SU(3) spin-nematic squeezing in a collective spin-1 atomic system

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Outline

- Introduction: Squeezing in Spin-1 BECs
- Lie Algebraic Classification of Squeezing in Spin-1 BECs
- Generation of squeezed states of spin-1 BECs and their limits of squeezing
- Summary
Spin Squeezing: Squeezing in Pseudo Spin-1/2 Collective Spin Systems

Coherent Spin State (CSS)

Squeezed Spin State (SSS) [1]

$N$ pseudo spin-$\frac{1}{2}$ particles

nonlinear interactions e.g. $H = \hbar \chi (\hat{S}_x \hat{S}_y + \hat{S}_y \hat{S}_x)$, $H = \hbar \chi \hat{S}_z$

Spin Squeezing in BECs

Spinor BECs are promising candidates to realize spin squeezing

- A pseudo spin-$\frac{1}{2}$ BEC can be prepared in a CSS, i.e., a Fock state
- Versatile tools for manipulation of BECs

Quantum fluctuations of (pseudo) magnetization below the standard quantum limit (SQL) [2]

Coherent States and Observables in Spin-1 BECs

\[ |\text{Fock}\rangle = \frac{1}{\sqrt{N!}} \sum_{m=-f}^{f} c_m \hat{a}_m^\dagger |\text{vac}\rangle \]

Spin state

the spin and the nematic tensor

\[ \hat{S}_\mu \equiv (S_\mu)_{mn} \hat{a}_m^\dagger \hat{a}_n \]
\[ \hat{N}_{\mu\nu} \equiv \frac{1}{2} (S_\mu S_\nu + S_\nu S_\mu)_{mn} \hat{a}_m^\dagger \hat{a}_n \]

Ferro

BA

Polar
Nematic Squeezing in Spin-1 BECs

Observation of spin-nematic squeezing in $^{87}$Rb [4] (theory [3])

Two mode squeezing [5] $m = -1 \quad 0 \quad 1$

Motivation

Motivation

- What observables can we spin-nematic squeeze?
- To what extent can we reduce quantum fluctuations?

Approach

- Lie algebraic classification of squeezing
- Squeezing limit for each class of squeezing via one-axis twisting
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Observables in Spin-1 BECs Generate su (3) Lie algebra

Independent observables in spin-1 BECs are

\[ \{ \hat{S}_x, \hat{S}_y, \hat{S}_z, \hat{Q}_{xy}, \hat{Q}_{yz}, \hat{Q}_{zx}, \hat{D}_{xy}, \hat{Y} \} \equiv \{ \Lambda_i \} \]

\( (i = 1, \cdots, 8) \)

\[ \hat{Q}_{\mu\nu} = 2\hat{N}_{\mu\nu} \quad \hat{D}_{\mu\nu} = \hat{N}_{\mu\mu} - \hat{N}_{\nu\nu} \]

\[ \hat{Y} = (-\hat{N}_{xx} - \hat{N}_{yy} + 2\hat{N}_{zz}) / \sqrt{3} \]

A set of generators of the su(3) Lie algebra

An arbitrary observable can be expressed as

\[ \hat{A} = \sum_{i=1}^{8} c_i \Lambda_i, \quad \sum_{i=1}^{8} c_i^2 = 1 \]
Lie Algebraic Classification of Squeezing

In the case of spin squeezing,

\[ [\hat{S}_\mu, \hat{S}_\nu] = i\epsilon_{\mu\nu\lambda} \hat{S}_\lambda \]

Redistribute quantum fluctuations

\[ \text{SU}(2) \{ \hat{S}_x, \hat{S}_y, \hat{S}_z \} \]

Classification of SU(2) subalgebras in the SU(3) Lie algebra

Classification of the structure constant

\[ [\hat{A}, \hat{B}] = ir^{C}_{AB} \hat{C}, \quad \lambda \equiv |r^{C}_{AB}| \]
Two Classes of Squeezing

Two types of SU(2) subalgebras can be derived from the root diagram.

Type 1 (\(\lambda=1\))

Unitarily equivalent to \(\{S_x, S_y, S_z\}\)
- Spin squeezing

Type 2 (\(\lambda=2\))

Unitarily equivalent to \(\{D_{yz}, Q_{yz}, S_x\}\)
- Nematic squeezing
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Type 1, One-Axis Twisting

Find \( \{S'_x, S'_y, S'_z\} \) such that a coherent spin state is polarized along \( x \) direction.

\[
\{\hat{S}'_x, \hat{S}'_y, \hat{S}'_z\} = \{\hat{U}_1 \hat{S}_x \hat{U}_1\dagger, \hat{U}_1 \hat{S}_y \hat{U}_1\dagger, \hat{U}_1 \hat{S}_z \hat{U}_1\dagger\}
\]

\[
|\text{Fock}\rangle = \frac{1}{\sqrt{N!}} \hat{b}^{\dagger N} |\text{vac}\rangle
\]

\[
\hat{b}^{\dagger} = \hat{U}_1(\alpha, \beta, \gamma, \varphi) e^{-i \frac{\pi}{2} \hat{S}_y} \hat{a}_{\dagger 1} e^{i \frac{\pi}{2} \hat{S}_y} \hat{U}_1\dagger(\alpha, \beta, \gamma, \varphi)
\]

Minimum quantum fluctuations of type-1 squeezing via one-axis twisting \( H_{\text{one}} = \chi S'_z \),

\[
\langle (\Delta \tilde{S})^2 \rangle_{\text{min}} \simeq \frac{1}{4} \left( \frac{9N}{4} \right)^{1/3}
\]

\[
\tilde{S} = \sin \nu \hat{S}'_y + \cos \nu \hat{S}'_z, \quad \nu \simeq \frac{1}{2} [\arctan (N\chi t) - \chi t]
\]
Generation of a Type-1 Squeezed State from a Polar State

\[ |\text{Fock}\rangle_{\text{polar}} = \frac{1}{\sqrt{N!}} \hat{a}_0^N |\text{vac}\rangle \]

\[ \hat{U}_1(0, 0, 0, -3\pi/4) \]

\[ \{ \hat{S}_x, \hat{S}_y, \hat{S}_z \} \quad \leftrightarrow \quad \{ \hat{S}'_x, \hat{S}'_y, \hat{S}'_z \} \]

\[ = \{ \hat{D}_{yz}, (-\hat{S}_y + \hat{Q}_{xy})/\sqrt{2}, -(\hat{S}_z + \hat{Q}_{zx})/\sqrt{2} \} \]

A one-axis twisting Hamiltonian is given by \( H_{\text{one}} = \chi(S_z + Q_{zx})^2/2 \).

\[ \langle (\Delta \tilde{S})^2 \rangle_{\text{min}} \approx \frac{1}{4} \left( \frac{9N}{4} \right)^{1/3} \]

\[ \tilde{S} = \sin \nu \hat{S}'_y + \cos \nu \hat{S}'_z, \quad \nu \approx \frac{1}{2} \left[ \arctan (N \chi t) - \chi t \right] \]
Type 2, One-Axis Twisting

Find \{D_{xy}'', Q_{xy}'', S_z''\} such that a coherent spin state is polarized along \(x\) direction.

\[
\{\hat{D}_{xy}'', \hat{Q}_{xy}'', \hat{S}_z''\} = \{\hat{U}_2 \hat{D}_{xy} \hat{U}_2^\dagger, \hat{U}_2 \hat{Q}_{xy} \hat{U}_2^\dagger, \hat{U}_2 \hat{S}_z \hat{U}_2^\dagger\}
\]

\[
\hat{b}^\dagger = \hat{U}_2 (\alpha, \beta, \gamma, \varphi + \pi/4) \hat{a}_1^\dagger \hat{U}_2^\dagger (\alpha, \beta, \gamma, \varphi + \pi/4)
\]

\[
\hat{b}^\dagger e^{-i\alpha \hat{S}_z} e^{-i\beta \hat{S}_y} e^{-i\gamma \hat{S}_z} e^{-i\varphi \hat{Q}_{xy}}
\]

A one-axis twisting Hamiltonian is given by \(H_{\text{one}} = \chi S_z''^2\).

\[
\langle (\Delta \tilde{Q})^2 \rangle_{\text{min}} = \frac{1}{2} (9N)^{1/3}
\]

\[
\tilde{Q} = \sin \nu \hat{Q}_{xy}'' + \cos \nu \hat{S}_z'' \quad \nu \simeq \frac{1}{2} \left[ \arctan (N \chi t) - 2\chi t \right]
\]

A Type-2 squeezed state can be generated from a ferro satate.
Summary

- SU(2)-type squeezing in spin-1 BECs can be categorized into two classes: type 1 and type 2.
- Type-1 squeezing involves spin squeezing and type-2 does nematic squeezing.
- Both types of squeezed states can be generated via one-axis twisting.
- In both cases, the minimum quantum fluctuations are proportional to $N^{1/3}$ whereas the proportionality coefficient of type-1 squeezing is a factor of $2^{-5/3}$ smaller than that of type-2 squeezing.
Appendix
The Ground-State Phase Diagram of Spin-1 BECs\cite{sadler2006, murata2007, kawaguchi2012}

\[ i\hbar \frac{\partial \psi_\mu}{\partial t} = \left[ -\frac{\hbar^2}{2M} \nabla^2 + U(\mathbf{r}) \right] \psi_\mu + \left[ -p(F_z)_{\mu\nu} + q(N_{zz})_{\mu\nu} \right] \psi_\nu \]

\[ + \left[ c_0 \delta_{\mu\nu} + c_1 f_\lambda(F_\lambda)_{\mu\nu} \right] \rho \psi_\nu. \]

\[ c_1 < 0 \quad p/(|c_1|\rho) \quad p = q \]
\[ p = -q \quad p^2 = q(q - 2|c_1|\rho) \]

\[ c_1 = 0 \quad p = q \quad p = -q \]
\[ p^2 = 2c_1\rho q \]

\[ c_1 > 0 \quad p = q + c_1\rho/2 \]
\[ p = -q - c_1\rho/2 \]

How to Generate SSSs

- Imprint squeezing of optical light onto atoms [a2]
- Non-linear spin interactions
  - One-axis twisting [a3]
    \[ \hat{H}_{\text{one}} = \hbar \chi \hat{S}_z^2 \]
  - Two-axis counter twisting [a2]
    \[ \hat{H}_{\text{two}} = \hbar \chi \left( \hat{S}_x \hat{S}_y + \hat{S}_y \hat{S}_x \right) / 2 \]
- Josephson couplings between two levels [a4]
- Interactions between coherent light and atoms [a6]

Root Vectors of the SU(3) Lie Algebra

\[ [\hat{X}_+, \hat{X}_-] = cX_z \]
\[ \rightarrow \{(\hat{X}_+ + \hat{X}_-)/2, (\hat{X}_+ - \hat{X}_-)/2i, X_z\} \]

\[ (\pm 2, 0), \quad \frac{1}{\sqrt{2}} \left( \hat{D}_{xy} \pm i\hat{Q}_{xy} \right) \propto \hat{a}_1^\dagger \hat{a}_{-1}, \ \hat{a}_{-1}^\dagger \hat{a}_1 \]

\[ (1, \pm \sqrt{3}), \quad \frac{1}{2} \left[ \hat{S}_x \pm \hat{Q}_{zx} + i \left( \hat{S}_y \pm \hat{Q}_{yz} \right) \right] \propto \hat{a}_1^\dagger \hat{a}_0, \ \hat{a}_0^\dagger \hat{a}_{-1} \]

\[ (-1, \pm \sqrt{3}), \quad \frac{1}{2} \left[ \hat{S}_x \mp \hat{Q}_{zx} - i \left( \hat{S}_y \mp \hat{Q}_{yz} \right) \right] \propto \hat{a}_{-1}^\dagger \hat{a}_0, \ \hat{a}_0^\dagger \hat{a}_1 \]