OIST OKINAWA INSTITUTE OF SCIENCE AND TECHNOLOGY GRADUATE UNIVERSITY

Lecture 1: Fundamental Concepts for Quantum Information

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Fundamental Concepts for QI

Quantum Information studies what you can do if you store **information** in a physical, **quantum system** and manipulate it with all the advantages and constraints of quantum mechanics.

We should therefore master the rules or quantum mechanics. Those can be condensed into 4 axioms, or **postulates**. They answer the following questions:

1. How do you describe a **system**?
2. How does a system **evolve**?
3. What happens when you do a **measurement**?
4. How do you **compose** different systems?

This lecture will focus on those points in the context of quantum information.

Fundamental Concepts for QI

Postulate 1: Associated to any isolated physical system is a complex vector space with inner product (that is, a Hilbert space) known as the *state space* of the system. The system is completely described by its *state vector*, which is a unit vector in the system's state space.

We write such a state $|\psi\rangle \in \mathcal{H}$

The smallest non trivial Hilbert space is C^2 and allows to describe a two-state system, called a *qubit*, with complex vectors of dimension 2. In matrix notations, he vector basis for a qubit is given by

$$|0\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \longrightarrow \quad |\psi\rangle = \alpha|0\rangle + \beta|1\rangle \equiv \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

Let us define the *Hermitian conjugate* of this qubit:

$$(|\psi\rangle)^\dagger \equiv \langle\psi| = \alpha^*\langle 0| + \beta^*\langle 1| \equiv \begin{pmatrix} \alpha^* & \beta^* \end{pmatrix}$$

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A Hilbert space is defined on a inner product. In the case of C^2, we have the scalar product

$$\langle\psi_1|\psi_2\rangle \equiv \begin{pmatrix} \alpha_1^* & \beta_1^* \end{pmatrix} \begin{pmatrix} \alpha_2 \\ \beta_2 \end{pmatrix} = \alpha_1^*\alpha_2 + \beta_2^*\beta_2$$

which allows us to *normalize* our state vectors

$$\langle\psi|\psi\rangle = 1 \iff |\alpha|^2 + |\beta|^2 = 1$$

There exist *operators* in the Hilbert space that act on the vector state.

$$A \equiv \sum_{i,j} a_{ij}|i\rangle\langle j| \qquad \langle i|A|j\rangle = a_{ij}$$

$$(|\phi_1\rangle\langle\phi_2|)\,|\psi\rangle = |\phi_1\rangle\langle\phi_2|\psi\rangle = \langle\phi_2|\psi\rangle|\phi_1\rangle$$

In C^2, they are 2x2 matrices $$A|\psi\rangle \equiv \begin{pmatrix} a_{0,0} & a_{0,1} \\ a_{1,0} & a_{1,1} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

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We define the *Hermitian conjugate* of the operators as

$$A^\dagger \equiv \sum_{i,j} a_{ij}^* |j\rangle\langle i| \quad \longleftrightarrow \quad A^\dagger \equiv (A^*)^T = \begin{pmatrix} a_{0,0}^* & a_{1,0}^* \\ a_{0,1}^* & a_{1,1}^* \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

as well as a few other useful things; an operator is *Hermitian* if it satisfies

$$A^\dagger = A$$

and it is *unitary* if

$$AA^\dagger = A^\dagger A = I$$

The *trace* of an operator is given by

$$\mathrm{Tr}(A) = \sum_i \langle i|A|i\rangle = \sum_i a_{ii}$$

The *commutator* and *anticommutator* of two operators are respectively defined as

$$[A, B] = AB - BA \qquad \{A, B\} = AB + BA$$

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For studying qubits, the *Pauli matrices* are absolutely **essential**. They are defined by

$$\sigma_0 \equiv \sigma_I \equiv I \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \sigma_1 \equiv \sigma_x \equiv X \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_2 \equiv \sigma_y \equiv Y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 \equiv \sigma_z \equiv Z \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Any hermitian operator in C^2 can be written as a real linear combination of these matrices. They obey the **commutation rules**

$$[\sigma_j, \sigma_k] = 2i \sum_{l=1}^{3} \epsilon_{jkl} \sigma_l$$

with ϵ_{jkl} the antisymmetric tensor.

Eigenvectors and eigenvalues of a matrix also play a crucial role in quantum mechanics. An *eigenvector* of an operator is a non zero vector such that

$$A|v\rangle = v|v\rangle$$

with *v* a complex number, called *eigenvalue*. Those may be found from the roots of the characteristic function

$$\det|A - vI| = 0$$

The diagonal representation of a matrix is given by

$$A = \sum_i v_i |v_i\rangle\langle v_i|$$

All operators may not, in general, possess such a form but Hermitian operators **always do**, and their eigenvalues are always **real**. When two eigenvalues are equal, their eigensystems are called *degenerate*.

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An important class of operators are called the *projectors*. Suppose *V* is a *d*-dimensional sub-vector space of *W*, we find an **orthonormal** basis of *V* and the projector on *V* is defined as

$$P = \sum_{i=1}^{d} |v_i\rangle\langle v_i|$$

The projector is *Hermitian*, and $P^2=P$.

It is possible to apply any **function** to an operator, the result is obtained by applying the function on the eigenvalues in the diagonal form of the operator,

$$f(A) = \sum_i f(v_i)|v_i\rangle\langle v_i|$$

With this, you can evaluate the exponential, logarithm, square root, etc, of an operator.

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Postulate 2: The evolution of a *closed* quantum system is described by a *unitary transformation*. That is, the state $|\psi\rangle$ of the system at time t_1 is related to the state $|\psi'\rangle$ of the system at time t_2 by a unitary operator U which depends only on the times t_1 and t_2,

$$|\psi'\rangle = U|\psi\rangle . \tag{2.84}$$

Postulate 2′: The time evolution of the state of a closed quantum system is described by the *Schrödinger equation*,

$$i\hbar \frac{d|\psi\rangle}{dt} = H|\psi\rangle. \tag{2.86}$$

In this equation, $\hbar$ is a physical constant known as *Planck's constant* whose value must be experimentally determined. The exact value is not important to us. In practice, it is common to absorb the factor $\hbar$ into H, effectively setting $\hbar = 1$. H is a fixed Hermitian operator known as the *Hamiltonian* of the closed system.

The specific form of the Hamiltonian depends on the physical system. In quantum information, we assume any quantum gates can be realized.

Postulate 3: Quantum measurements are described by a collection $\{M_m\}$ of *measurement operators*. These are operators acting on the state space of the system being measured. The index m refers to the measurement outcomes that may occur in the experiment. If the state of the quantum system is $|\psi\rangle$ immediately before the measurement then the probability that result m occurs is given by

$$p(m) = \langle\psi|M_m^\dagger M_m|\psi\rangle \,, \tag{2.92}$$

and the state of the system after the measurement is

$$\frac{M_m|\psi\rangle}{\sqrt{\langle\psi|M_m^\dagger M_m|\psi\rangle}} \,. \tag{2.93}$$

The measurement operators satisfy the *completeness equation*,

$$\sum M_m^\dagger M_m = I \,. \tag{2.94}$$

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The notion of *projective measurement* of an *observable* can be used to replace the measurement operators with some advantages.

Projective measurements: A projective measurement is described by an *observable*, M, a Hermitian operator on the state space of the system being observed. The observable has a spectral decomposition,

$$M = \sum_m m P_m , \tag{2.102}$$

where P_m is the projector onto the eigenspace of M with eigenvalue m. The possible outcomes of the measurement correspond to the eigenvalues, m, of the observable. Upon measuring the state $|\psi\rangle$, the probability of getting result m is given by

$$p(m) = \langle\psi|P_m|\psi\rangle . \tag{2.103}$$

Given that outcome m occurred, the state of the quantum system immediately after the measurement is

$$\frac{P_m|\psi\rangle}{\sqrt{p(m)}} . \tag{2.104}$$

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Although the result of each measurement is in general **unpredictable**, supposing that you can measure the same state many time, you will find a **predictable** average value, given by

$$\begin{aligned}\mathbf{E}(M) &= \sum_m m\, p(m) \\ &= \sum_m m \langle\psi|P_m|\psi\rangle \\ &= \langle\psi|\left(\sum_m mP_m\right)|\psi\rangle \\ &= \langle\psi|M|\psi\rangle.\end{aligned}$$

along with the variance of the distribution of your measurements

$$\begin{aligned}[\Delta(M)]^2 &= \langle(M - \langle M\rangle)^2\rangle \\ &= \langle M^2\rangle - \langle M\rangle^2.\end{aligned}$$

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Postulate 4: The state space of a composite physical system is the tensor product of the state spaces of the component physical systems. Moreover, if we have systems numbered 1 through n, and system number i is prepared in the state $|\psi_i\rangle$, then the joint state of the total system is $|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_n\rangle$.

Where the $\otimes$ represents a tensor product, which joins different vector spaces. Some of its properties are

$$(\alpha|\psi\rangle) \otimes (|\phi_1\rangle + |\phi_2\rangle) = \alpha|\psi\rangle \otimes |\phi_1\rangle + \alpha|\psi\rangle \otimes |\phi_2\rangle$$

$$(A \otimes B)(|\psi\rangle \otimes |\phi\rangle) = A|\psi\rangle \otimes B|\phi\rangle$$

in matrix notations, we have the Kronecker product

$$M = A \otimes B = \begin{pmatrix} a_{11} & \dots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mm} \end{pmatrix} \otimes B = \begin{pmatrix} a_{11}B & \dots & a_{1m}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \dots & a_{mm}B \end{pmatrix}$$

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Sometimes, the state of the system is only known probabilistically. In those cases, we have to use the *density matrix*

$$\rho \equiv \sum_{i=1}^{n} p_i |\psi_i\rangle\langle\psi_i| \qquad p_i \geqslant 0 \; \forall i, \;\; \sum_{i=1}^{n} p_i = 1$$

with no constraint on *n*. If only one p_i is non zero, we recover a *pure state*. Note that this probability distribution is very different from a coherent superposition in a pure state.

The normalization of a density matrix is obtained through a trace

$$\mathrm{Tr}(\rho) = \sum_i p_i \,\mathrm{Tr}(|\psi_i\rangle\langle\psi_i|) = \sum_i p_i = 1$$

and the eigenvalues are always positive or zero.

The density matrix evolves according to

$$\rho' = U\rho U^\dagger$$

as for measurements we have

$$p(m) = \sum_i p_i p(m|i) = \sum_i p_i \mathrm{Tr}(P_m|\psi_i\rangle\langle\psi_i|) = \mathrm{Tr}(P_m\rho)$$

$$\rho^m = \frac{P_m\rho P_m}{\mathrm{Tr}(P_m\rho)}$$

$$\langle M\rangle = \sum_m m p(m) = \sum_m m\mathrm{Tr}(P_m\rho) = \mathrm{Tr}(\rho M)$$