M. Tomari 2023-03-30, Mini workshop "Various aspects of singularities"

"Some sketch for today's talk"

$f_{[L]} \in \mathbb{C}\{x, y\}$

$\operatorname{ord} f_{[L]} = 4$, Laufer type curves

Case of reducible $f_{[L]}$

$$f_{[L]} \sim (y^2 - x^{2k+1})\{(y^2 - x^{2k+1}) + \alpha x^a y^b\}$$

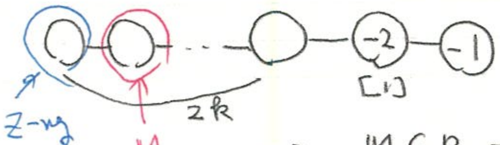
generic constant

$a \geq 0,\ b \geq 0,\ a, b \in \mathbb{Z}$

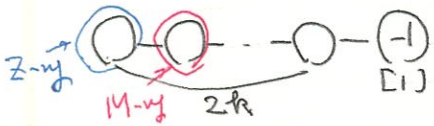
$2a + (2k+1)b = 4k + 2 + s$, $s \geq 0$

Resolution graph of $z^2 = x f_{[L]}$

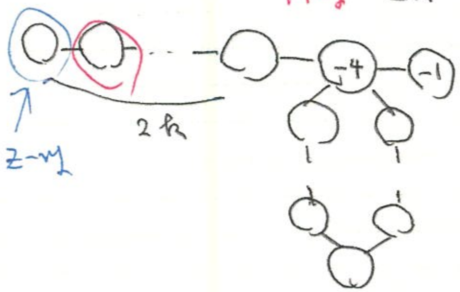
$s = 0$ MSGE = Minimal Covering Res.



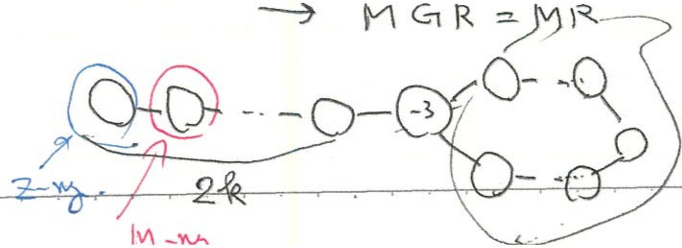
$\longrightarrow$ MGR = MR



$s \geq 1$



$\longrightarrow$ MGR = MR



$2s-1$ rat'l curves

$f_{[L]} \in \mathbb{C}\{x, y\}$

$\operatorname{ord} f_{[L]} = 4$, Laufer type curves

Case $f_{[L]}$: irreducible Puiseux Pairs $\{(2, n_1), (2, n_2)\}$

$$\begin{cases} x = t^4 & 4 = 2 \cdot 2 = m_1 \cdot m_2 \\ y = t^{2n_1} + t^{n_2} & 2n_1 < n_2 \end{cases}$$

$n_1 = 2k+1$, $n_2 = 4k + 2\lambda + 3$, $k \geq 1$, $\lambda \geq 0$.

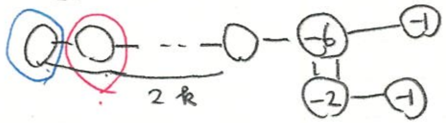
$$f_{[L]} \underset{\text{equisingular}}{=} (y^2 - x^{2k+1})^2 + \alpha x^a y^b$$

generic constant.

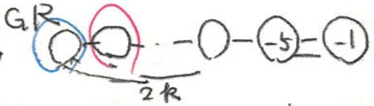
$2a + (2k+1)b = 8k + 2\lambda + 5$

Resolution Graph of $z^2 = x \cdot f_{[L]}$

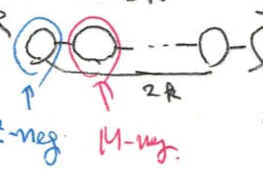
$\lambda = 0$ Minimal Cov. Res. = MSGE-resolution



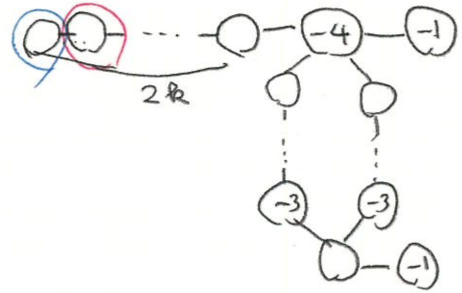
$\longrightarrow$ MGR



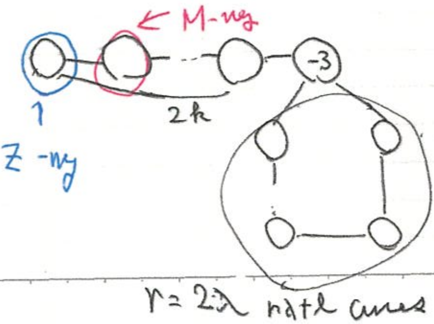
$\longrightarrow$ MR



$\lambda \geq 1$ MSGE



$\longrightarrow$ MGR = MR



$r = 2\lambda$ rat'l curves

$f_{[c]} \in \mathbb{C}\{x, y\}$

ord $f_{[c]} = 6$, contact type curve

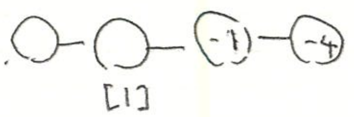
Case $f_{[c]}$ reducible

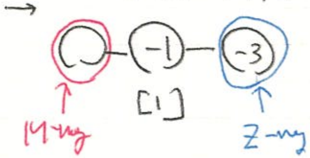
$f_{[c]} = (y^3 - x^4)\{(y^3 - x^4) + \alpha x^a y^b\}$ ← generic constant

$3a + 4b = 12 + \mu$, $\mu \geqq 0$.

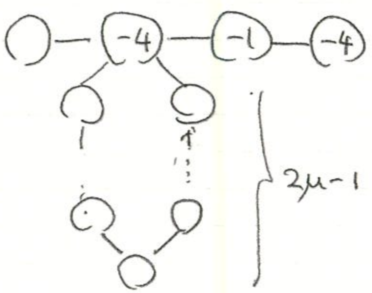
resolution graph of $z^2 = y f_{[c]}$

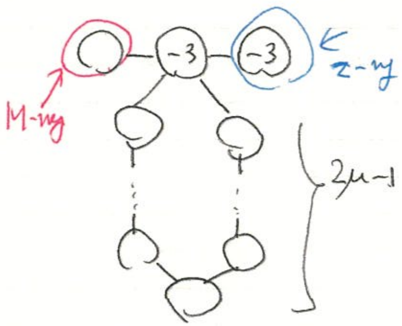
$\mu = 0$ MSGE → MGR = MR





$\mu \geqq 1$ MSGE → MGR = MR





$f_{[c]} \in \mathbb{C}\{x, y\}$

ord $f_{[c]} = 6$ contact type

Case $f_{[c]}$: irreducible

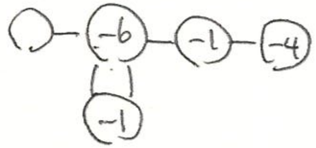
$$\begin{cases} x = t^6 \\ y = t^8 + t^{m_2}, \quad m_2 > 8, \ \& \ m_2 : \text{odd} \end{cases}$$

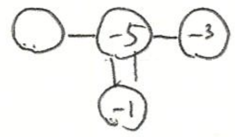
$m_2 = 2\lambda + 9 \quad \lambda \geqq 0$

$f_{[c]} = (y^3 - x^4)^2 + \alpha x^a y^b$, $3a + 4b = 25 + 2\lambda$

resolution graph of $z^2 = y \cdot f_{[c]}$

$\lambda = 0$ MSGE → MGR





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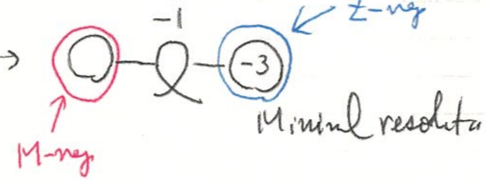
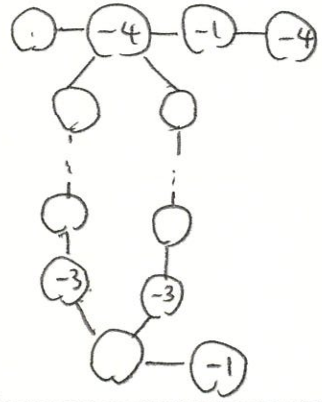
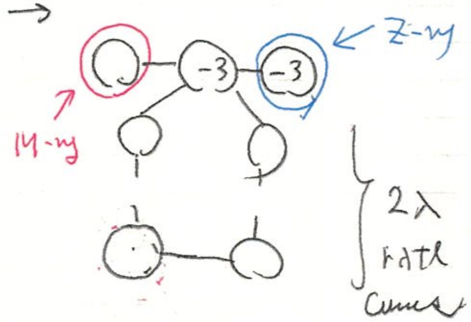


$\lambda \geqq 1$ MSGE → MGR





S.S.-T. Yau. Amer. J. Math (1979) Vol 101. pp 761–812

From Hypersurface weighted dual graphs of normal singularities of surface

Table 1 The Weighted Dual Graphs for Weakly Elliptic Singularities with $Z \cdot Z = -1$.[a]

	Dual Graph	$A_* \cdot A_*$	Equation
(1)	$\underline{El} + A_l$	$\underline{-1}$	$z^2 = y^3 + x^{6+6l}$
(2)	$\underline{N_0} + A_l$	$\begin{Bmatrix} -1, r=0 \\ -3, r>1 \end{Bmatrix}$	$z^2 = (y + x^{2+2l})(y^2 + x^{r+5+4l})$
(3)	$\underline{Cu} + A_l$	$\underline{-1}$	$z^2 = y^3 + x^{7+6l}$
(4)	$\underline{Ta} + A_l$	$-2, \underline{-3}$	$z^2 = y^3 + x^{5+4l}y$
(5)	$\underline{Tr} + A_l$	$-2, -2, \underline{-3}$	$z^2 = y^3 + x^{8+6l}$
(6)	$A_{1,****} + A_l$	$-2, -2, -2, \underline{-3}$	$z^2 = y^3 + x^{9+6l}$
(7)	$A_{n,****} + A_l, n \geq 2$	$-2, -2, -2, \underline{-3}$	$z^2 = (y + x^{3+2l})(y^2 + x^{n+5+4l})$
(8)	$D_{4,***} + A_l$	$-2, -2, \underline{-3}$	$z^2 = y^3 + x^{10+6l}$
(9)	$E_{6,**} + A_l$	$-2, \underline{-3}$	$z^2 = y^3 + x^{7+4l}y$
(10)	$E_{8,*} + A_l$	$\underline{-3}$	$z^2 = y^3 + x^{11+6l}$

[a] $l \geq 0$.

This corresponds to the curve of degree 3 "odd type".

Thm (with T. Tomaru) $z^2 = f(x,y)$ double pt

$M > Z$ on the minimal resolution

$\Updownarrow$

$M > Z$ on the minimal good resolution

$\Updownarrow$

$f(x,y)$ has the Laufer decomposition

i.e. $f(x,y) = f_{[L]} \cdot f_{[c]} \cdot f_{[o]}$ as follows;

(i) If $f_{[L]} \neq 1$, $f_{[L]}$ is of Laufer type & $T_0(f_{[o]})_{red} \not\subset T_0(f_{[L]})_{red}$

(ii) If $f_{[c]} \neq 1$, $f_{[c]}$ is of contact type & $T_0(f_{[o]})_{red} = T_0(f_{[c]})_{red}$

~~(iii) $f_{[o]}$ is of odd type and $f_{[L]} f_{[c]} \neq 1$.~~

Def $f(x,y) \in \mathbb{C}\{x,y\}$ is of Laufer type
$\overset{\text{def}}{\Longleftrightarrow}$ $z^2 = (ax+by) f(x,y)$ is of $Z^2 = -1$

Def $f(x,y) \in \mathbb{C}\{x,y\}$ is of contact type
$\overset{\text{def}}{\Longleftrightarrow}$ $T_0(f)$ is a line say l,
$m_l(f_j) < 2m_1(f_j)$ $\forall f_j \mid f$ irred comp
$z^2 = l f(x,y)$ is of $Z^2 = -1$

Def $f(x,y) \in \mathbb{C}\{x,y\}$ is odd type
$\Longleftrightarrow$ ord f : odd
&
ord $f = 1$, or $M^2 = -1$ on MGR
i.e. $M = Z$ on MGR