Statistical Approach to Neural Learning and Population Coding
— From Mathematical Neuroscience

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Extended Abstract

## 1. Mathematical Neuroscience

The brain is an organ of storing, organizing and processing information. There are many approaches to understand the mechanisms of the brain,—microscopic and macroscopic as well as bottom-up and top-down approaches. Mathematical neuroscience is a top-down approach, where capabilities of information processing in brain-like systems are elucidated by mathematical formulations. This approach mostly treats abstract models of neural networks in order to capture the essential features of their behaviors.

The present lecture is a brief introduction to mathematical neuroscience. In the former part, we study classic theories of mathematical neuroscience, including topics of statistical characteristics of signal transformations in a layer of neurons, stochastic equations for learning and self-organization, dynamics of neural fields and their self-organization.

The latter part is devoted to more advanced topics, where information geometry is used. Population coding, Fisher information, and singular structures in population coding are discussed, and synchronization of firing is shown to give rise to higher-order interactions, without entering technical details.

## 2. Two types of signal transformation by layered networks

We begin with a very simple one-layer neural network consisting of McCulloch-Pitts neurons. Input signals are represented by a vector $\boldsymbol{x} = (x_1, \cdots, x_n)$, where $n$ is the number of input fibers; and output signals are represented by another vector $\boldsymbol{z} = (z_1, \cdots, z_m)$, where $z_i$ is the output of the $i$-th neuron. The input-output relations is simply represented by

$$z_i = f\left(\sum w_{ij} x_j - h\right)$$

or in the matrix-vector notation

$$\boldsymbol{z} = f(W\boldsymbol{x} - \boldsymbol{h})$$

Here the matrix $W = (w_{ij})$ is the connection matrix, $w_{ij}$ being the synaptic efficacy of the $i$-th neuron from the $j$-th input line. Here, $f$ is a nonlinear sigmoid function, and is the unit step function in the case of McCulloch-Pitts neurons.

Statistical neurodynamics studies the macroscopic behaviors of such networks. We consider two different types of connections: One is a random network where $w_{ij}$ is randomly and independently assigned. The other is the associative network, where $W$ is given by the outer product of a number of input-output pairs to be memorized. The pairs are randomly generated, so that this is also a random network. Their macroscopic behaviors have been studied well. Here we remark their stabilities.

The former random network has such characteristics that, when an input signal is perturbed slightly, the output signal expands the difference. Such a network can be used to study the difference among local similar patterns. The transmission of information in mossy fibers to granule cells systems are believed to have such characteristics. On the other hand, the associative network reduces the perturbation of inputs, so that it stabilizes the memory traces. This is believed to be used in the hippocampus.

## 3. Dynamics of neural fields

We study the dynamics of neural fields, which is used for decoding stimuli in population coding. Let us show a simple equation of the following form

$$\tau \frac{\partial u(\xi,t)}{\partial t} = -u(\xi,t) + \int w(\xi,\xi') f[u(\xi',t)] d\xi' + s(\xi,t)$$

where $u(\xi,t)$ denotes the average membrane potential at location $\xi$ at time $t$. The activity of neurons at around $\xi$ at $t$ is given by

$$z(\xi,t) = f[u(\xi,t)]$$

and $w(\xi,\xi')$ is the intensity of recurrent or feedback connection from neurons at location $\xi'$ to those of location $\xi$.

The dynamics has the so-called line attractor without any outside stimuli $s(\xi,t)$, and the local excitation occurs depending on the stimuli distribution. This can be used to decode information. We give a very simple but rigorous mathematical study of such systems.

4. Stochastic equations of neural learning

When an ensemble of neurons receive input signals from an information source, each neuron changes its synaptic efficacies. This is a prototype of learning and self-organization. We consider the following general learning equation of generalized Hebb type,

$$\tau \frac{ds_i(t)}{dt} = -s_i(t) + crx_i(t)$$

where $r$ is called the learning signal and $s_i$ are synaptic weights. This is a stochastic difference equation, where input signals $\boldsymbol{x}(t)$ are given randomly from the information source.

When inhibitory effects exist, the learning dynamics of a neuron is stabilized such that it becomes a representative of a specific signal in the source. This is a prototype of self-organization. When neurons in the ensemble are mutually connected, the system regulates the entire behaviors. A typical case is seen in the dynamics of self-organizing neural fields.

It should be remarked that the inhibitory effects are given, not only by the inhibitory neurons but also depression effects in STDP (spike timing dependent plasticity).

5. Neural firing and higher-order correlations

When a pool of neurons are stimulated, we have firing patterns of neurons. Let us denote such a pattern by a vector $\boldsymbol{x} = (x_1, \cdots, x_n)$. When excitation is stochastic, we specify its probability by $p(\boldsymbol{x})$. The probability tells us the nature of the neuron pool. A simple statistical quantity is the firing rate of each neuron, which represents the firing probability. The next one is the correlation of firing between two neurons. How about the third-order and higher-order interactions of firing?

We use information geometry to elucidate the structure of various orders of interactions. The interactions are decomposed orthogonally, and the generalized Pythagorian theorem holds.

6. Synchronization in population of neurons and higher-order interactions

Neurons sometimes fire synchronously, to make the effect stronger. When neurons are excited independently, we cannot see any synchronization, so that mutual interactions must exist in the case of synchrony. We show that higher-order interactions, not merely the second order, are necessary to generate synchrony, by using a very simple model of a pool of neurons.

## 7. Fisher information in population coding and singularity

We study the Fisher information in a one-dimensional neural field by using the Fourier domain. Various characteristics of population coding become clear by this approach. We then study the case where multiple stimuli are represented at the same time.

The Fisher information degenerates in such a case, and topological singularity emerges. There are difficulties in decoding information in such a case. We discuss this problem from the point of view of the standard statistical theory of estimation, as well as the Bayesian point of view. The singularity plays a major pathological role in information processing.