to \mathcal{M}, \quad t \mapsto x^i(t). \tag{A.4}$$

Suppose this curve goes through p at $t = 0$. We can associate a tangent vector at p with C in the following way:

$$V_C = \dot{x}^i(0) \frac{\partial}{\partial x^i} \in \mathrm{T}_p(\mathcal{M}) \text{ where } \dot{x}^i = \frac{\mathrm{d}x^i}{\mathrm{d}t}. \tag{A.5}$$



If we let V_C act on a function f we can see that it is simply the total derivative of f along the curve at p:

$$V_C(f) = \dot{x}^i \frac{\partial}{\partial x^i} f(x) \bigg|_{x=0} = \frac{\mathrm{d}x^i}{\mathrm{d}t} \frac{\partial f}{\partial x^i} \bigg|_{x=0} = \frac{\mathrm{d}f}{\mathrm{d}t} \bigg|_p . \tag{A.6}$$

In physics, often C is the trajectory of a particle or other dynamical object, and V_C represents its velocity vector.

Definition. Let X be a vector field, and $p \in \mathcal{M}$. The *integral curve* of X through p is defined as the curve through p whose tangent vector at every point q (on the curve) is X_q.

Let λ be an integral curve of X with $\lambda(0) = p$, and (x^μ) be a coordinate chart. Then we have

$$\frac{\mathrm{d}x^\mu(\lambda(t))}{\mathrm{d}t} = X^\mu(x^\alpha(\lambda(t))) \quad \text{and} \quad x^\mu(\lambda(0)) = x^\mu_p . \tag{A.7}$$

ODE theory guarantees the existence and uniqueness of a solution. Therefore the integral curve of X through $p \in \mathcal{M}$ exists and is unique.

Example. Let $X = \frac{\partial}{\partial x^1} + x^1 \frac{\partial}{\partial x^2}$, and $x^\mu(p) = (0, \ldots, 0)$. Then we have

$$\frac{\mathrm{d}x^1}{\mathrm{d}t} = 1, \quad \frac{\mathrm{d}x^2}{\mathrm{d}t} = x^1, \tag{A.8}$$

which has solution

$$x^1 = t, \quad x^2 = \frac{1}{2}t^2, \quad x^i = 0 \text{ for } i > 2. \tag{A.9}$$

Definition. The *tangent bundle* of a manifold $\mathcal{M}$ is the union of all of its tangent spaces:

$$\mathrm{T}\mathcal{M} = \bigcup_{p \in \mathcal{M}} \mathrm{T}_p\mathcal{M}. \tag{A.10}$$

The tangent bundle is an example of a fibre bundle, which is informally a space which is 'locally' a product space:

Definition. A *fibre bundle* is a *total space* E, a *base space* B, a map $\pi : E \to B$, and a *fibre* F, such that for every $x \in B$, and every sufficiently small neighbourhood U of x, there is a homeomorphism $\varphi : U \times F \to \pi^{-1}(U)$ obeying $\pi \circ \varphi(y, a) = y$ for all $y \in U$, $a \in F$.

Informally, the preimage $\pi^{-1}(U)$ of the open set $U \subset B$ looks like the product space $U \times F$. The preimage $\pi^{-1}(x)$ of a point $x \in B$ is known as the fibre over x.

In the case of the tangent bundle, we have $E = \mathrm{T}\mathcal{M}$, $B = \mathcal{M}$, $F = \mathbb{R}^n$, and $\pi : \mathrm{T}\mathcal{M} \to \mathcal{M}$ is defined via

$$V \in \mathrm{T}_p\mathcal{M} \implies \pi(V) = p. \tag{A.11}$$

In fact, the tangent bundle is a vector bundle:

Definition. A *vector bundle* is a fibre bundle whose fibre is a vector space, and for which the map φ given above can be chosen such that $V \mapsto \varphi(x, V)$ gives a linear isomorphism between F and $\pi^{-1}(x)$.

Essentially, a vector bundle has vector spaces for fibres, and the 'local product structure' is consistent with the linearity of the vector spaces.

Definition. A *section* of a fibre bundle (E, B, π, F) is a map $\sigma : B \to E$ such that $\pi \circ \sigma : B \to B$ is the identity. A *local section* is a map $\sigma : U \to E$ such that $\pi \circ \sigma : U \to U$ is the identity for some open set $U \subset B$.

In other words, a section of a bundle is the choice of an element $\sigma(x)$ in each fibre $\pi^{-1}(x)$ of the bundle. A local section is the same, but restricted to a subset of the base space. The space of all sections of a bundle is denoted $\Xi(E)$.

Sections (local or otherwise) of the tangent bundle are known as *vector fields*.

Definition. The *Lie bracket* of two vector fields V_1, V_2 is their commutator $[V_1, V_2]$. In other words, the action of this object on a function f is

$$[V_1, V_2](f) = V_1(V_2(f)) - V_2(V_1(f)). \tag{A.12}$$

The Lie bracket of two vector fields is also a vector field. To see this, let us write V_1, V_2 in terms of their components, so

$$V_{1,2} = V_{1,2}^i \frac{\partial}{\partial x^i} \tag{A.13}$$

for some functions $V_{1,2}^i$, called the components of $V_{1,2}$. Then we have

$$V_1(V_2(f)) - V_2(V_1(f)) = V_2^i \frac{\partial}{\partial x^i}\left(V_1^j \frac{\partial f}{\partial x^j}\right) - V_1^i \frac{\partial}{\partial x^i}\left(V_2^j \frac{\partial f}{\partial x^j}\right) \tag{A.14}$$

$$= \left(V_2^i \frac{\partial V_1^j}{\partial x^i} - V_2^i \frac{\partial V_2^j}{\partial x^i}\right)\frac{\partial f}{\partial x^j}. \tag{A.15}$$

So $[V_1, V_2]$ is indeed a vector field with components

$$[V_1, V_2]^j = V_2^i \frac{\partial V_1^j}{\partial x^i} - V_2^i \frac{\partial V_2^j}{\partial x^i}. \tag{A.16}$$

A.3 Tensors and forms

Any real vector space F has an associated dual space F^* comprised of all linear maps from F to $\mathbb{R}$.

Definition. The cotangent space to $\mathcal{M}$ at p, denoted $\mathrm{T}_p^*\mathcal{M}$, is the n-dimensional vector space dual to the tangent space $\mathrm{T}_p\mathcal{M}$. Elements of $\mathrm{T}_p^*\mathcal{M}$ are called *1-forms* or *covectors*.

Given a set of coordinates x^i near p, and the associated basis $\frac{\partial}{\partial x^i}$ of the tangent space $\mathrm{T}_p\mathcal{M}$, we have a 'dual basis' $\mathrm{d}x^i$ for T_p^* obeying

$$\mathrm{d}x^i\left(\frac{\partial}{\partial x^j}\right) = \delta_j^i = \begin{cases} 1 \text{ if } i = j, \\ 0 \text{ otherwise.} \end{cases} \tag{A.17}$$

A general 1-form $\alpha \in \mathrm{T}_p^*\mathcal{M}$ can be expanded in this basis as $\alpha = \alpha_i \, \mathrm{d}x^i$.

Definition. The *cotangent bundle* is the union of all the cotangent spaces:

$$\mathrm{T}^*\mathcal{M} = \bigcup_{p \in \mathcal{M}} \mathrm{T}_p\mathcal{M}. \tag{A.18}$$

The cotangent bundle is a vector bundle over $\mathcal{M}$. It is the 'dual bundle' to $\mathrm{T}\mathcal{M}$. A section of $\mathrm{T}^*\mathcal{M}$ is known as a 1-form field.

Given two real vector spaces F_1, F_2, we can construct a third vector space F_{12} by taking their 'tensor product',

$$F_{12} = F_1 \otimes F_2. \tag{A.19}$$

We can similarly take tensor products of vector bundles. The fibre of the resulting vector bundle is defined as the tensor product of the fibres of the two starting vector bundles. For example, $\mathrm{T}\mathcal{M} \otimes \mathrm{T}^*\mathcal{M}$ is a vector bundle over $\mathcal{M}$ whose fibre over $p \in \mathcal{M}$ is $\mathrm{T}_p\mathcal{M} \otimes \mathrm{T}^*_p\mathcal{M}$. This is known as a $(1,1)$-tensor bundle.

More generally, we can construct a vector bundle from the tensor product of r tangent bundles and s cotangent bundles, for example

$$\underbrace{\mathrm{T}\mathcal{M} \otimes \cdots \otimes \mathrm{T}\mathcal{M}}_{r} \otimes \underbrace{\mathrm{T}^*\mathcal{M} \otimes \cdots \otimes \mathrm{T}^*\mathcal{M}}_{s}. \tag{A.20}$$

This is a (r,s)-tensor bundle, and its elements are (r,s)-tensors. Its sections are (r,s)-tensor fields. (r,s) is called the rank of a tensor. Usually we will not explicitly give r and s, and just call these objects *tensors*.

Definition. An r-form α is a totally antisymmetric $(0,r)$-tensor. In other words, at $p \in \mathcal{M}$ it is a map

$$\underbrace{\mathrm{T}_p\mathcal{M} \otimes \cdots \otimes \mathrm{T}_p\mathcal{M}}_{r} \to \mathbb{R} \tag{A.21}$$

that aquires a minus sign under the interchange of any two of its arguments. The bundle of r-forms is sometimes denoted $\Omega^r\mathcal{M}$.

Definition. Let η be a p-form and ω a q-form. Their 'wedge' product (or 'exterior' product) $\eta \wedge \omega$ is a $(p+q)$-form defined by

$$\eta \wedge \omega = \frac{(p+q)!}{p!q!}\mathcal{A}(\eta \otimes \omega), \tag{A.22}$$

where $\mathcal{A}$ is an operator that totally antisymmetrises the given tensor.

One can show that $\eta \wedge \omega = (-1)^{pq}\omega \wedge \eta$, so $\eta \wedge \eta = 0$ if p is odd. Also, the wedge product is associative:

$$(\eta \wedge \omega) \wedge \chi = \eta \wedge (\omega \wedge \chi) \tag{A.23}$$

Given a dual basis $\{\mathrm{d}x^i\}$, the p-forms

$$\mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p} = p!(\mathrm{d}x^{[i_1} \otimes \cdots \otimes \mathrm{d}x^{i_p]}) \tag{A.24}$$

form a basis for the space of p-forms. We will often expand a p-form in terms of its components in this basis as

$$\eta = \frac{1}{p!}\eta_{i_1 \dots i_p}\, \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_p}\,. \tag{A.25}$$

The wedge product can be written in terms of these components as

$$(\eta \wedge \omega)_{i_1 \dots i_p j_1 \dots j_q} = \frac{(p+q)!}{p!q!}\eta_{[i_1 \dots i_p}\omega_{j_1 \dots j_q]}. \tag{A.26}$$

Definition. The *exterior derivative* of a p-form η is a $(p+1)$-form $\mathrm{d}\eta$ given by

$$(\mathrm{d}\eta)_{\mu_1\ldots\mu_{p+1}} = (p+1)\partial_{[\mu_1}\eta_{\mu_2\ldots\mu_{p+1}]} \tag{A.27}$$

Definition. The *gradient* of a function f is its exterior derivative $\mathrm{d}f = \frac{\partial f}{\partial x^i}\,\mathrm{d}x^i$.

The exterior derivative operator d maps $\Omega^p\mathcal{M}$ to $\Omega^{p+1}\mathcal{M}$. It can be shown that:

- $\mathrm{d}(\mathrm{d}\eta) = 0$.
- $\mathrm{d}(\eta\wedge\omega) = \mathrm{d}\eta\wedge\omega + (-1)^p\eta\wedge\mathrm{d}\omega$ (η a p-form).
- $\mathrm{d}(\phi^*\eta) = \phi^*\,\mathrm{d}\eta$.

Definition. A p-form η is *closed* if $\mathrm{d}\eta = 0$, and *exact* if there exists a $(p-1)$-form ω such that $\eta = \mathrm{d}\omega$.

Note that if η is exact, then it is closed. The converse is not always true in spaces with non-trivial topology. For a quick counterexample, consider the plane with the origin removed, $\mathbb{R}^2\setminus\{0\}$, with the standard Cartesian coordinates x, y. The 1-form

$$\alpha = \frac{x\,\mathrm{d}y - y\,\mathrm{d}x}{x^2+y^2} \tag{A.28}$$

satisfies $\mathrm{d}\alpha = 0$, so it is closed. However, it is not exact. We do have $\alpha = \mathrm{d}\beta$ where $\tan\beta = y/x$, but tan does not have a unique inverse, and it is not possible to pick a β that is continuous everywhere in $\mathbb{R}^2\setminus\{0\}$, so at best $\alpha = \mathrm{d}\beta$ only holds locally. This is a general property, and we have:

Lemma (Poincaré lemma)**.** *If η is closed, then for all points $r\in\mathcal{M}$ there exists a neighbourhood $\mathcal{O}$ of r and a $(p-1)$-form ω such that $\eta = \mathrm{d}\omega$ in $\mathcal{O}$.*

In other words, if η is closed, then it is locally exact.

A.4 Diffeomorphisms and Lie derivatives

Definition. Let $\mathcal{M},\mathcal{N}$ be differentiable manifolds of dimension m, n respectively. A function $\phi:\mathcal{M}\to\mathcal{N}$ is *smooth* iff $\psi_A\circ\phi\psi_\alpha^{-1}:\mathbb{R}^m\to\mathbb{R}^n$ is smooth for all charts ψ_A on $\mathcal{N}$ and ψ_α on $\mathcal{M}$.

Definition. Let $\phi:\mathcal{M}\to\mathcal{N}$, $f:\mathcal{N}\to\mathbb{R}$ be smooth. The *pullback* of f by ϕ is the map $\phi^*f:\mathcal{M}\to\mathbb{R}, p\mapsto\phi^*f(p) = f(\phi(p))$.

Definition. The *pushforward* of a curve $\lambda: I\to\mathcal{M}$ is $\phi_*\lambda = \phi\circ\lambda: I\to\mathcal{N}, t\to\phi(\lambda(t))$.

Definition. Let $p\in\mathcal{M}$, and let $X\in\mathcal{T}_p(\mathcal{M})$ be the tangent vector of a curve $\lambda: I\to\mathcal{M}$. The *pushforward* of X by ϕ is $\phi_*X\in\mathcal{T}_{\phi(p)}(\mathcal{N})$, defined as the tangent of $\phi_*\lambda$ at $\phi(x)$.

Lemma. *Let $X\in\mathcal{T}_p(\mathcal{M})$ and $f:\mathcal{N}\to\mathbb{R}$. Then $(\phi_*X)(f) = X(\phi^*f)$.*

Proof. Let $\lambda(0) = p$. We have

$$(\phi_*X)(f)|_{\phi(p)} = \left[\frac{\mathrm{d}}{\mathrm{d}t}(f\circ(\phi\circ\lambda))(t)\right]_{t=0} = \left[\frac{\mathrm{d}}{\mathrm{d}t}((f\circ\phi)\circ\lambda)(t)\right]_{t=0} = X(\phi^*f). \tag{A.29}$$

□

Definition. Let $\phi : \mathcal{M} \to \mathcal{N}$ be smooth, $p \in \mathcal{M}$ and $\eta \in \mathcal{T}^*_{\phi(p)}(\mathcal{N})$. The *pullback* of η by ϕ is $\phi^*\eta \in \mathcal{T}^*_p(\mathcal{M})$, defined by $(\phi^*\eta)(X) = \eta(\phi_* X)$ for all $X \in \mathcal{T}_p(\mathcal{M})$.

Lemma. *Let $f : \mathcal{N} \to \mathbb{R}$. The gradient of f at $\phi(p)$ is $\mathrm{d}f \in \mathcal{T}^*_{\phi(p)}(\mathcal{N})$. We have $\phi^*(\mathrm{d}f) = \mathrm{d}(\phi^* f)$.*

Proof. Let $X \in \mathcal{T}_p(\mathcal{M})$. Then $(\phi^*(\mathrm{d}f))(X) = \mathrm{d}f\,(\phi_* X) = (\phi_* X)(f) = X(\phi^* f) = [\mathrm{d}(\phi^* f)](X)$. □

Let x^μ be coordinates on $\mathcal{M}$ and y^α coordinates on $\mathcal{N}$. $\phi : \mathcal{M} \to \mathcal{N}$ defines a map $x^\mu \mapsto y^\alpha(x^\mu)$. One can show that for a vector $X \in \mathcal{T}_p(\mathcal{M})$ we have

$$(\phi_* X)^\alpha = \left.\frac{\partial y^\alpha}{\partial x^\mu}\right|_p X^\mu, \tag{A.30}$$

and for a 1-form $\eta \in \mathcal{T}^*_P(\mathcal{N})$ we have

$$(\phi^*\eta)_\mu = \left.\frac{\partial y^\alpha}{\partial x^\mu}\right|_p \eta_\alpha. \tag{A.31}$$

Note that we have not required that ϕ is invertible, so we only have:

	pushforward	pullback
functions	✗	✓
curves	✓	✗
vectors	✓	✗
1-forms	✗	✓

$p \in \mathcal{M}$ was arbitrary, so the pushforward similarly applies to vector fields and the pullback to covector fields.

We can define the pullback of a $(0, s)$ tensor S by

$$(\phi^* S)(X_1, \dots, X_s) = S(\phi_*(X_1), \dots, \phi_*(X_s)) \quad \text{for all} \quad X_1, \dots, X_s \in \mathcal{T}_p(\mathcal{M}). \tag{A.32}$$

Similarly we can define the pushforward of a $(r, 0)$ tensor T by

$$(\phi_* T)(\eta_1, \dots, \eta_r) = T(\phi^*\eta_1, \dots, \phi^*\eta_r) \quad \text{for all} \quad \eta_1, \dots, \eta_r \in \mathcal{T}^*_p(\mathcal{M}). \tag{A.33}$$

These have components given by

$$(\phi^* S)_{\mu_1 \dots \mu_s} = \left.\frac{\partial y^{\alpha_1}}{\partial x^{\mu_1}}\right|_p \cdots \left.\frac{\partial y^{\alpha_s}}{\partial x^{\mu_s}}\right|_p S_{\alpha_1 \dots \alpha_s}, \tag{A.34}$$

$$(\phi_* T)^{\alpha_1 \dots \alpha_r} = \left.\frac{\partial y^{\alpha_1}}{\partial x^{\mu_1}}\right|_p \cdots \left.\frac{\partial y^{\alpha_r}}{\partial x^{\mu_r}}\right|_p S^{\mu_1 \dots \mu_r}. \tag{A.35}$$

Example. Let $\mathcal{M} = S^2$, $\mathcal{N} = \mathbb{R}^3$. Use (θ, ϕ) spherical coordinates on S^2 and let $\phi : \mathcal{M} \to \mathcal{N}, p(\theta, \phi) \mapsto y^\alpha = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. Let g be a Euclidean metric on $\mathbb{R}^3$. $g_{\alpha\beta} = \delta_{\alpha\beta}$ in cartesian coordinates, and the pullback of g onto S^2 is $(\phi^* g)_{\mu\nu} = \mathrm{diag}(1, \sin^2\theta)$.

Definition. A map $\phi : \mathcal{M} \to \mathcal{N}$ is a *diffeomorphism* if ϕ is 1-to-1, onto, smooth, and has a smooth inverse. $\mathcal{M}$ and $\mathcal{N}$ must have the same dimension.

Definition. Let $\phi : \mathcal{M} \to \mathcal{N}$ be a diffeomorphism and T a (r, s) tensor on $\mathcal{M}$. The *pushforward* of T under ϕ is the (r, s) tensor on $\mathcal{N}$ given by

$$\phi_* T(\eta_1, \dots, \eta_r, X_1, \dots, X_s) = T(\phi^* \eta_1, \dots, \phi^* \eta_r, (\phi^{-1})_* X_1, \dots, (\phi^{-1})_* X_s) \tag{A.36}$$

for all $\eta_i \in \mathcal{T}^*_{\phi(p)}(\mathcal{N})$ and $X_i \in \mathcal{T}_{\phi(p)}(\mathcal{N})$. The *pullback* of S an (r, s) tensor on $\mathcal{N}$ is defined to be $\phi^* S = (\phi^{-1})_* S$.

Pushforwards and pullbacks enable us to compare tensors at different $p, q \in \mathcal{M}$.

Definition. A diffeomorphism $\phi : \mathcal{M} \to \mathcal{M}$ is a *symmetry transformation* of a tensor field T if $\phi_* T = T$ everywhere. An *isometry* is a symmetry transformation of the metric.

Definition. Let X be a vector field on a manifold $\mathcal{M}$. Define $\phi_t : \mathcal{M} \to \mathcal{M}, p \mapsto q =$ a point a parameter distance t along the integral curve of X through p.

For small enough t, ϕ_t can be shown to be a diffeomorphism. ϕ_0 is the identity, and we have $\phi_s \circ \phi_t = \phi_{s+t}$, $\phi_{-t} = (\phi_t)^{-1}$. If ϕ_t is a diffeomorphism for all $t \in \mathbb{R}$, then we can define for ap.pdfll $p \in \mathcal{M}$ the curve $\lambda_p : \mathbb{R} \to \mathcal{M}, t \mapsto \phi_t(p)$.

Definition. The *Lie derivative* of a tensor T along a vector field X at $p \in \mathcal{M}$ is

$$(\mathcal{L}_X T)_p = \lim_{t \to 0} \frac{[(\phi_{-t})_* T]_p - T_p}{t}. \tag{A.37}$$

$\mathcal{L}_X$ maps (r, s) tensor fields to (r, s) tensor fields. If α and β are constants, then we have $\mathcal{L}_X(\alpha S + \beta T) = \alpha \mathcal{L}_X S + \beta \mathcal{L}_X T$.

Definition. Let Σ be a $(n-1)$-dimensional hypersurface in $\mathcal{M}$, and X be a vector field that is nowhere tangent to Σ. Let x^i, $i = 1, \dots, n-1$, be coordinates on Σ. Assign to $q \in \mathcal{M}$ coordinates (t, x^i) such that q is a parameter distance t along the integral curve of X through x^i on Σ. For sufficiently small t, (t, x^i) is a coordinate chart; these are *adapted coordinates.*

Note that integral curves of X have fixed x^i and parameter t, so we can write $X = \frac{\partial}{\partial t}$. The diffeomorphism ϕ_t sends points p with $x^\mu = (t_p, x^i)$ to q with $y^\mu = (t_p + t, x^i)$, and we have $\frac{\partial y^\mu}{\partial x^\nu} = \delta^\mu_\nu$. Now consider an (r, s) tensor T in these coordinates. We have

$$[((\phi_t)_* T)^{\mu \dots}{}_{\nu \dots}]_{\phi_t(p)} = \frac{\partial y^\mu}{\partial x^\rho} \dots \frac{\partial y^\sigma}{\partial x^\nu} \dots [T^{\rho \dots}{}_{\sigma \dots}]_p = [T^{\mu \dots}{}_{\nu \dots}]_p. \tag{A.38}$$

Hence

$$[((\phi_{\pm t})_* T)^{\mu \dots}{}_{\nu \dots}]_p = [T^{\mu \dots}{}_{\nu \dots}]_{\phi_{\mp t}(p)}, \tag{A.39}$$

and so at p with (t_p, x^i), in this chart we have

$$(\mathcal{L}_X T)^{\mu \dots}{}_{\nu \dots} = \lim_{t \to 0} \frac{1}{t} [T^{\mu \dots}{}_{\nu \dots}(t_p + t, x^i) - T^{\mu \dots}{}_{\nu \dots}(t_p, x^i)] = \frac{\partial}{\partial t} T^{\mu \dots}{}_{\nu \dots}(t, x^i) \bigg|_{t_p}. \tag{A.40}$$

It follows that we have a Leibniz rule

$$\mathcal{L}_X(S \otimes T) = (\mathcal{L}_X S) \otimes T + S \otimes (\mathcal{L}_X T), \tag{A.41}$$

and also that $\mathcal{L}_X$ commutes with contraction.

It is useful to write down an explicit coordinate expression for the Lie derivative of a general tensor:

$$(\mathcal{L}_X T)^{\mu\dots}{}_{\nu\dots} = X^\alpha \partial_\alpha T^{\mu\dots}{}_{\nu\dots} - (\partial_\alpha X^\mu) T^{\alpha\dots}{}_{\nu\dots} - \dots + (\partial_\nu X^\alpha) T^{\mu\dots}{}_{\alpha\dots} + \dots \tag{A.42}$$

There is also a very useful expression for the Lie derivative of a differential form, called Cartan's magic formula:

$$\mathcal{L}_X \omega = \iota_X \, \mathrm{d}\omega + \mathrm{d}(\iota_X \omega) \, . \tag{A.43}$$

In this expression, ι_X is the 'contraction' operator or 'interior product', defined such that

$$(\iota_X \omega)_{\nu\rho\dots} = X^\mu \omega_{\mu\nu\rho\dots}. \tag{A.44}$$

This operator takes p-forms to $(p-1)$-forms.

A.5 Integration

Lemma. *Let ω be a n-form and $\{f^\mu = \mathrm{d}x^\mu\}$ a dual basis for an n-dimensional manifold $\mathcal{M}$. Then there exists a function h such that $\omega = h f^1 \wedge \dots f^n$.*

Definition. An *orientation* of an n-dimensional manifold $\mathcal{N}$ is a smooth nowhere vanishing n-form η, up to an equivalence given by $\eta \sim \eta'$ iff there exists a function $h > 0$ such that $\eta' = h\eta$.

Definition. A coordinate chart x^μ on $\mathcal{N}$ is *right-handed* relative to a given orientation η if there exists a function $h > 0$ such that $\eta = h \, \mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^\mu$.

Definition. Let $\psi = x^\mu : \mathcal{O} \subset \mathcal{N} \to \mathbb{R}^n$ be a right-handed coordinate chart, and ω a n-form. Then the integral of ω over $\mathcal{O}$ is

$$\int_{\mathcal{O}} \omega = \int_{\psi(\mathcal{O}) \subset \mathbb{R}^n} \omega_{1\dots n} \, \mathrm{d}x^1 \dots \mathrm{d}x^n \, . \tag{A.45}$$

To integrate over regions with more than one chart, we add these expressions patchwise.

This expression for the integral can be shown to be chart independent.

Definition. A diffeomorphism $\phi : \mathcal{N} \to \mathcal{N}$ is *orientation preserving* if $\phi^* \eta$ is equivalent to η for all orientations η.

If ϕ is orientation preserving, it can be shown that $\int_{\mathcal{N}} \phi^* \omega = \int_{\mathcal{N}} \omega$.

A.6 Submanifolds, Stokes' theorem

Let $\mathcal{M}, \mathcal{N}$ be orientable manifolds of dimensions $m < n$.

Definition. An *embedding* $\phi : \mathcal{M} \to \mathcal{N}$ is a smooth, 1-to-1 map such that for all $p \in \mathcal{M}$ there exists a neighbourhood $\mathcal{O}$ such that $\phi^{-1} : \phi(\mathcal{O}) \to \mathcal{M}$ is smooth. If $m = n - 1$, then $\phi(\mathcal{M})$ is a *hypersurface*.

Definition. Let $\phi[\mathcal{M}]$ be m-dimensional, and let η be an m-form on $\mathcal{N}$. Then we define the integral of η over $\phi[\mathcal{M}]$ as

$$\int_{\phi[\mathcal{M}]} \eta = \int_{\mathcal{M}} \phi^* \eta. \tag{A.46}$$

Note that if $\eta = \mathrm{d}\omega$ then we have

$$\int_{\phi[\mathcal{M}]} \mathrm{d}\omega = \int_{\mathcal{M}} \mathrm{d}\phi^* \eta \,. \tag{A.47}$$

Definition. Let $\frac{1}{2}\mathbb{R}^n = \{(x^1, \dots, x^n) \in \mathbb{R} \text{ s.t. } x^1 \leq 0\}$. A *manifold with boundary* $\mathcal{N}$ is identical to a manifold, except for that the charts now map to $\frac{1}{2}\mathbb{R}^n$. Its *boundary* is $\partial\mathcal{N} = \{p \in \mathcal{N} \text{ s.t. } x^1(p) = 0\}$, and is $(n-1)$-dimensional. $(x^2, \dots, x^n)$ is *right-handed* on $\partial\mathcal{N}$ if $(x^1, \dots, x^n)$ is right-handed on $\partial\mathcal{M}$.

Theorem (Stokes' theorem)**.** *Given an n-dimensional orientable manifold $\mathcal{N}$ with boundary $\partial\mathcal{N}$ and $(n-1)$-form η, we have*

$$\int_{\mathcal{N}} \mathrm{d}\eta = \int_{\partial\mathcal{N}} \eta. \tag{A.48}$$

Definition. $X \in \mathcal{T}_p(\mathcal{N})$ is *tangent* to $\phi(\mathcal{M})$ if there exists a curve in $\phi(\mathcal{M})$ with tangent X. $\tilde{n} \in \mathcal{T}_p^*(\mathcal{N})$ is *normal* to $\phi(\mathcal{M})$ if $\tilde{n}(X) = 0$ for all X tangent to $\phi(\mathcal{M})$.

A.7 Metric, volume and curvature

Definition. A metric g on a manifold $\mathcal{M}$ is a non-degenerate symmetric $(0,2)$-tensor.

Given a metric $g_{\mu\nu}$, we define its inverse $g^{\mu\nu}$ to be the unique symmetric $(2,0)$-tensor satisfying $g^{\mu\nu} g_{\nu\rho} = \delta^\mu_\rho$. We can lower and raise tensor indices by contracting with $g_{\mu\nu}$ and $g^{\mu\nu}$ respectively.

Definition. The volume form ϵ is defined in terms of a local set of coordinates $x^0, x^1, \dots x^{n-1}$ as

$$\epsilon = \sqrt{|\det(g_{\mu\nu})|}\, \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \dots \wedge \mathrm{d}x^{n-1} \,. \tag{A.49}$$

Definition. An (r,s)-tensor density of weight m is an object with the same index structure as an (r,s)-tensor, but whose components under a diffeomorphism $\phi : x^\mu \mapsto y^\alpha(x^\mu)$ transform as

$$(\phi_* T)^{\alpha\dots}{}_{\beta\dots} = \left(\det\left(\frac{\partial x^\rho}{\partial y^\gamma}\right)\right)^m \frac{\partial y^\alpha}{\partial x^\mu} \dots \frac{\partial x^\nu}{\partial y^\beta} \dots T^{\mu\dots}{}_{\nu\dots} \,. \tag{A.50}$$

An (r,s)-pseudotensor density of weight m is similar but obeys

$$(\phi_* T)^{\alpha\dots}{}_{\beta\dots} = \operatorname{sign}\left(\det\left(\frac{\partial x^\sigma}{\partial y^\delta}\right)\right)\left(\det\left(\frac{\partial x^\rho}{\partial y^\gamma}\right)\right)^m \frac{\partial y^\alpha}{\partial x^\mu} \dots \frac{\partial x^\nu}{\partial y^\beta} \dots T^{\mu\dots}{}_{\nu\dots} \,. \tag{A.51}$$

An (r,s)-pseudotensor is just an (r,s)-pseudotensor of weight 0. A p-pseudoform is an antisymmetric $(0,p)$-pseudotensor. A pseudoscalar is a $(0,0)$-pseudotensor.

Example. The volume form is a n-pseudoform.

Definition. Given a metric $g_{\mu\nu}$, the Christoffel symbols are

$$\Gamma^\mu_{\nu\rho} = \frac{1}{2} g^{\mu\sigma}\Big(\partial_\rho g_{\nu\sigma} + \partial_\nu g_{\rho\sigma} - \partial_\sigma g_{\nu\rho}\Big). \tag{A.52}$$

These are the components of the Levi-Civita connection, which is *not* a tensor.

Definition. The covariant derivative of a (r,s)-tensor $T^{\mu\dots}{}_{\nu\dots}$ is a $(r,s+1)$ tensor

$$\nabla_\rho T^{\mu\dots}{}_{\nu\dots} = \partial_\rho T^{\mu\dots}{}_{\nu\dots} + \Gamma^\mu_{\rho\sigma} T^{\sigma\dots}{}_{\nu\dots} + \dots - \Gamma^\sigma_{\rho\nu} T^{\mu\dots}{}_{\sigma\dots} - \dots \,. \tag{A.53}$$

The Levi-Civita connection is the unique torsion-free (i.e. $\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\nu\mu}$) connection on $\mathrm{T}\mathcal{M}$ obeying $\nabla_\mu g_{\nu\rho} = 0$.

Definition. The Riemann curvature tensor $R^\mu{}_{\nu\rho\sigma}$ is defined such that

$$R^\mu{}_{\nu\rho\sigma} X^\rho Y^\sigma Z^\nu = X^\rho \nabla_\rho (Y^\sigma \nabla_\sigma Z^\mu) - Y^\sigma \nabla_\sigma (X^\sigma \nabla_\sigma Z^\mu). \tag{A.54}$$

In terms of the Christoffel symbols, we have

$$R^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} - \partial_\sigma \Gamma^\mu_{\nu\rho} + \Gamma^\mu_{\rho\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\mu_{\sigma\lambda}\Gamma^\lambda_{\nu\rho}. \tag{A.55}$$

The Ricci curvature is $R_{\nu\sigma} = R^\mu{}_{\nu\mu\sigma}$. The Ricci scalar is $R = g^{\nu\sigma} R_{\nu\sigma}$. The Einstein tensor is $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$.

Sometimes it is useful to use the notation $\nabla_X = X^a \nabla_a$.

A.8 Hodge star

Given a metric g, there is a natural inner product on p-forms/p-pseudoforms given by

$$(\alpha, \beta) = \frac{1}{p!} \alpha^{\mu_1 \dots \mu_p} \beta_{\mu_1 \dots \mu_p}. \tag{A.56}$$

Definition. The Hodge star or Hodge dual operator $*$ maps p-forms (or p-pseudoforms) to $(n-p)$-pseudoforms (respectively $(n-p)$-forms), and is defined by

$$\lambda \wedge * \mu = (\lambda, \mu) \epsilon \tag{A.57}$$

for all pairs of p-forms/p-pseudoforms λ, μ.

In components, we have

$$(*\lambda)_{\nu_1 \dots \nu_{n-p}} = \frac{(-1)^t}{p!} \epsilon_{\nu_1 \dots \nu_{n-p}}{}^{\mu_1 \dots \mu_p} \lambda_{\mu_1 \dots \mu_p}, \tag{A.58}$$

where t is the number of minus signs in the signature of the metric; usually the signature is $-+++\dots$ so $t = 1$. Taking the Hodge dual twice gives the original form up to a minus sign:

$$**\lambda = (-1)^{t + p(n-p)} \lambda. \tag{A.59}$$

The Hodge dual of a 0-form f (i.e. a function) is

$$* f = f \epsilon. \tag{A.60}$$

A.9 Useful formulae

We have defined several operators acting on forms:

$$\mathcal{L}_X, \quad \iota_X, \quad \mathrm{d}, \quad *, \tag{A.61}$$

where X is a vector field. These operators have the following useful commutation and anti-commutation relations:

$$[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X,Y]}, \tag{A.62}$$
$$[\mathcal{L}_X, \iota_Y] = \iota_{[X,Y]}, \tag{A.63}$$
$$[\mathcal{L}_X, \mathrm{d}] = 0, \tag{A.64}$$
$$\{\iota_X, \mathrm{d}\} = \mathcal{L}_X, \tag{A.65}$$
$$\{\iota_X, \iota_Y\} = 0, \tag{A.66}$$
$$[\mathcal{L}_X, *] = \sum_\mu (\mathrm{d}x^\mu \wedge)\iota_{h_\mu} *, \tag{A.67}$$

where in the last line the vectors h_μ are defined such that

$$(h_\mu)^\nu = \nabla_\mu X^\nu + \nabla^\nu X_\mu - \frac{1}{2}\delta^\nu_\mu \nabla_\rho X^\rho. \tag{A.68}$$