[QI;MP]

Solve Ordinary Differential Equations Using MATLAB

Yongping Zhang

13 Oct 2013

OIST

Outline

- What is Ordinary Differential Equations (ODEs)
- Motivation to solve ODEs
- ODE solvers in MATLAB
- Application of ODE solvers

Ordinary Differential Equations

ODE: a equation with the function of only one variable and its derivatives

$$F(x, \frac{\partial y}{\partial x}, \frac{\partial^2 y}{\partial x^2}, \cdots \frac{\partial^{n-1} y}{\partial x^{n-1}}, \frac{\partial^n y}{\partial x^n}) = 0 \qquad y = y(x)$$

n order ODE

Linear ODE:

$$\frac{\partial^2 y}{\partial x^2} = 8x\frac{\partial y}{\partial x} + x^3 y$$

Nonlinear ODE:

$$\frac{\partial^2 y}{\partial x^2} = 8x(\frac{\partial y}{\partial x})^2 + y^3$$

Partial Differential Equations: many variables and corresponding derivatives

$$F(x, t, \frac{\partial y}{\partial x}, \frac{\partial y}{\partial t}, \frac{\partial^2 y}{\partial x^2}, \frac{\partial^2 y}{\partial t^2}, \cdots) = 0$$

Motivation to solve ODEs

[QI;MP]

- ODEs broadly exist in different contexts Biology, Physics, Chemistry, etc.

PRL **111,** 053603 (2013) PHYSICAL REVIEW LETTERS

Signatures of Nonlinear Cavity Optomechanics in the Weak Coupling Regime

K. Børkje,[1] A. Nunnenkamp,[2] J. D. Teufel,[3] and S. M. Girvin[4]

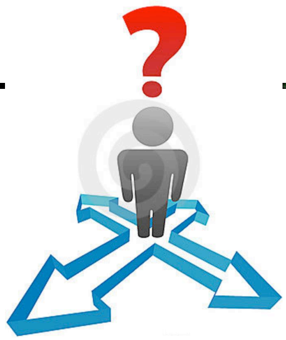
$$\dot{\hat{a}} = -\left(\frac{\kappa}{2} - i\Delta_p\right)\hat{a} - i(G + g\hat{a})(\hat{c} + \hat{c}^\dagger) + \sqrt{\kappa}\hat{a}_{\mathrm{in}},$$

$$\dot{\hat{c}} = -\left(\frac{\gamma}{2} + i\omega_m\right)\hat{c} - iG(\hat{a} + \hat{a}^\dagger) - ig\hat{a}^\dagger\hat{a} + \sqrt{\gamma}\hat{c}_{\mathrm{in}}.$$

- Usually ODEs are very complication to get analytical solutions, numerical calculation is necessary

- MATLAB provides several integrators for ODEs, which are very powerful, efficient and accuracy.

MATLAB ODEs Solvers



ode23 - Solve Nonstiff differential equation; Low order method

ode45- Solve Nonstiff differential equation; Medium order method

ode23t- Solve moderately stiff ODEs

ode113 - Solve nonstiff differential equations; variable order method

ode15i - Solve fully implicit differential equations, variable order method

ode15s - Solve stiff differential equations and DAEs; variable order method

ode23s - Solve stiff differential equations; low order method

ode45 is most popular for most of ODEs with higher accuracy and efficiency.

Use MATLAB Help

Practice (1): solve a single first order

[QI;MP]

$$\frac{\partial y}{\partial t} = -5y$$

Exact solution

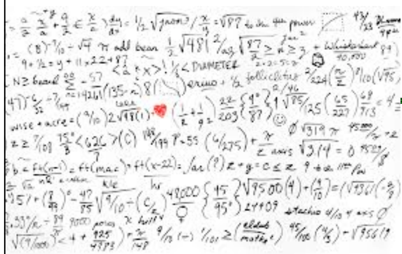
$$y(t) = y(0)\exp(-5t)$$

It is a initial value problem

Initial value of y: y(t=0)

needs Initial and last time

The equation as well!



How does ode45 work

[QI;MP]

Fourth order Runge-Kutta method. See wiki

$$\frac{dy}{dt} = f(y,t)$$

$$t_1, t_2, t_3, t_4, t_5, \cdots, t_{n-1}, t_n, t_{n+1}, \cdots$$

$$y_1, y_2, y_3, y_4, y_5, \cdots, y_{n-1}, y_n, y_{n+1}, \cdots$$

If we know $t_n - -y_n$ what is y_{n+1}

$$k_1 = f(y_n, t_n)$$

$$k_2 = f(y_n + \frac{h}{2}k_1, t_n + \frac{h}{2})$$

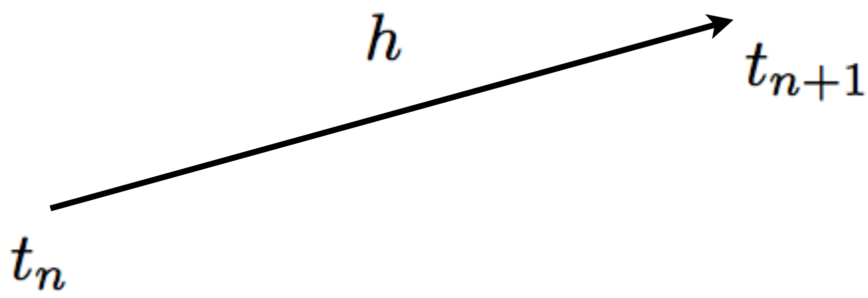
$$k3 = f(y_n + \frac{h}{2}k_2, t_n + \frac{h}{2})$$

$$k4 = f(y_n + hk_2, t_n + h)$$

$$h = t_{n+1} - t_n$$

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

How does ode45 work



Default: Relative error=10^(-3)

If $|y_{n+1} - y_n| >$ Relative error

Then change the time step h to more smaller value and do the Runge-Kutta again

ode45 changes each time step by estimating the error, and decides whether the time step is too large: automatically adapt the time step

```
tspan = 0:deltat:70;
```

deltat is meaningful just for the output results

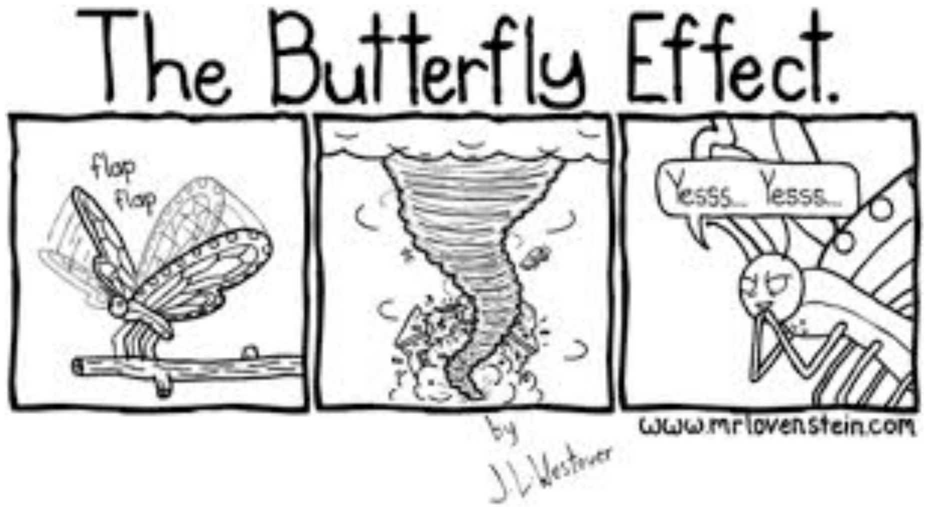
Practice (2): Three coupled first order equations

Edward Lorentz (in 1963) modeled the earth's atmosphere

$$\frac{dx}{dt} = -px + py$$

$$\frac{dy}{dt} = rx - y - xz$$

$$\frac{dz}{dt} = xy - bz$$

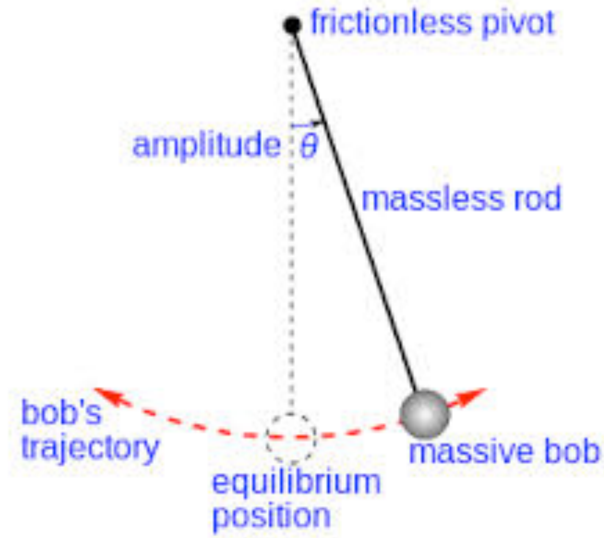


- The Matlab ode solvers will generally be better than anything you would program yourself

Practice (3): A second order equation-Pendulum

[QI;MP]

$$\frac{d^2\theta}{dt^2} = -\sin(\theta)$$



$$\frac{d\theta}{dt} = x$$
$$\frac{dx}{dt} = -\sin(\theta)$$

$$\theta = y(1), x = y(2)$$

$$\frac{dy(1)}{dt} = y(2)$$
$$\frac{dy(2)}{dt} = -\sin[y(1)]$$