Introduction to ring-theoretic properties of geometric ideals

by K. Yoshida (j.w. T. Okuma, K.-i. Watanabe, M. E. Rossi)

Throughout this talk, let

$\star$ (A, m) be an excellent normal local domain of $\dim A = 2$, $A \supset R = k \cong A/m$, not regular

$\Rightarrow$ for any m-primary, integrally closed ideal $I \subset A$,
$\exists$ res. of sing. $f: X \to \operatorname{Spec} A$ and $\exists$ anti-nef cycle Z on X s.t.
I can be represented by Z on X as follows:

$$I = H^0(X, \mathcal{O}_X(-Z)), \quad I\mathcal{O}_X = \mathcal{O}_X(-Z)$$

(We denote it by $I = I_Z$).

- **Main aim of this talk** is to introduce several ring-theoretic properties of "geometric" ideals in our context.

§1 Preliminaries

§2 p_g-ideal, good ideal elliptic ideal

§3 Normal tangent cone

Notation and Terminologies

- **Reduction number**

$A : \star$, $I = I_Z$

- Q : minimal reduction of I

$\underset{\text{def}}{\Longleftrightarrow}$ $Q = (a, b) \subset I$, $I^{n+1} = QI^n$ ($^\exists n \geq 0$)

- $\bar{I} = \{ z \in A \mid {}^\exists n, {}^\exists c_i \in I^i \text{ s.t. } z^n + c_1 z^{n-1} + \cdots + c_n = 0 \}$.

is called the integral closure of I

- $r_Q(I) = \min\{ n \mid I^{n+1} = QI^n \}$

$r(I) = \min\{ r_Q(I) \mid Q : \text{min red of } I \}$

is called the reduction number (exponent).

• Geometric genus

$p_g(A) := \dim_k H^1(\mathcal{O}_X)$ is called geometric genus of A

Note : indep. on the choice of res. of sing.

$R = \bigoplus_{n\geq 0} R_n$, $R_0 = k$, $m = R_+$, $A = R_m$

$\Rightarrow p_g(A) = \dim_k H^2_m(R)_{\geq 0}$

In particular, if $\boxed{A \text{ is Gor}}$, then. $\boxed{p_g(A) = \sum_{n=0}^{a(A)} \dim_k R_n}$

$a(A) = \max\{n \mid [H^2_m(A)]_n \neq 0\}$

For instance, if $R = k[x,y,z]/(x^3+y^4+z^6)$, then

$\deg x = 4$, $\deg y = 3$, $\deg z = 2$

$a(A) = -\{\deg x + \deg y + \deg z\} + \deg f = -(4+3+2)+12 = 3$

$\therefore p_g(A) = 3$

deg	0	1	2	3
	1		z	y

• Singularities

A : rat'l sing $\overset{\text{def}}{\Longleftrightarrow} p_g(A) = 0$

A : strongly elliptic sing $\overset{\text{def}}{\Longleftrightarrow} p_g(A) = 1$

♦ A : Gor. $p_g(A) = 2 \Rightarrow A$: "elliptic sing".

- $\underline{q(I) = \dim H^1(\mathcal{O}_X(-Z))}$

For $\forall I = I_Z$, $\forall n \in \mathbb{Z}_{\geq 1}$, we put

$$q(nI) := \dim_k H^1(\mathcal{O}_X(-nZ)).$$

$$\rightsquigarrow \quad p_g(A) = q(0A) \geq q(1 \cdot I) \geq q(2I) \geq \cdots \quad = q(\infty I)$$

$$\boxed{\ell_A\left(\overline{I^{n+1}} / Q\overline{I^n}\right) = \{q((n-1)I) - q(nI)\} - \{q(nI) - q((n+1)I)\}}$$

$$nr(I) = \min\{n \in \mathbb{Z} \mid q((n-1)I) - q(nI) = q(nI) - q((n+1)I)\}$$

$$\bar{r}(I) = \min\{n \in \mathbb{Z} \mid q((n-1)I) = q(nI)\}$$

- $\underline{\text{Normal red. number of } A}$

$\bar{r}(A) := \sup\{\bar{r}(I) \mid I = I_Z \subset A\}$.

the $\underline{\text{normal red. number of } A}$.

$\boxed{\text{Problem 1}}$ Find an upper bound for $\bar{r}(A)$!

- $\underline{\text{Kato's Riemann-Roch formula}}$

For $I = I_Z$. $\quad \ell_A(A/\overline{I}) + q(\overline{I}) = \chi(Z) + p_g(A)$,

where $\quad \chi(Z) = -\dfrac{Z^2 + KZ}{2}$

• Blow-up algebras

$\mathcal{F} = \{I_n\}_{n \in \mathbb{Z}}$: a filtration of ideals

(i) $I_0 = A$, $I_1 \subseteq \mathfrak{m}$. $I_n = A$ ($\forall n \leq -1$)

(ii) $I_n \supset I_{n+1}$ ($\forall n \in \mathbb{Z}$)

(iii) $I_m I_n \subseteq I_{m+n}$ ($\forall m, n \in \mathbb{Z}$)

★ (iv) $I_{n+1} = I I_n$ ($\forall n \gg 0$) $\Rightarrow$ $\mathcal{F}$ is I-good filtration

In our situation, $\mathcal{F} = \{\overline{I^n}\}$ is I-good filtration.

$\bar{R}(I) = R(\mathcal{F}) = \sum_{n \geq 0} I_n t^n$: the Rees algebra

$\bar{R}'(I) = R'(\mathcal{F}) = \sum_{n \in \mathbb{Z}} I_n t^n$: the extended ———

$\bar{G}(I) = G(\mathcal{F}) = \bigoplus_{n \geq 0} I_n / I_{n+1}$: the associated graded ring.

⇓

normal tangent cone of I

• Brieskorn hypersurface

$2 \leq a \leq b \leq c$: integers, $K = \bar{K}$: field char $K = p \nmid abc$

$A = K[x, y, z]_{(x,y,z)} / (x^a + y^b + z^c)$ or $\hat{A} = K[[x, y, z]] / (x^a + y^b + z^c)$

$\mathfrak{m} = (x, y, z)A \supset Q = (y, z)A$, $\hat{\mathfrak{m}} = \mathfrak{m}\hat{A} = (x, y, z)\hat{A}$

A or $\hat{A}$ is called the Brieskorn hypersurface of type (a, b, c).

Theorem ([OWY4, Thm 3.1])

$$\boxed{\begin{array}{l}(1)\ \overline{\mathfrak{m}^n} = \overline{Q^n} = Q^n + xQ^{n-n_1} + x^2 Q^{n-n_2} + \cdots + x^{a-1} Q^{n-n_{a-1}} \\ (2)\ F(\mathfrak{m}) = \mathrm{nr}(\mathfrak{m}) = \underline{n_{a-1}} \\ (3)\ \overline{G}(\mathfrak{m}) \text{ is Cohen-Macaulay.} \\ (4)\ e_0(\overline{G}) = a\end{array}}$$

- <u>p_g-ideals</u>

Theorem

$I = I_Z$ に対して TFAE

(1) $\overline{F}(Z) = 1$

(2) $q(Z) = p_g(A)$

(3) $\overline{I^n} = I^n$ $(\forall n \geq 1)$, $I^2 = QI$.

(4) $\overline{G}(Z)$: CM with $a(\overline{G}(Z)) < 0$

(5) $\overline{R}(I)$: CM.

Then I is called a <u>p_g-ideal</u>.

- <u>good ideals</u>

$I = I_Z$: <u>good</u> $\overset{\text{def}}{\iff}$ $I^2 = QI$ and $Q : I = I$

◇ Any A admits a good, p_g-ideal. [OWY3].

- elliptic sing.

  A : elliptic singularity

$\underset{\text{def}}{\Longleftrightarrow}$ $\chi(D) \geq 0$ ($^\forall D$: cycle) and $\chi(F) = 0$ for $^\exists F > 0$.

Thm (Okuma) A : elliptic sing $\Rightarrow$ $\bar{r}(A) \leq 2$

- elliptic ideal

TFAE.

(1) $\bar{r}(I) = 2$.

(2) $p_g(A) > q(I) = q(\infty I)$

(3) $\bar{G}(I)$: CM with $a(\bar{G}(I)) = 0$

Then I is called an elliptic ideal.

- Strong elliptic sing. and strong elliptic ideal

A : strong elliptic singularity $\underset{\text{def}}{\Longleftrightarrow}$ $p_g(A) = 1$.

TFAE

(1) $\bar{r}(I) = 2$. $\ell_A(\overline{I^2}/QI) = 1$

(2) $q(I) = q(\infty I) = p_g(A) - 1$

(3) $\bar{G}(I)$: CM with $a(\bar{G}(I)) = 0$. $\ell_A([H^2_M(\bar{G}(I))]_0) = 1$.

This ideal I is called a strongly elliptic ideal.

- Some properties of blow-up algebras

Proposition

(1) TFAE (a) $R(I)$: normal
(b) $R'(I)$: normal
(c) $\overline{I^n} = I^n$ ($\forall n \geq 1$)

(2) $\overline{G}(I) \cong \overline{R'}(I) / t^{-1}\overline{R'}(I)$.

In particular, $\overline{G}(I)$: CM (resp. Gor) $\Longleftrightarrow$ $\overline{R'}(I)$: CM (resp. Gor)

(3) $\overline{G}(I)$: CM $\Longleftrightarrow$ $Q \cap \overline{I^n} = Q\overline{I^{n-1}}$ ($\forall n \geq 2$)

(Varabrega-Valla type theorem)

(4) $Q \cap \overline{I^2} = QI$ (Huneke-Itoh)

- <u>CMness of $\overline{G}(I)$</u>

Theorem (CMness of $\overline{G}(I)$)

(1) $\overline{F}(I) = 1 \Rightarrow \overline{G}(I) = G(I)$ is <u>CM.</u>

(2) $\overline{F}(I) = 2 \Rightarrow \overline{G}(I)$ is <u>CM.</u>

(3) Given $r \geq 3$, $\exists I = I_Z$ s.t. $\overline{F}(I) = r$, $\overline{G}(I)$ is <u>not CM.</u>

- Gorness of $\overline{G}(I)$.

Assume A is Gorenstein and $I = I_Z$

Thm When $F(I)=1$, $\overline{G}(I)$ is Gor. $\Longleftrightarrow$ I is good.

Thm Assume $F(I)=2$, Then TFAE

(1) $\overline{G}(I)$ is Gorenstein.

(2) $Q : I = Q + \overline{I^2}$.

(3) $\ell_A(A/I) = \ell_A(\overline{I^2}/QI)$.

(4) $\chi(Z)=0$ i.e. $Z^2+KZ=0$.

When this is the case, $\ell_A(A/I) \leq p_g(A)$.

Theorem (Gorness of $\overline{G}(I)$ for $F(I) \geq 3$)

Put $F(I)=r$.

Suppose $\overline{G}(I)$ is CM. Put $L_n = Q + \overline{I^n}$ ($\forall n \geq 1$).

Then TFAE

(1) $\overline{G}(I)$ is Gorenstein.

(2) $Q : L_n = Q + L_{r+1-n}$ for $\forall n = 1, 2, \cdots, \lceil \frac{r}{2} \rceil$.

(3) $\ell_A(A/L_n) + \ell_A(A/L_{r+1-n}) = e_0(I)$ for $\forall n = 1, 2, \cdots, \lceil \frac{r}{2} \rceil$.

When this is the case,

(a) $(r-1)Z^2 + KZ = 0$. (b) $\ell_A(A/I) \leq p_g(A) + 2 - r$.

Theorem ([OW78])

A : Brieskorn hyp. $r = F(m) = n_{a-1} = \lfloor \frac{(a-1)b}{a} \rfloor$ TFAE.

(1) $\overline{G}(m)$ is Gorenstein

(2) $(r-1)Z^2 + KZ = 0$, where $\begin{cases} KZ = d + ab - b - 2a \\ Z^2 = -a \end{cases}$.

(3) $b' \equiv 1 \pmod{a'}$.

1) When $d = a$, (i.e. $a | b$),

$$\overline{G}(m) = \begin{cases} K[X, Y, Z]/(X^a + Y^b + Z^c) & \text{if } b = c \\ K[X, Y, Z]/(X^a + Y^b) & \text{if } b < c \end{cases}$$

2) When $d = 1$, $b \equiv 1 \pmod a$,

$$\overline{R}'(m) = K[yt, zt, \underline{xt^{n_1}}, t^{-1}] \cong K[X, Y, Z, U]/(X^a + Y^b + Z^c U^{c-b+1})$$

$$\therefore \overline{G}(m) = K[X, Y, Z]/(X^a) \cong G(m)$$ hypersurface.

3) When $1 < d < a$, $b' \equiv 1 \pmod{a'}$,

$$\overline{R}'(m) = K[\underline{xt^{n_1}}, yt, zt, \underline{x^{a'}t^{b'}}, t^{-1}]$$
$$\cong K[X, Y, Z, W, U]/(X^{a'} - WU, W^d + Y^b + Z^c U^{c-b})$$

$$\therefore \overline{G}(m) = \begin{cases} K[X, Y, Z, W]/(X^{a'}, W^d + Y^b + Z^b) & \text{if } b = c \\ K[X, Y, Z, W]/(X^{a'}, W^d + Y^b) & \text{if } b < c \end{cases}$$

complete intersection

Ex. $A = k[[x,y,z]]/(x^3+y^5+z^5)$

$\overline{G} = \overline{G}(m) = k[X,Y,Z,W]/(X^2, XW, W^2+XY^5+XZ^3)$

$\overline{G}/(Y,Z)\overline{G} = k[X,W]/(X^2, XW, W^2)$: not Gor

Ex $A = \mathbb{C}[[x,y,z]]/(x^2+y^4+z^{4(m+1)})$

$\Rightarrow \{ I \mid \overline{F}(I)=2,\ \overline{G}(I) : \text{Gor} \}$

$= \{ (x,y,z^i) \mid 1 \leq i \leq m+1 \}$ ← finite set.

[GI] Shiro Goto and Shin-ichiro Iai, *Embeddings of certain graded rings into their canonical modules*, J. Algebra **228** (2000), no. 1, 377–396.

[GIW] Shiro Goto, Sin-ichiro Iai, and Kei-ichi Watanabe, *Good ideals in Gorenstein local rings*, Trans. Amer. Math. Soc. **353** (2001), no. 6, 2309–2346.

[GN] Shiro Goto and Koji Nishida, *The Cohen-Macaulay and Gorenstein Rees algebras associated to filtrations*, Mem. Am. Math. Soc., vol. 526, Providence, RI: American Mathematical Society (AMS), 1994.

[GW] Shiro Goto and Kei-ichi Watanabe, *On graded rings. I*, J. Math. Soc. Japan **30** (1978), no. 2, 179–213.

[Hun] Craig Huneke, *Hilbert functions and symbolic powers*, Michigan Math. J. **34** (1987), no. 2, 293–318.

[It] Shiroh Itoh, *Integral closures of ideals generated by regular sequences*, J. Algebra **117** (1988), no. 2, 390–401.

[Ka] Masahide Kato, *Riemann-Roch theorem for strongly pseudoconvex manifolds of dimension* 2, Math. Ann. **222** (1976), no. 3, 243–250.

[Ok1] Tomohiro Okuma, *Numerical Gorenstein elliptic singularities*, Math. Z. **249** (2005), no. 1, 31–62.

[Ok2] ______, *Cohomology of ideals in elliptic surface singularities*, Illinois J. Math. **61** (2017), no. 3–4, 259–273.

[ORWY] Tomohiro Okuma, Maria Evelina Rossi, Kei-ichi Watanabe, and Ken-ichi Yoshida, *Normal Hilbert coefficients and elliptic ideals in normal two-dimensional singularities*, Nagoya Math. J. **248** (2022), 779–800.

[OWY1] Tomohiro Okuma, Kei-ichi Watanabe, and Ken-ichi Yoshida, *Good ideals and p_g-ideals in two-dimensional normal singularities*, Manuscripta Math. **150** (2016), no. 3-4, 499–520.

[OWY2] ______, *Rees algebras and p_g-ideals in a two-dimensional normal local domain*, Proc. Amer. Math. Soc. **145** (2017), no. 1, 39–47.

[OWY3] ______, *A characterization of two-dimensional rational singularities via core of ideals*, J. Algebra **499** (2018), 450–468.

[OWY4] ______, *Normal reduction numbers for normal surface singularities with application to elliptic singularities of Brieskorn type*, Acta Math. Vietnam. **44** (2019), no. 1, 87–100.

[OWY5] ______, *The normal reduction number of two-dimensional cone-like singularities*, Proc. Amer. Math. Soc. **149** (2021), no. 11, 4569–4581.

[OWY7] ___, Gorensteinness for normal tangent cones of elliptic ideals, submitted.

Ring theoretic properties of "geometric" ideals

2023. 3. 30

@ 東大シンポジウム

§1. Notation. Preliminaries.

$\star \begin{cases} (A,\mathfrak{m}) : \text{excellent normal local dom of } \dim A = 2 \\ A \supset k = \bar{k} \cong A/\mathfrak{m}. \end{cases}$, not regular

$\Rightarrow$ for any $\mathfrak{m}$-primary, integrally closed ideal $I \subset A$,
$\exists$ res. of sing. $f : X \to \operatorname{Spec} A$ and $\exists$ anti-nef cycle Z on X s.t.

$$I = H^0(X, \mathcal{O}_X(-Z)), \quad I\mathcal{O}_X = \mathcal{O}_X(-Z)$$

We denote it by $I = I_Z$.

- **Normal reduction numbers**

Dfn 1.1 (Normal red. number)

$nr(I) := \min\{ r \in \mathbb{Z}_+ \mid \overline{I^{r+1}} = Q\overline{I^r} \}$.

$\bar{r}(I) := \min\{ r \in \mathbb{Z}_+ \mid \overline{I^{n+1}} = Q\overline{I^n} \ ({}^\forall n \geq r) \}$

Note :
- $\overline{I^{r+1}} = Q\overline{I^r} \not\Rightarrow \overline{I^{r+2}} = Q\overline{I^{r+1}}$
- " $\overline{I^{n+1}} = Q\overline{I^n}$ " is independent on Q. (by Huneke)
- In general, $nr(I) \leq \bar{r}(I)$.

${}^\forall g \geq 2$, $\exists (A,\mathfrak{m}) : \star$ $\exists I = I_Z$ s.t. $1 = nr(I) < \bar{r}(I) = g+1$. (OWY5)

$\bar{r}(I)$ is called the normal red. number of I.

Dfn 1.2 ($q(nI)$)

For $\forall I = I_Z$, $\forall n \in \mathbb{Z}_{\geq 1}$, we put

$$q(nI) := \dim_k H^1(\mathcal{O}_X(-nZ)).$$

$$\rightsquigarrow \; p_g(A) := q(0A) \geq q(1\cdot I) \geq q(2I) \geq \cdots \quad = q(\infty I)$$

↑ the geometric genus of A.

Prop 1.3 ($F(I)$ and $q(nI)$)

(1) $\ell_A\left(\overline{I^{n+1}}/Q\overline{I^n}\right) = \left[\{q((n-1)I) - q(nI)\} - \{q(nI) - q((n+1)I)\}\right]$

(2) $F(I) = \min\{n \in \mathbb{Z} \mid q((n-1)I) = q(nI)\}$

Dfn 1.4 (normal red. number of A)

$$F(A) := \sup\{F(I) \mid I = I_Z \subset A\}.$$

the normal red. number of A.

$\boxed{\text{Problem 1}}$ Find an upper bound for $F(A)$!

Thm 1.5 $F(A) \leq p_g(A) + 1$. (More sharp bound ...)

Ex 1.6 A: rat'l sing (i.e. $p_g(A) = 0$) $\Longleftrightarrow F(A) = 1$.

(This means) A: rat'l sing. $I = I_Z \Rightarrow \overline{I^2} = I^2 = QI$ (Lipman)

- **Kato's Riemann-Roch formula**

Thm 1.7 (Kato's R-R)

For $I = I_Z$, $\ell_A(A/I) + q(I) = \chi(Z) + p_g(A)$,

where $\chi(Z) = -\dfrac{Z^2 + KZ}{2}$

- **Blow-up algebras**

Let $\mathcal{F} = \{I_n\}_{n \in \mathbb{Z}}$ be a filtration of ideals

In our situation, $\mathcal{F} = \{\overline{I^n}\}$ is I-good filtration.

$\bar{R}(I) = R(\mathcal{F}) = \sum_{n \geq 0} I_n t^n$: the Rees algebra

$\bar{R}'(I) = R'(\mathcal{F}) = \sum_{n \in \mathbb{Z}} I_n t^n$: the extended ______

$\bar{G}(I) = G(\mathcal{F}) = \bigoplus_{n \geq 0} I_n / I_{n+1}$: the associated graded ring.

$\Downarrow$

normal tangent cone of I

$\boxed{\text{Problem 2}}$

Investigate several ring-theoretic properties!

(CMness, Gorness etc.)

• Brieskorn hypersurface

* $2 \leq a \leq b \leq c$: integers, $K = \overline{K}$: field $\operatorname{char} K = p \nmid abc$.

$$A = K[x,y,z]_{(x,y,z)} / (x^a + y^b + z^c)$$

$\mathfrak{m} = (x,y,z)A \supset Q = (y,z)A$,

A or $\widehat{A}$ is called the Brieskorn hypersurface of type (a,b,c).

Put $\overline{G} := \overline{G}(\mathfrak{m}) \cong \overline{G}(\widehat{\mathfrak{m}}) = \bigoplus_{n \geq 0} \overline{\mathfrak{m}^n} / \overline{\mathfrak{m}^{n+1}}$.

$$d = \gcd(a,b), \quad a' = \frac{a}{d}, \quad b' = \frac{b}{d}$$

$$n_k = \left\lfloor \frac{kb}{a} \right\rfloor \quad ({}^\forall k = 1, 2, \cdots, a-1).$$

Thm 1.8 ([OWY4, Thm 3.1])

(1) $\overline{\mathfrak{m}^n} = \overline{Q^n} = Q^n + xQ^{n-n_1} + x^2 Q^{n-n_2} + \cdots + x^{a-1} Q^{n-n_{a-1}}$

(2) $F(\mathfrak{m}) = \operatorname{nr}(\mathfrak{m}) = n_{a-1}$

(3) $\overline{G}(\mathfrak{m})$ is Cohen-Macaulay.

(4) $e_0(\overline{G}) = a$

$\boxed{\text{Problem 3}}$ How about any $I = \overline{I}$?

§2. p_g-ideals, good ideals, elliptic ideals

- $I = I_Z \Longrightarrow q(I) \leq p_g(A)$

Thm 2.1 (characterization of p_g-ideals)

For $I = I_Z$, TFAE

(1) $\bar{r}(I) = 1$

(2) $q(I) = p_g(A)$

(3) $\overline{I^n} = I^n$ ($\forall n \geq 1$), $I^2 = QI$

(4) $\overline{G}(I)$: CM with $a(\overline{G}(I)) < 0$

(5) $\overline{R}(I)$: CM.

Then I is called a p_g-ideal.

Ex 2.2

$$A = \mathbb{R}[x,y,z]/(x^a + y^b + z^c)$$

(1) A : rat'l sing $\Longleftrightarrow$ $(a,b,c) = (2,2,n), (2,3,3), (2,3,4), (2,3,5)$

(2) $\mathfrak{m}$: p_g-ideal $\Longleftrightarrow$ $(a,b) = (2,2), (2,3)$.

Dfn 2.3 (good ideal)

$I = I_Z$: good $\overset{\Longleftrightarrow}{\text{def}}$ $I^2 = QI$ and $Q : I = I$

(When A is Gorenstein, I : good $\Longleftrightarrow$ $e_0(I) = 2 \cdot \ell_A(A/I)$.)

Moreover, if I is a p_g-ideal, then
I : good $\Longleftrightarrow$ $KZ = 0$. ($\Rightarrow \overline{G}(I)$: Gor.)

Theorem 2.4 ([OWY1, OWY3])

Any 2-dim excellent normal local domain ($\neq$ RLR) $\supset k = \bar{k}$ admits a good, p_g-ideal.

◇ elliptic ideal

Dfn 2.5 (elliptic ideal)

TFAE.

(1) $\bar{r}(I) = 2$

(2) $p_g(A) > q(I) = q(\infty I)$

(3) $\bar{G}(I)$: CM with $a(\bar{G}(I)) = 0$

Then I is called an elliptic ideal.

Theorem 2.6 (Okuma) $\cancel{\Longleftarrow}$

A : elliptic sing $\Longrightarrow$ $\bar{r}(A) \leq 2$ (See also Okuma's talk)

Ex 2.7 $A = k[[x,y,z]]/(x^a + y^b + z^c)$

(1) A : elliptic

$\Longleftrightarrow$ $(a,b,c) = (2,3,n),\ n \geq 6$; $= (3,3,n),\ n \geq 3$

$= (2,4,n),\ n \geq 4$; $= (3,4,n),\ 4 \leq n \leq 5$.

$= (2,5,n),\ 5 \leq n \leq 9$

(2) $\mathfrak{m}$: elliptic $\Longleftrightarrow$ $(a,b) = \cancel{(2,3)}, (2,4), (2,5), (3,3), (3,4)$

Ex 2.8

(1) A : elliptic sing $\Rightarrow$ $\forall I = I_Z$ is either elliptic or p_g-ideal

(2) $\forall I = I_Z$: not p_g-ideal $n \gg 0$ $\Rightarrow$ $\overline{I^n}$ is elliptic

(For instance, if A is Brieskorn hyp, then
$\overline{F}(\mathfrak{m}) \geq 2$, $n \geq \overline{F}(\mathfrak{m}) - 1$ $\Rightarrow$ $\overline{\mathfrak{m}^n}$ is elliptic)

Ex 2.9 $A = k[[x.y.z]]/(x^a + y^b + z^c)$

(1) $(a,b,c) = (2,3,6) \Rightarrow \mathfrak{m}$: p_g-ideal

(2) $(a,b,c) = (2,4,4) \Rightarrow \mathfrak{m}$: elliptic. not normal

(3) $(a,b,c) = (3,3,3) \Rightarrow \mathfrak{m}$: elliptic. normal.

§3. Normal tangent cone

Let me remind you the following criterion:

(3.1) Valabrega-Valla (type) theorem

$\overline{G}(I)$: CM $\Longleftrightarrow$ $Q \cap \overline{I^n} = Q\overline{I^{n-1}}$ ($^\forall n \geq 2$)

Huneke-Itoh theorem

$Q \cap \overline{I^2} = QI$.

Thm 3.2 (CMness of $\overline{G}(I)$)

(1) $\overline{F}(I) = 1 \Rightarrow \overline{G}(I) = G(I)$ is CM.

(2) $\overline{F}(I) = 2 \Rightarrow \overline{G}(I)$ is CM.

(3) Given $r \geq 3$, $^\exists I = I_Z$ s.t. $\overline{F}(I) = r$, $\overline{G}(I)$ is not CM.

(Note: $\overline{G}(I)$: CM $\Rightarrow$ $\mathrm{nr}(I) = \overline{F}(I)$)

Now assume A is Gorenstein and $I = I_Z$.

Theorem 3.3 (Gor'ness of $\overline{G}(I)$ for $F(I)=1$)

Put $F(I)=1$.

Then $\overline{G}(I)$ is Gorenstein $\Longleftrightarrow$ I is good

Theorem 3.4 (Gor'ness of $\overline{G}(I)$ for $F(I)=2$).

Put $F(I)=2$. Then TFAE

(1) $\overline{G}(I)$ is Gorenstein.

(2) $Q : I = Q + \overline{I^2}$.

(3) $\ell_A(A/I) = \ell_A(\overline{I^2}/QI)$.

(4) $\chi(Z)=0$ i.e. $Z^2+KZ=0$.

When this is the case $\ell_A(A/I) \le p_g(A)$.

Cor. 3.5

If $p_g(A) \le 2$ or $\overline{F}(m) \le 2$, then $\overline{G}(m)$ is Gorenstein.

e.g. $A = \mathbb{C}[[x,y,z]]/(x^3+y^5+z^5)$.

Then $F(m) = p_g(A) = 3$.

$\overline{G}(m)$ is not Gorenstein (See also Ex 3.8).

Theorem 3.6 (Gorness of $\overline{G}(I)$ for $r(I) \geq 3$)

If $\overline{G}(I)$ is Gorenstein and $r(I) = r$

$\Longrightarrow$ $(r-1)\bar{e}_2 + K\bar{e}_2 = 0$.

and $\ell_A(A/I) \leq p_g(A) + 2 - r$.

We want to prove (1) $\Rightarrow$ (a) only here.

Put $B = \overline{G}(I)/(a^t, b^t)\overline{G}(I) \cong \underset{B_0}{\underset{\|}{A/I}} \oplus \underset{B_1}{\underset{\|}{I/Q+\overline{I^2}}} \oplus \underset{B_2}{\underset{\|}{\dfrac{Q+\overline{I^2}}{Q+\overline{I^3}}}} \oplus \cdots \oplus \underset{B_r}{\underset{\|}{\dfrac{Q+\overline{I^r}}{Q}}}$

and $b_k = \ell_A(B_k)$.

Then $\ell_A\left(\overline{I^n}/Q\overline{I^{n-1}}\right) = \ell_A\left(\overline{I^n}/Q \cap \overline{I^n}\right) = \ell_A\left(Q+\overline{I^n}/Q\right) = b_n + b_{n+1} + \cdots + b_r$

$b_2 + b_3 + \cdots + b_r = \ell_A\left(\overline{I^2}/QI\right) = [p_g(A) - q(I)] - [q(I) - q(2I)]$

$b_3 + \cdots + b_r = \ell_A\left(\overline{I^3}/Q\overline{I^2}\right) = [q(I) - q(2I)] - [q(2I) - q(3I)]$

$\vdots$

$b_r = \ell_A\left(\overline{I^r}/Q\overline{I^{r-1}}\right) = [q((r-2)I) - q((r-1)I)]$

By summing up, we have

$b_2 + 2b_3 + \cdots + (r-2)b_{r-1} + (r-1)b_r = p_g(A) - q(I)$

$= \ell_A(A/I) - \chi(Z)$

Kato's RR $\nearrow$

3-6

$\overline{G}(I)$: Gor $\Rightarrow$ B : Gor $\Rightarrow$ $b_k = b_{r-k}$ ($^\forall k = 0, 1, \cdots, r$)

By subtracting $b_r = b_0 = \ell_A(A/I)$ from both sides,

we have

$$b_2 + 2b_3 + \cdots + (r-2)b_{r-1} + (r-2)b_r = -\chi(Z)$$

$$\therefore 2b_2 + 4b_3 + \cdots + 2(r-2)b_{r-1} + 2(r-2)b_r = -2\cdot\chi(Z) = Z^2 + KZ.$$

For simplicity, assume $r = \text{odd} = 2m+1$.

Since the sum of coeff of b_k and b_{r-k} $= 2(k-1) + 2(r-k-1)$

$= 2(r-2)$,

$$(\text{LHS}) = (r-2)(b_0 + b_1 + b_2 + \cdots + b_{r-1} + b_r) = (r-2)\ell_A(B)$$

$$= (r-2)e_0(I)$$

$$= -(r-2)Z^2.$$

Hence $(r-1)Z^2 + KZ = 0$,

as required. ※

Problem

$$\left.\begin{array}{l} \underline{(r-1)Z^2 + KZ = 0} \\ \overline{G}(I) : \text{CM} \\ A : \text{Gor} \end{array}\right\} \Rightarrow \overline{G}(I) : \text{Gor}?$$

Theorem 3.7 ([OWY8])

A : Brieskorn hyp. $r = F(m) = n_{a-1} = \left\lfloor \frac{(a-1)b}{a} \right\rfloor$ TFAE.

$d = \gcd(a,b)$, $a' = \frac{a}{d}$, $b' = \frac{b}{d}$

(1) $\overline{G}(m)$ is Gorenstein

(2) $(r-1)Z^2 + KZ = 0$, where $\begin{cases} KZ = d + ab - b - 2a \\ Z^2 = -a. \end{cases}$

(3) $b' \equiv 1 \pmod{a'}$.

Suppose $b < c$.

1) When $d = a$, (i.e. $a \mid b$),

$$\overline{G}(m) = K[X,Y,Z]/(X^a + Y^b)$$

2) When $d = 1$, $b \equiv 1 \pmod{a}$,

$$\overline{R}'(m) = K[yt, zt, xt^{n_1}, t^{-1}] \cong K[X,Y,Z,U]/(X^a + Y^b + Z^c U^{c-b+1})$$

$$\therefore \overline{G}(m) = K[X,Y,Z]/(X^a) \cong G(m)$$ hypersurface.

3) When $1 < d < a$, $b' \equiv 1 \pmod{a'}$,

예 $$\overline{R}'(m) = K[xt^{n_1}, yt, zt, x^{a'}t^{b'}, t^{-1}]$$
$$\cong K[X,Y,Z,W,U]/(X^{a'} - WU, W^d + Y^b + Z^c U^{c-b})$$

$$\therefore \overline{G}(m) = K[X,Y,Z,W]/(X^{a'}, W^d + Y^b)$$

complete intersection

Example 3.8 [$\overline{G}(m)$ is not Gor.]

$a = 3$
$b = c = 5$

$\Downarrow$ $\overline{G}(m)$ is NOT Gorenstein.

$A = \mathbb{C}[x,y,z]/(x^3+y^5+z^5)$

$n_1 = \lfloor \frac{5}{3} \rfloor = 1$, $n_2 = \lfloor \frac{10}{3} \rfloor = 3 = r$.

$\therefore \overline{m^n} = Q^n + xQ^{n-1} + x^2Q^{n-3}$

$m = (x,y,z)$

$\overline{m^2} = Q^2 + xQ + (x^2) = m^2$

$\overline{m^3} = Q^3 + xQ^2 + (x^2) = (x^2) + m^3$

$\overline{m^4} = Q^4 + xQ^3 + x^2Q = Q\overline{m^3}$

$\therefore \overline{R'(m)} = A[xt, yt, zt, \underline{x^2t^3}, t^{-1}] = \mathbb{C}[\overset{X}{xt}, \overset{Y}{yt}, \overset{Z}{zt}, \overset{W}{x^2t^3}, \overset{U}{t^{-1}}]$

$WU - X^2 = 0$, $XW + Y^5U + Z^5U = 0$.

$W^2 + XY^5 + XZ^5 = 0$

f.i.2. $\overline{R'(m)} \cong \mathbb{C}[X,Y,Z,W,U]/(X^2-UW,\ XW+Y^5U+Z^5U,\ W^2+XY^5+XZ^5)$

$\overline{G}(m) \cong \overline{R'(m)}/t^{-1}\overline{R'(m)} \cong \mathbb{C}[X,Y,Z,W]/(X^2, XW, W^2+XY^5+XZ^5)$

$\left(\overline{G}/(Y,Z)\overline{G} = \mathbb{C}[X,W]/(X^2, XW, W^2) \quad \therefore \ell(\overline{G}/(Y,Z)\overline{G}) = 3 = a \right)$