OIST OKINAWA INSTITUTE OF SCIENCE AND TECHNOLOGY GRADUATE UNIVERSITY

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Lecture 2: Correlated Quantum Systems

Steve Campbell

Composite Quantum Systems

How do we describe a system of two (or more) quantum particles?

> **Postulate 4**: The state space of a composite physical system is the tensor product of the state spaces of the component physical systems. Moreover, if we have systems numbered 1 through n, and system number i is prepared in the state $|\psi_i\rangle$, then the joint state of the total system is $|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_n\rangle$.

One can understand the need for the tensor product by returning to the superposition principle. If A (B) is a subsystem spanning the Hilbert space $\{|0\rangle_A, |1\rangle_A\}$ $(\{|0\rangle_B, |1\rangle_B\})$ Then the linearity of quantum mechanics allows any superposition to also be valid, therefore we can write the state of A (B) as:

$$|\psi_A\rangle = \alpha_A|0\rangle_A + \beta_A|1\rangle_A \quad (|\psi_B\rangle = \alpha_B|0\rangle_B + \beta_B|1\rangle_B)$$

Intuition then says the joint state of system A and B should be described by the product $|\psi_A\rangle|\psi_B\rangle$. Allowing for the above quantum superpositions in this product naturally leads to the state of AB as:

$$|\Psi_{AB}\rangle = \alpha_A\alpha_B|0\rangle_A|0\rangle_B + \beta_A\alpha_B|1\rangle_A|0\rangle_B + \alpha_A\beta_B|0\rangle_A|1\rangle_B + \beta_A\beta_B|1\rangle_A|1\rangle_B$$

This tells us the state of the composite system lives in the tensor product of the Hilbert spaces:

$$\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}_{AB}$$

Composite Quantum Systems

This innocent looking result has quite remarkable implications. As we can describe the system using the tensor product of the subsystem Hilbert spaces, again by the linearity of quantum mechanics any linear combination of basis states is permitted for example:

$$|\Phi^+_{AB}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle_{AB} + |11\rangle_{AB}\right)$$

This state however has a most peculiar property. Despite simply being a linear superposition of possible states for the composite system (an exact analogy to $|\psi_A\rangle$) it cannot be expressed as a product of distinct single particle states, i.e.

$$|\Phi^+_{AB}\rangle \neq |\psi_A\rangle|\psi_B\rangle$$

"When two systems, of which we know the states by their respective representatives, enter into temporary physical interaction due to known forces between them, and when after a time of mutual influence the systems separate again, then they can no longer be described in the same way as before, that is, by endowing each of them with a representative of its own. I would not call that one but rather the characteristic trait of quantum mechanics, the one that enforces its entire departure from classical lines of thought. By the interaction the two representatives have become entangled."

Erwin Schrödinger

Quantum Entanglement

Mathematically defining entanglement is quite straight forward, if a state cannot be written in the form:

$$|\psi_{AB}\rangle = |\psi_A\rangle|\psi_B\rangle$$

Then it is an entangled state of systems A and B. There is a significant body of work devoted to developing a mathematical framework to understand quantum entanglement.

However the essence of what it 'means' to be entangled is still not understood, except to say that any attempt to describe the total system by only examining the subsystems will fail.

The collective system is more than the sum of its parts.

The main problem with understanding entanglement is the attempts to draw an analogue to something familiar (knots etc.) but entanglement is a purely quantum mechanical phenomena and so no such description is possible. Only when we abandon the notion that two (spatially separated) particles are distinct do we start to understand entangled systems.

Quantum Entanglement

When one wants to deal with understanding and exploiting quantum entanglement, often changing picture to the density matrix formalism is extremely useful. For a pure quantum state $|\Phi^+\rangle = 1/\sqrt{2}\,(|00\rangle + |11\rangle)$ the density matrix can be expressed as

$$\varrho_{AB} = |\Phi^+_{AB}\rangle\langle\Phi^+_{AB}| = $$

$$=$$

In particular, the density operator of a composite quantum system allows one to be able to determine the individual measurement statistics arising from studying any subsystem by using the partial trace.

$$\mathrm{Tr}_B(\varrho_{AB}) = \sum_i {}_B\langle i|\varrho_{AB}|i\rangle_B$$

with i are the basis vectors of subsystem B. This gives us

$$\rho_A = \mathrm{Tr}_B(\varrho_{AB}) = $$

$$=$$

Quantum Entanglement

Now we are in a position to begin to appreciate the essence of quantum entanglement and in particular how it relates to information. The original state $|\Phi^+_{AB}\rangle$ is a pure state (i.e. it is expressed in terms of a single state vector).

However the reduced state of subsystem A is given by the density operator

$$\begin{aligned}\rho_A &= \frac{1}{2}\left(|0\rangle_A\langle 0| + |1\rangle_A\langle 1|\right)\\ &= \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2}\end{pmatrix}\end{aligned}$$

Here we have a statistical mixture of the possible basis states of subsystem A. In fact, for this example by 'ignoring' system B we are unable to say anything (useful) about system A.

This can be interpreted as the 'information lost' by only examining the reduced state. (von Neumann) Entropy gives us a quantifiable answer

$$\begin{aligned}S(\varrho) &= -\varrho\log_2\varrho\\ &= -\sum \lambda_i \log_2 \lambda_i\end{aligned}$$

Where λ_i are the eigenvalues of ϱ and by convention we define $0\log_2 0 = 0$.

Quantum Entanglement

If we calculate the von Neumann entropy for the state $\varrho_{AB} = |\Phi^+_{AB}\rangle\langle\Phi^+_{AB}|$, since the state is pure it has a single eigenvalue of 1 hence

$$\begin{aligned} S(\varrho_{AB}) &= 1 \log_2 1 \\ &= 0 \end{aligned}$$

We can understand this to tell us that we are in possession of all the information available about the composite system. However looking at what happens for the reduced state

$$\begin{aligned} S(\rho_A) &= 2\left(\frac{1}{2}\log_2\frac{1}{2}\right) \\ &= 1 \end{aligned}$$

The information lost has now increased to its maximum value indicating that by examining the reduced state we have lost all information about the composite system.

The ‘information lost’ gives us a feel for what we mean when stating the composite system is more than the sum of its parts. The maximum value for von Neumann entropy tells us the original state was a maximally entangled state.

Bell's Theorem

The main problem with understanding non-locality is the attempts to draw an analogue to something familiar (knots etc.) but entanglement is a purely quantum mechanical phenomena and so no such description is possible. Only when we abandon the notion that two (spatially separated) particles are distinct do we start to understand non-local systems.

The problem is common experience tells us that the properties of a system are intrinsic. (Think: "We don't need to look at the moon to know its there") but as we have just seen looking at the reduced state of entangled systems we cannot ascribe an exact state to them only a probability.

Einstein, Podolsky and Rosen asserted that any complete physical theory should fulfill a sufficient condition that a physical property must be an element of reality, in the sense that it should be possible to predict with certainty the value of said property before measurement.

According to this condition, quantum mechanics is not a complete theory. Einstein believed there were "hidden variables" in entangled states that were needed to complete the physical description of the system.

Bell's Theorem

Enter John S. Bell who proposed a thought experiment to test the validity of EPR's arguments.

Remarkably Bell's theorem is not a result about quantum theory.

BELL'S THOUGHT EXPERIMENT

Charlie prepares two particles. It doesn't matter how he prepares the particles, just that he is capable of repeating the experimental procedure which he uses. Once he has performed the preparation, he sends one particle to Alice, and the second particle to Bob.

Once Alice receives her particle, she performs a measurement on it. Imagine that she has available two different measurement apparatuses, so she could choose to do one of two different measurements. These measurements are of physical properties which we shall label P_Q and P_R, respectively. Alice doesn't know in advance which measurement she will choose to perform. Rather, when she receives the particle she flips a coin or uses some other random method to decide which measurement to perform. We suppose for simplicity that the measurements can each have one of two outcomes, +1 or −1. Suppose Alice's particle has a value Q for the property P_Q. Q is assumed to be an objective property of Alice's particle, which is merely revealed by the measurement. Similarly, let R denote the value revealed by a measurement of the property P_R.

Similarly, suppose that Bob is capable of measuring one of two properties, P_S or P_T , once again revealing an objectively existing value S or T for the property, each taking value +1 or −1. Bob does not decide beforehand which property he will measure, but waits until he has received the particle and then chooses randomly. The timing of the experiment is arranged so that Alice and Bob do their measurements at the same time (or, to use the more precise language of relativity, in a causally disconnected manner). Therefore, the measurement which Alice performs cannot disturb the result of Bob's measurement (or vice versa), since physical influences cannot propagate faster than light.

Bell's Theorem

Now if we are interested in considering what happens with the following expression:

$$\begin{aligned} & QR + RS + RT - QT \\ = \; & (Q + R)S + (R - Q)T \end{aligned}$$

Since the values of Q, R, S, and T can only be ± 1 it follows directly that

$$QR + RS + RT - QT = \pm 2$$

Letting $p(q, r, s, t)$ be the probability that before the measurements are performed the system in a state where $Q = q, \quad R = r, \quad S = s, \quad T = t$. We can express the above quantity in terms of expectation values acquired over many repeats of the experiment:

$$|\mathbf{E}(QS) + \mathbf{E}(RS) + \mathbf{E}(RT) - \mathbf{E}(QT)| \leq 2$$

This is Bell's (CHSH) inequality.

Now, if Alice and Bob get together and examine the outcomes of their measurements EPR reasoning insists the above inequality must hold.

But for entangled states like $|\Phi^+_{AB}\rangle$ we will see we can violate this inequality!

Bell's Theorem

The observed violation means that something must be wrong in the original assumptions:

(1) The assumption that the physical properties P_Q, P_R, P_S, P_T have definite values Q, R, S, T which exist independent of observation. This is sometimes known as the assumption of *realism*.

(2) The assumption that Alice performing her measurement does not influence the result of Bob's measurement. This is sometimes known as the assumption of *locality*.

These assumptions constitute what we call local-realism. According to quantum theory one (or both) of these assumptions must be wrong.

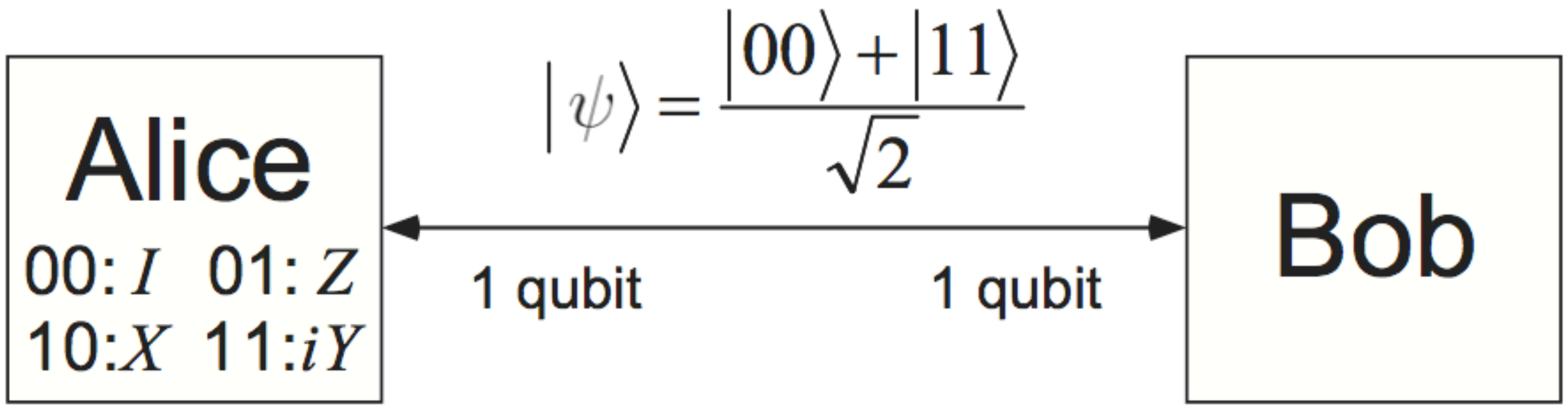
Most experts (Bell included) advocate that it is the 'local' assumption that is incorrect. Allowing us to explain such phenomena by abandoning the idea that spatially separated implies distinct systems.

But currently....we have no idea.

Superdense Coding

A concrete example of how entangled states might be useful can be found in one of the most basic areas of information processing: coding.

Imagine Alice wishes to send two bits of classical information to Bob. If they share an entangled state this can be achieved by sending only a single bit of quantum information, a ‘qubit’.



By applying single qubit operations to the system in her possession Alice can change the state of the whole system. By agreeing on a convention of what measurement implies each classical string the coding protocol can achieve its goal.

Superdense Coding

The resulting states (encoded as follows) come from the 4 possible operations Alice performs

$$00: \quad I \ \ |\psi\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

$$01: \quad Z \ \ |\psi\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}}$$

$$10: \quad X \ \ |\psi\rangle = \frac{|10\rangle + |01\rangle}{\sqrt{2}}$$

$$11: \quad iY \ |\psi\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

Once Alice sends the qubit in her possession to Bob, he performs a Bell measurement to determine what type of state he has. Once identified he knows what message was sent.

Quantum Teleportation

Correlated quantum systems offer the ability to many things that are classically impossible, like teleportation!

It relies on two reasonable assumptions: (1) Alice and Bob share an entangled pair and (2) Alice has no knowledge of the state she wishes to teleport but can send classical communications.

Note: if she knew the state then the teleportation would be superfluous. But in order to determine an arbitrary state she would require, in principle, an infinite amount of measurements

If Alice and Bob share the entangled state $|\Phi^+_{AB}\rangle = 1/\sqrt{2}\left(|00\rangle_{AB} + |11\rangle_{AB}\right)$ and the unknown state to be sent is $|\psi_t\rangle = \left(\alpha|0\rangle_t + \beta|1\rangle_t\right)$ the total state is

$$|\psi_t\rangle|\Phi^+_{AB}\rangle =$$

$$=$$

Quantum Teleportation

She now projects the two qubits in her possession onto one of the 4 possible 'Bell states'

$$|\Phi^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle \pm |11\rangle\right)$$
$$|\Psi^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle \pm |10\rangle\right)$$

Leaving the state of Bobs qubit in one of the following

$${}_{tA}\langle\Phi^{\pm}|\varphi\rangle_{tAB} = \alpha|0\rangle_B \pm \beta|1\rangle_B$$
$${}_{tA}\langle\Psi^{\pm}|\varphi\rangle_{tAB} = \beta|0\rangle_B \pm \alpha|1\rangle_B$$

And it should be clear, that up to a suitable local operation the original unknown state has been teleported to Bob.

However the state of Bobs qubit is conditional on knowing what projection Alice has performed. It can be shown that without this information Bobs qubit will be in a completely mixed state.