Marked singularities, their moduli spaces, distinguished bases and Stokes regions

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Plan of the 1st talk

- Isolated hypersurface singularity, topology, Milnor lattice
- Universal unfolding, F-manifold
- Gauss-Manin connection, Fourier-Laplace transformation, Brieskorn lattice
- Marked singularities, their moduli spaces (Teichmüller spaces)
- μ-constant monodromy group
- Period map, Torelli type conjectures and results

Partly joint work with Falko Gauss.

Plan of the 2nd talk

- Global unfoldings of simple and simple elliptic singularities
- Lyashko-Looijenga map locally and globally
  (ADE: Looijenga, $\tilde{E}_k$: Jaworski, Hertling-Roucairol)
- Distinguished bases, Stokes matrices
- Stokes regions, Theorem: a bijection
  (Interpretation: a Torelli result at semisimple points)
- ADE: Approach of Looijenga and Deligne '73/'74
- ADE and $\tilde{E}_k$: Approach of Hertling '07/'18

Partly joint work with Céline Roucairol.

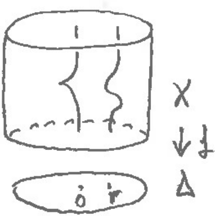
Isolated hypersurface singularity

$f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ holomorphic, isolated singularity at 0,

$$Q_f := \mathcal{O}_{\mathbb{C}^{n+1},0}/(\frac{\partial f}{\partial x_i}) \text{ Jacobi algebra}, \quad \mu := \dim Q_f \text{ Milnor number.}$$

Choose a good representative $f : X \to \Delta$,

$\Delta =$(very small disk in $\mathbb{C}$),
$X =$ (small ball in $\mathbb{C}^{n+1}$) $\cap f^{-1}(\Delta)$,
$X_\tau = f^{-1}(\tau) \subset X$ for $\tau \in \Delta$.



For $\tau \in \Delta^*$, the Milnor fibre X_τ is homotopy equivalent to $\bigvee_\mu S^n$.

The Milnor lattice is $Ml(f) := H_n^{(\text{red if } n=0)}(X_r, \mathbb{Z}) \cong \mathbb{Z}^\mu$ (some $r > 0$)

Milnor lattice

On $Ml(f)$ we have the monodromy Mon (quasiunipotent),
the intersection form I ($(-1)^n$-symmetric),
the Seifert form L (unimodular).

L determines Mon and I by

$$L(Mon(a), b) = (-1)^{n+1}L(b, a), \quad I(a, b) = -L(a, b) + (-1)^{n+1}L(b, a).$$

$$G_{\mathbb{Z}}(f) := \mathrm{Aut}(Ml(f), Mon, I, L) = \mathrm{Aut}(Ml(f), L).$$

Well known: $Mon_{\mathbb{C}}, Mon_{\mathbb{R}}, I_{\mathbb{R}}, L_{\mathbb{R}}$.
Fairly well known: $I_{\mathbb{Z}}$.
Badly known: $Mon_{\mathbb{Z}}, L_{\mathbb{Z}}$.

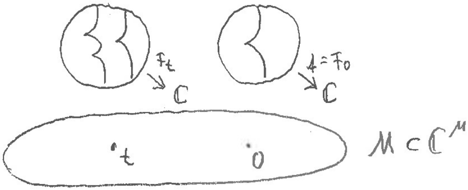
Universal unfolding

Choose $m_1, \dots, m_\mu \in \mathcal{O}_{\mathbb{C}^{n+1},0}$ s.t. $[m_1], \dots, [m_\mu] \in Q_f$ is a basis of Q_f. Define $F : X \times M \to \mathbb{C}$ by

$$F = F(x,t) = F_t(x) = f(x) + \sum_{i=1}^{\mu} m_i t_i : X \times M \to \mathbb{C},$$

where $M \subset \mathbb{C}^\mu$ is an open neighborhood of 0.

Theorem: F is a *universal unfolding*, it *induces* any unfolding of f.



$$\sum_{x \in \mathrm{Crit}(F_t)} \mu(F_t, x) = \mu$$

μ-constant stratum, modality, Arnold's classification

For generic $t \in M$, F_t has μ A_1-singularities
(i.e. $x_0^2 + ... + x_n^2$ up to coordinate changes).
Their values under F are locally *canonical coordinates* $u_1, ..., u_\mu$.

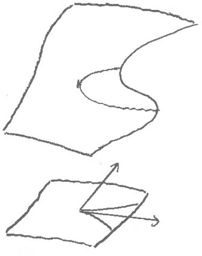
$$\begin{aligned} M \supset S_\mu &:= \{t \in M \,|\, F_t \text{ has only one singularity } x^0 \\ &\qquad \text{and } F_t(x^0) = 0\} \quad \mu\text{-constant stratum} \\ \text{modality of } (f) &:= \dim S_\mu. \end{aligned}$$

Arnold '72: classification of all singularities (up to coordinate changes) with modality in $\{0, 1, 2\}$.

mod $(f) = 0$: A-series, D-series, E_6, E_7, E_8.
mod $(f) = 1$: $\widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8, T_{pqr}$, 14 exceptional types.
mod $(f) = 2$: 14+6 exceptional types, 8 series.

Structure on M: Multiplication

$$C := \mathrm{Crit}(F) := \{(x,t) \in X \times T \mid \frac{\partial F}{\partial x_0}, \dots, \frac{\partial F}{\partial x_n} = 0\}.$$



$$\begin{array}{ll} C \subset X \times M & \text{smooth} \\ \downarrow \pi & \text{finite, flat of degree } \mu \\ M & \end{array}$$

$$\begin{array}{rcl} \mathcal{T}_M & \xrightarrow{\cong} & \pi_* \mathcal{O}_C \\ \dfrac{\partial}{\partial t_i} & \mapsto & \left[\dfrac{\partial F}{\partial t_i}\right] = [m_i] \\ T_t M & \overset{\cong}{\mapsto} & \bigoplus_{x \in \mathrm{Crit}(F_t)} Q_{(F_t, x)} \\ \text{multiplication } \circ & \leftarrow & \text{multiplication} \\ \text{unit field } e & \leftarrow & [1] \\ \text{Euler field } E & \leftarrow & [F] \end{array}$$

F-manifold

$(M, \circ, e, E)$ is an *F-manifold with Euler field* (Def. H-Manin '98):

M a complex manifold.
$\circ$ a hol. commutative and associative multiplication on the hol. tangent bundle TM with $e\circ = \mathrm{id}$.
An integrability condition for hol. vector fields $X, Y \in \mathcal{T}_M$:

$$\mathrm{Lie}_{X\circ Y}(\circ) = X \circ \mathrm{Lie}_Y(\circ) + Y \circ \mathrm{Lie}_X(\circ). \qquad (*)$$

And $\mathrm{Lie}_E(\circ) = 1 \cdot \circ$.

Implications of the integrability condition $(*)$

(1) For $t \in M \quad T_t M \cong \bigoplus_{x \in \mathrm{Crit}(F_t)} Q_{(F_t,x)}$
is the unique decomposition of $T_t M$ into local algebras.

$(*) \Rightarrow$ It extends to a local decomposition

$$(M,t) = \prod_{x \in \mathrm{Crit}(F_t)} (M^{(x)}, 0) \qquad \text{of F-manifolds.}$$

(2) $C \cong$ (analytic spectrum of $(M, \circ, e)$) $\subset T^*M$.

$(*) \iff$ it is a Lagrange subvariety (in the gen. semisimple case).

<u>Theorem</u> (Arnold '72/Hörmander '71):

Anal. sp. smooth $\iff$ the F-manifold comes from a singularity.

2 additional structures on M (details not this time)

(I) Gauss-Manin conn. and an idea of Kyoji Saito (early 80ies) and a trick + choice of Morihiko Saito '83

$\Rightarrow$ a holomorphic flat metric g on M

s.t. $(M, \circ, e, E, g)$ becomes a *Frobenius manifold with Euler field*
= an F-manifold with Kyoji Saito's flat structure.

(II) Gauss-Manin conn. and a trick of S. Cecotti & C. Vafa '91

$\Rightarrow$ a natural hermitian pos. def. metric h on $M(r \cdot f)$ for $|r| \gg 0$

s.t. the hol. sectional curvature is ≤ 0 everywhere and < 0 near S_μ except for the direction e (Liana David & H 15).

Gauss-Manin connection of a universal unfolding F

Discriminant $\mathcal{D} := F \times \mathrm{id}(C) \subset \mathbb{C} \times M$.

Flat cohomology bundle $\bigcup_{(\tau,t)\in\mathbb{C}\times M-\mathcal{D}} H^n(F_t^{-1}(\tau),\mathbb{C})$.

$\exists$ canonical extension to a hol. vector bundle H^{GM} on $\mathbb{C} \times M$ via hol. differential forms: $\omega \in \Omega^{n+1}_{X\times M/M} \rightsquigarrow$ the section $s^{GM}[\omega]$ with

$$\langle s^{GM}[\omega](\tau, t), \delta(\tau, t)\rangle := \int_{\delta(\tau,t)} \frac{\omega}{dF_t},$$

here $\delta(\tau, t) \subset F_t^{-1}(\tau) \subset X \times \{t\}$ is a (vanishing) cycle.

The Gauss-Manin conn. ∇^{GM} has a logarithmic pole along $\mathcal{D}$.

A partial Fourier-Laplace transformation

A partial Fourier-Laplace transformation $\rightsquigarrow$

a hol. vector bundle H^{osc} on $\mathbb{C} \times M$ with sections $s^{osc}[\omega]$ with (in the case of ADE or $\tilde{E}_k$)

$$\langle s^{osc}[\omega](z,t), \Gamma(\frac{z}{|z|},t)\rangle := \int_{\Gamma(\frac{z}{|z|},t)} e^{-F_t/z}\omega,$$

here $\Gamma(\frac{z}{|z|},t) \subset X \times \{t\}$ a Lefschetz thimble (in direction $\frac{z}{|z|}$)

Also:
∇^{osc} flat conn. with a pole of Poincaré rank 1 along $\{0\} \times M$.
$H^{osc}_{\mathbb{Z}} \to \mathbb{C}^* \times M$ a flat $\mathbb{Z}$-lattice bundle dual to a bundle generated by (hom. classes) of Lefschetz thimbles.
P a flat pairing (from intersecting Lefschetz thimbles).

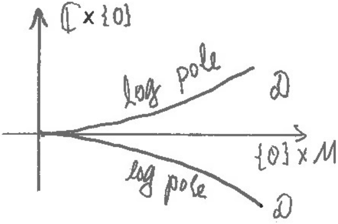
TEZP-structure $\sim$ non-commutative Hodge structure

$(H^{osc} \to \mathbb{C} \times M, \nabla^{osc}, H_{\mathbb{Z}} \to \mathbb{C}^* \times M, P)$ = a "TEZP"-structure,

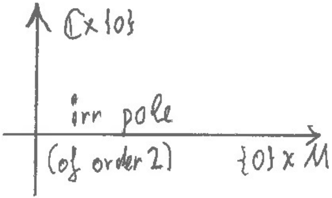
(Twistor - Extension - $\mathbb{Z}$-lattice - Pairing)

S. Cecotti & C. Vafa: tt^* geometry '91+'93.
H '02 rephrased it as TERP-structure.
Sabbah '05, Sevenheck '05 and Mochizuki '08 studied it, too.
Katzarkov-Kontsevich-Pantev '08 rephrased it as
non-commutative Hodge structure.

$(H^{GM} \to \mathbb{C} \times M, \nabla^{GM})$



$(H^{osc} \to \mathbb{C} \times M, \nabla^{osc})$



Brieskorn lattice

Brieskorn lattice

$$H_0''(f) := H^{GM}|_{(\mathbb{C},0)\times\{0\}} = \frac{\Omega^{n+1}_{X,0}}{df \wedge d\Omega^{n-1}_{X,0}}$$
$$= \text{(inverse Fourier-Laplace transform of } TEZP(0)),$$

free $\mathbb{C}\{\tau\}$-module of rank μ, free $\mathbb{C}\{\{\partial_\tau^{-1}\}\}$-module of rank μ.

$H^\infty := \{$global flat multivalued sections in $H^{osc}|_{\mathbb{C}\times\{0\}}\}$,

μ-dim $\mathbb{C}$-vector space, $H^\infty \supset H^\infty_{\mathbb{R}} \supset H^\infty_{\mathbb{Z}}$,
with monodromy $Mon = Mon_{ss} \cdot Mon_u$, $Mon_u = e^N$, N nilpotent,
and a polarizing form S.

PMHS and nilpotent orbit of PHS

Brieskorn lattice and (Kashiwara-Malgrange) $V^\bullet$-filtration
$\rightsquigarrow$ a polarized mixed Hodge structure $(F^\bullet, W_\bullet, S, N)$ on H^∞
with automorphism Mon_{ss}.

Varchenko '80, M. Saito & Scherk-Steenbrink '82, polarization H '97.

Observation Cecotti-Vafa '91, H '02:
Real structure & flat structure on $H^{osc}|_{\mathbb{C}^* \times M}$
$\rightsquigarrow$ *twin* along $\{\infty\} \times M$ of extension along $\{0\} \times M$.
$\rightsquigarrow$ bundle $\tilde{H}^{osc} \to \mathbb{P}^1 \times M$, real analytic in t, hol. in z.

Observations H '02:
PMHS $\leftrightsquigarrow$ nilpotent orbit of PHS (Cattani-Kaplan-Schmid '85)
$\rightsquigarrow$ $(H^\infty, H^\infty_{\mathbb{R}}, S, F^\bullet(r \cdot f_0))$ is a (pure) PHS for $|r| \gg 0$.
$\rightsquigarrow$ $\widetilde{H}^{osc}|_{\mathbb{P}^1 \times \{t\}}(r \cdot f_0)$ trivial for $t \in M$.

Torelli type conjecture

The tuple $(Ml(f), \text{Seifert form } L, H_0''(f)) \sim TEZP(0)$ is rich.

It is a generalization of a polarized mixed Hodge structure and can be seen as a *non-commutative Hodge structure*.

Torelli type conjecture (H '91): Up to isomorphism, it determines the germ f up to hol. coordinate changes.

H since '91: Proof for special families. Infinitesimal Torelli result. Generic Torelli result. Version for *marked singularities*.

$\rightsquigarrow$ Study families of singularities.
Study period maps and the action of the group $G_{\mathbb{Z}}$.

Torelli result

Theorem (H '92+'93): The Torelli type conjecture is true for all singularities with modality ≤ 2 and for the families containing the singularities $\sum_{i=0}^{n} x_i^{a_i}$ with $\gcd(a_i, a_j) = 1$ for $i \neq j$.

Proofs by calculations of two types:

(1) Period maps for families with the Gauss-Manin connection.

(2) $G_{\mathbb{Z}} := \mathrm{Aut}(Ml(f), L) =?$,
and its action on a classifying space for Brieskorn lattices.

Marked singularities

Fix a singularity f_0.

Definition (H '11)

(a) Its μ-homotopy class is

$$\{\text{singularities } f \mid \exists \text{ a } \mu\text{-constant family connecting } f \text{ and } f_0\}.$$

(b) A marked singularity is a pair $(f, \pm\rho)$ with f as in (a) and

$$\rho : (Ml(f), L) \stackrel{\cong}{\to} (Ml(f_0), L).$$

$M_\mu^{mar}(f_0)$ and $M_\mu(f_0)$

Definition (H '11)

(c) Two marked singularities $(f_1, \pm\rho_1)$ and $(f_2, \pm\rho_2)$ are right equivalent $(\sim_R)$

$\iff \exists$ biholomorphic $\varphi : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n+1}, 0)$ s.t.

$$\begin{array}{ccc} (\mathbb{C}^{n+1},0) & \stackrel{\varphi}{\to} & (\mathbb{C}^{n+1},0) \\ \downarrow f_1 & & \downarrow f_2 \\ \mathbb{C} & = & \mathbb{C} \end{array} , \quad \begin{array}{ccc} Ml(f_1) & \stackrel{\varphi_{hom}}{\to} & Ml(f_2) \\ \downarrow \rho_1 & & \downarrow \pm\rho_2 \\ Ml(f_0) & = & Ml(f_0) \end{array}$$

(d)

$$M_\mu^{mar}(f_0) \stackrel{\text{as a set}}{:=} \{(f, \pm\rho) \text{ as above}\}/ \sim_R .$$

(e) $\sim_R$ for f gives

$$M_\mu(f_0) := \{f \text{ in the } \mu\text{-homotopy class of } f_0\}/ \sim_R .$$

Results on $M_\mu^{mar}(f_0)$ and $M_\mu(f_0)$

Theorem ((a) H '99, (b)-(d) H '11)

(a) $M_\mu(f_0)$ can be constructed as an analytic geometric quotient.

(b) $M_\mu^{mar}(f_0)$ can be constructed as an analytic geometric quotient.

(c) $G_{\mathbb{Z}}(f_0)$ acts properly discontinuously on $M_\mu^{mar}(f_0)$ via

$$\psi \in G_{\mathbb{Z}}(f_0) : [(f, \pm\rho)] \mapsto [(f, \pm\psi \circ \rho)].$$

$$M_\mu(f_0) = M_\mu^{mar}(f_0)/G_{\mathbb{Z}}(f_0).$$

(d) Locally $M_\mu^{mar}(f_0)$ is isomorphic to a μ-constant stratum. Locally $M_\mu(f_0)$ is isomorphic to a (μ-constant stratum)/(a finite group).

$M_\mu^{mar}(f_0)$ for the singularities with modality 0, 1, 2

(Joint work with Falko Gauss '15+'17)

Singularity family	Isom class of $M_\mu^{mar}(f_0)$
ADE sing	point
$\widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$ = simple ell sing	$\mathbb{H}$
T_{pqr}	$\mathbb{C}$
exceptional unimodal sing	$\mathbb{C}$
exceptional bimodal sing	$\mathbb{C}^2$
quadrangle sing	$(\mathbb{H} - (\text{discrete set})) \times \mathbb{C}$
series, generic, e.g. $E_{3,p}$ with $18 \not| p$	$\mathbb{C}^* \times \mathbb{C}$
subseries, e.g. $E_{3,p}$ with $18 \| p$	countably many copies of $\mathbb{C}^* \times \mathbb{C}$

Analogue of Teichmüller space for Riemann surfaces,
but in general (?) not contractible and ∞ many components.

μ-constant monodromy group $G^{mar}(f_0)$

Mather '68: $\mu(f,0) < \infty \quad \Rightarrow \quad f \sim_{\mathcal{R}} j_{\mu+1} f := (\mu+1)$-jet of f.

Choose f_0 with isol sing, $\mu := \mu(f_0)$, $G_{\mathbb{Z}} := G_{\mathbb{Z}}(f_0)$.

$$
\begin{aligned}
J(f_0) &:= \text{component of } \{g \in \mathbb{C}[x]_{\deg \leq \mu+1} \,|\, \mu(g) = \mu\} \\
&\quad\ \text{which contains } j_{\mu+1} f_0, \quad (\text{big } \mu\text{-constant family}) \\
\pi_1(J(f_0)) &\to G_{\mathbb{Z}} \\
G^{mar}(f_0) &:= \langle \text{image in } G_{\mathbb{Z}}, \pm \mathrm{id} \rangle \subset G_{\mathbb{Z}} \\
&= \{g \in G_{\mathbb{Z}} \,|\, \pm g \text{ transversal mon of a } \mu\text{-constant family}\}.
\end{aligned}
$$

Lemma

G^{mar} = subgroup of $G_{\mathbb{Z}}$ which acts on the component of $M_\mu^{mar}(f_0)$ which contains $[(f_0, \pm \mathrm{id})]$. Thus

$$G_{\mathbb{Z}}/G^{mar}(f_0) \stackrel{1:1}{\longleftrightarrow} \{\text{components of } M_\mu^{mar}(f_0)\}.$$

Classifying space D_{BL} for Brieskorn lattices, period map

Fix a sing f_0. $\rightsquigarrow$ A classifying space D_{BL} for Brieskorn lattices with PMHS with same invariants (spectral pairs) as $H_0''(f_0)$ (H 99).

$$\begin{array}{lcl} D_{BL} & \longleftarrow & \mathbb{C}^{N_1} \\ \downarrow & & \\ D_{PMHS} & \longleftarrow & \mathbb{C}^{N_2} \\ \downarrow & & \\ \prod_i D_{PHS_i} & & \end{array}$$

Hol period map $\quad BL : M_\mu^{mar}(f_0) \to D_{BL}, \quad f \mapsto$ marked $H_0''(f)$.

Theorem (M. Saito '89 weaker statement, H '01)

Infinitesimal Torelli result: BL is an immersion.

Torelli type conjectures

Conjecture A (H '91): $H_0''(f)$ determines f up to $\sim_R$.
Equiv. (H 00): The period map

$$BL/G_{\mathbb{Z}} : M_\mu(f_0) \to D_{BL}(f_0)/G_{\mathbb{Z}}(f_0), \quad [f] \mapsto H_0''(f) \text{ mod isom},$$

is injective.

Conjecture B (H '11): $BL : M_\mu^{mar}(f_0) \to D_{BL}$ is injective.

Lemma (H '11): $B \Rightarrow A$.

Theorem (A: H '92+'93, B: Gauss+H '11+'15+'17)

A and B are true for the singularities with modality ≤ 2 and for the Brieskorn-Pham singularities with coprime exponents.

Present and future (?) methods

Present methods:

(i) Determination of $G_{\mathbb{Z}}$ and G^{mar} (both difficult).

(ii) Gauss-Manin connection calculations for the period map BL (rather easy, classical).

Future (?) methods:

(iii) Thicken M_μ^{mar} to a germ of a μ-dim F-manifold M^{mar} along M_μ^{mar} which is everywhere locally the base of a universal unfolding. Glue it with a global space of semisimple Stokes regions.

(iv) Extend Torelli type conjectures to points beyond M_μ^{mar}, especially to semisimple points.
There Stokes structure instead of PMHS and H_0''.
Compare semisimple points and points in M_μ^{mar}.

Global unfolding of the simple singularities (ADE)

From now on, most of the time only the simple singularities (ADE) and the simple elliptic singularities ($\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$, 1-par-families) are considered. There I have results on (iii) and (iv).

Each ADE-singularity $f(x)$ has a global unfolding

$$F(x,t) = F_t(x) = f(x) + \sum_{i=1}^{\mu} m_i t_i,$$

$m_i \in \mathbb{C}[x_0, ..., x_n]$ suitable monomials, $t = (t_1, ..., t_\mu) \in M = \mathbb{C}^\mu$.

Here $M = M^{alg} = M^{mar} = \mathbb{C}^\mu$ is a thickening of $M_\mu^{mar} = \{pt\}$.

Therefore $G_{\mathbb{Z}}$ acts on M^{mar}, and (Theorem H '18:)

$$\{\pm \mathrm{id}\} \hookrightarrow G_{\mathbb{Z}} \twoheadrightarrow \mathrm{Aut}(M^{mar}, \circ, e, E)$$

Global unfolding of the simple elliptic singularities

$\exists$ Legendre families f_{t_μ} with $t_\mu \in \mathbb{C} - \{0,1\}$.

Jaworski '86: $\exists$ a global unfolding $F = f_{t_\mu} + \sum_{i=1}^{\mu-1} m_i t_i$ with

$$M^{alg} = \mathbb{C}^{\mu-1} \times (\mathbb{C} - \{0,1\}),$$

and $F = F(x,t) = F_t(x)$ is locally universal.

Its universal covering $M^{mar} := \mathbb{C}^{\mu-1} \times \mathbb{H}$.

is a thickening of $M^{mar}_\mu \cong \mathbb{H}$.

Therefore $G_\mathbb{Z}$ acts on M^{mar}, and (Theorem H '18:)

$$\{\pm \mathrm{id}\} \hookrightarrow G_\mathbb{Z} \twoheadrightarrow \mathrm{Aut}(M^{mar}, \circ, e, E)$$

Caustic and Maxwell stratum

Let M be the base space of a universal unfolding F of a sing. f.

$$M \supset \mathcal{K}_3 := \{t \in M \,|\, F_t \text{ has not } \mu \ A_1\text{-singularities}\} \quad \text{caustic.}$$

$$M \supset \mathcal{K}_2 := \overline{\{t \in M \,|\, F_t \text{ has } \mu \ A_1\text{-singularities,}} \\ \overline{\text{but } < \mu \text{ critical values}\}} \quad \text{Maxwell stratum.}$$

$$M \supset \mathcal{K}_3 \supset S_\mu := \{t \in M \,|\, F_t \text{ has only one singularity } x^0 \\ \text{and } F_t(x^0) = 0\} \quad \mu\text{-constant stratum.}$$

$\mathcal{K}_3$ and $\mathcal{K}_2$ are (irreducible) hypersurfaces.

On $M - \mathcal{K}_3$ the critical values $u_1, ..., u_\mu$ are locally *canonical* coordinates, there the multiplication is semisimple.

Lyashko-Looijenga map locally

Let M be the base space of a universal unfolding F of a sing. f.

$$
\begin{array}{rccc}
 & t & \mapsto & \text{critical values of } F_t \quad \text{mod } \mathrm{Sym}_\mu \\
LL: & M & \to & \mathbb{C}^\mu / \mathrm{Sym}_\mu \\
 & \cup & & \cup \\
 & \mathcal{K}_3 \cup \mathcal{K}_2 & \to & \text{discriminant} =: \mathcal{D}_{LL}
\end{array}
$$

It is locally biholomorphic on $M - (\mathcal{K}_3 \cup \mathcal{K}_2)$,

branched of order 3 along $\mathcal{K}_3$ and of order 2 along $\mathcal{K}_2$.

(Looijenga '74, Lyashko '74 (published '79+'84))

Lyashko-Looijenga map globally for ADE

ADE-singularities:

<u>Theorem</u> (Looijenga '74): $LL^{alg} : M^{alg} \to \mathbb{C}^\mu / \operatorname{Sym}_\mu$ is a branched covering of order

$$\deg LL^{alg} = \frac{\mu!}{\prod_{i=1}^{\mu} \deg_{\mathbf{w}} t_i} = \frac{\mu! N^{\mu}_{Coxeter}}{|W|}.$$

Here $\mathbf{w} = (w_0, ..., w_n) \in (\mathbb{Q} \cap (0, \frac{1}{2}])^{n+1}$ is a weight system with $\deg_{\mathbf{w}} x_j = w_j$, $\deg_{\mathbf{w}}(f) = 1$, $\deg_{\mathbf{w}} t_i = 1 - \deg_{\mathbf{w}} m_i$.

The restriction

$$LL^{alg} : M^{alg} - (\mathcal{K}_3^{alg} \cup \mathcal{K}_2^{alg}) \to \mathbb{C}^\mu / \operatorname{Sym}_\mu - \mathcal{D}_{LL}$$

is a covering.

Lyashko-Looijenga map globally for $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$

Simple elliptic singularities:

Theorem (Jaworski '86+'88): The restriction

$$LL^{alg} : M^{alg} - (\mathcal{K}_3^{alg} \cup \mathcal{K}_2^{alg}) \to \mathbb{C}^\mu / \operatorname{Sym}_\mu - \mathcal{D}_{LL}$$

is a covering (of finite degree).

Theorem (H-Roucairol '07/'18): $\exists$ partial compactification

$$\begin{array}{ccccc} M^{orb} & \supset & M^{alg} & \leftarrow & \mathbb{C}^{\mu-1} \\ \downarrow & & \downarrow & & \\ \mathbb{P}^1 & \supset & \mathbb{C} - \{0;1\} & & t \end{array}$$

to an orbibundle s.t. $LL^{orb} : M^{orb} \to \mathbb{C}^\mu / Sym_\mu$
is (almost) a branched covering, except that 0-section $\to \{0\}$.

Degree of LL^{alg} for $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$

Jaworski's methods do not allow to calculate $\deg LL^{alg}$.
The calculation of the orbibundles M^{orb} allows it.

Theorem (H-Roucairol '07/'18):

$$\deg LL^{orb} = \deg LL^{alg} = \frac{\mu! \cdot \frac{1}{2} \cdot \sum_{i=2}^{\mu-1} \frac{1}{\deg_{\mathbf{w}} t_i}}{\prod_{i=2}^{\mu-1} \deg_{\mathbf{w}} t_i}.$$

Here $\mathbf{w} = (w_0, ..., w_n) \in (\mathbb{Q} \cap (0, \frac{1}{2}])^{n+1}$ is a weight system with $\deg_{\mathbf{w}} x_j = w_j$, $\deg_{\mathbf{w}}(f) = 1$, $\deg_{\mathbf{w}} t_i = 1 - \deg_{\mathbf{w}} m_i$.

Intersection form and vanishing cycles

Simple elliptic singularities with $n \equiv 0 \mod 4$:
The intersection form I is positive semi-definite.
$\rightsquigarrow$ For any n and any two vanishing cycles δ_1, δ_2 with $\delta_1 \neq \pm\delta_2$.
$I(\delta_1, \delta_2) \in \{0, \pm 1, \pm 2\}$.

Theorem (Jaworski '88, H-Roucairol '18) Consider a path in $M^{alg} - (\mathcal{K}_3^{alg} \cup \mathcal{K}_2^{alg})$ tending to a generic point in $\mathcal{K}_3^{alg} \cup \mathcal{K}_2^{alg} \cup$ (fibers of M^{orb} above $0, 1, \infty$) such that u_i and u_{i+1} come together.

$$\begin{array}{rcl} I(\delta_i, \delta_{i+1}) = & & \text{generic point in} \\ 0 & \leftrightarrow & \mathcal{K}_2, \\ \pm 1 & \leftrightarrow & \mathcal{K}_3, \\ \pm 2 & \leftrightarrow & \text{fibers of } M^{orb} \text{ above } 0, 1, \infty \end{array}$$

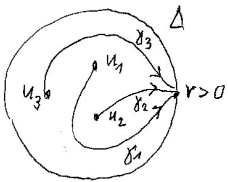
Analogous result for ADE with $I(\delta_i, \delta_{i+1}) \in \{0, \pm 1\}$.

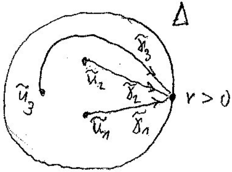
Distinguished bases

Let M be the base space of a universal unfolding F of a sing f.

Choose $t \in M - (\mathcal{K}_3 \cup \mathcal{K}_2)$,

choose a *distinguished system of paths* $\gamma_1, ..., \gamma_\mu$ in Δ:





Push vanishing cycles to $r > 0, r \in \partial\Delta$:

$$\delta_1, ..., \delta_\mu \in Ml(f) \cong H_n(F_t^{-1}(r), \mathbb{Z})$$

$\underline{\delta} = (\delta_1, ..., \delta_\mu)$ is a *distinguished basis* of the Milnor lattice,

it is unique up to signs: $(\pm\delta_1, ..., \pm\delta_\mu)$.

Stokes matrices and Coxeter-Dynkin diagrams

A distinguished basis $\rightsquigarrow$ a *Stokes matrix* S,

$$S := (-1)^{\frac{(n+1)(n+2)}{2}} \cdot L(\underline{\delta}^{tr}, \underline{\delta})^t = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}.$$

$S \longleftrightarrow$ Coxeter-Dynkin diagram (CDD) of $\underline{\delta}$:

Numbered vertices $1, ..., \mu$,
the line between i and j is weighted by s_{ij} (no line if $s_{ij} = 0$).

<u>Theorem</u> (Gabrielov, Lazzeri, Lê '73): All CDD's are connected.

Numbers $|\mathcal{B}|$ and $|\{\text{Stokes matrices}\}|$

$$\mathcal{B} := \{\text{all distinguished bases in } MI(f)\},$$

($\mathcal{B}$ up to signs) $= \mathcal{B}/\{\pm 1\}^{\mu}$,

The braid group Br_{μ} acts on $\mathcal{B}$, $\mathcal{B}$ is one orbit of $Br_{\mu} \ltimes \{\pm 1\}^{\mu}$.

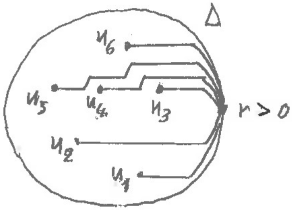
$\mathcal{B}$ comes from one t, many $(\gamma_1, ..., \gamma_{\mu})$.

f	$\lvert\mathcal{B}(f)\rvert$	$\lvert\{\text{Stokes matrices}\}\rvert$	S_{ij}
ADE	finite	finite	$\in \{0, \pm 1\}$
$\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$	infinite	finite	$\in \{0, \pm 1, \pm 2\}$
any other sing.	infinite	infinite	unbounded

(Last line: Ebeling '18)

Stokes regions

But now: many t, one special $(\gamma_1, ..., \gamma_\mu)$:



Now S is a *Stokes matrix* of the *TEZP*-structure of F_t.
Get a map

$$\begin{array}{rcl} LD : M - (\mathcal{K}_3 \cup \mathcal{K}_2) & \to & \mathcal{B}/\{\pm 1\}^\mu \\ t & \mapsto & (\underline{\delta} \text{ (mod signs) from these paths)} \end{array}$$

The connected components of the fibers are *Stokes regions*, the boundaries are *Stokes walls*.

Theorem: a bijection

LD induces

$$\widetilde{LD} : \{\text{Stokes regions}\} \to \mathcal{B}/\{\pm 1\}^\mu.$$

Theorem (Looijenga+Deligne '74 for ADE,
H-Roucairol '07/'18 for $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$)
$\widetilde{LD}^{mar} : \{\text{Stokes regions in } M^{mar}\} \to \mathcal{B}/\{\pm 1\}^\mu$ *is a bijection.*

Interpretation: $M^{mar} - (\mathcal{K}_3^{mar} \cup \mathcal{K}_2^{mar})$ is an *atlas of Stokes data*.

Corollary

$$\frac{\widetilde{LD}^{mar}}{G_{\mathbb{Z}}} : \frac{\{\text{Stokes regions in } M^{mar}\}}{G_{\mathbb{Z}}} \to \frac{\{\text{Stokes matrices}\}}{\{\pm 1\}^\mu}$$

is a bijection.

ADE: Looijenga '73

Looijenga 73: $M^{alg} \cong \mathbb{C}^\mu$. $LL^{alg} : M^{alg} \to \mathbb{C}^\mu / Sym_\mu$ is a branched covering of order $\frac{\mu!}{\prod_{i=1}^{\mu} \deg_{\mathbf{w}} t_i}$.

$$\rightsquigarrow \quad LL^{alg}(\text{one Stokes region}) \stackrel{1:1}{\to} (\mathbb{C}^\mu / Sym_\mu - \mathcal{D}_{LL}),$$

$$\rightsquigarrow \quad \deg LL^{alg} = |\{\text{Stokes regions in } M^{alg}\}|$$

and LL^{alg} branched covering $\rightsquigarrow$ $\widetilde{LD}^{alg}$ is surjective.

For A_μ $\widetilde{LD}^{alg}$ is injective. Question 73: Also for D_μ, E_μ?

ADE: Deligne '74

In the case $n \equiv 0 \mod 4$, $(MI(f), I)$ is the root lattice of type ADE.

Deligne 74: In that case

$$\mathcal{B} = \{\text{bases } \underline{\delta} \text{ of } MI(f) \mid I(\delta_i, \delta_i) = 2,\ s_{\delta_1} \circ \ldots \circ s_{\delta_\mu} = Mon\}$$

and

$$|\mathcal{B}/\{\pm 1\}^\mu| = \ldots = \deg LL^{alg}.$$

$\rightsquigarrow \widetilde{LD}^{alg} : \{\text{Stokes regions in } M^{alg}\} \to \mathcal{B}/\{\pm 1\}^\mu$ is a bijection.

ADE: Hertling '18

New argument for $\widetilde{LD}^{mar}$ injective:

Let A and B be Stokes regions in M^{mar} with $\widetilde{LD}(A) = \widetilde{LD}(B)$.

$\rightsquigarrow \quad CDD(A) = CDD(B)$ and $S(A) = S(B)$.

$$\rightsquigarrow \quad \exists!\text{ deck trf. } \psi_M : \begin{array}{ccccccc} A & & \mapsto & & B & & \\ M & & \to & & M & \supset & \mathcal{K}_2 \cup \mathcal{K}_3 \\ & \searrow & & \swarrow & & & \downarrow \\ & & \mathbb{C}^\mu / Sym_\mu & & & \supset & \mathcal{D}_{LL} \end{array}$$

Proof with: $s_{ij} \in \{0, \pm 1\}$,
$s_{ij} = 0 \leftrightarrow \mathcal{K}_2, \quad s_{ij} = \pm 1 \leftrightarrow \mathcal{K}_3.$

ADE: Hertling '18

Recall the Theorem (H '18): $\mathrm{Aut}(M^{mar}, \circ, e, E) \cong G_{\mathbb{Z}}/\{\pm \mathrm{id}\}$.

$\rightsquigarrow \psi_M$ comes from an element $\psi_{hom} \in G_{\mathbb{Z}}$ with

$$\widetilde{LD}(B) = \psi_{hom}(\widetilde{LD}(A)).$$

Now $\widetilde{LD}(A) = \widetilde{LD}(B) \Rightarrow \psi_{hom} = \pm \mathrm{id} \Rightarrow \psi_M = \mathrm{id} \Rightarrow A = B$.

The proof for $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$ is analogous.

Tables for ADE

Sing	$\deg LL^{alg}$	$N_{Coxeter}$	$\lvert G_{\mathbb{Z}}\rvert$	$\lvert\{\text{Stokes matrices}\}\rvert$
A_μ	$(\mu+1)^{\mu-1}$	$\mu+1$	$2(\mu+1)$	$2^{\mu-1}\cdot(\mu+1)^{\mu-2}$
D_4	$2\cdot 3^4$	6	36	$2^3\cdot 3^2$
$D_\mu\ (\mu\geq 5)$	$2(\mu-1)^\mu$	$2(\mu-1)$	$4(\mu-1)$	$2^{\mu-1}\cdot(\mu-1)^{\mu-1}$
E_6	$2^9\cdot 3^4$	12	24	$2^{12}\cdot 3^3$
E_7	$2\cdot 3^{12}$	18	18	$2^7\cdot 3^{10}$
E_8	$2\cdot 3^5\cdot 5^7$	30	30	$2^8\cdot 3^4\cdot 5^6$

$\deg LL^{alg}$ $(=|\mathcal{B}/\{\pm 1\}^\mu|)$: Looijenga '74,
$N_{Coxeter}$: classical,
$|G_{\mathbb{Z}}|$: I.S. Lifshits '81, Yu Jianming '90, H '11,
$|\{\text{Stokes matrices}\}|$ $(=2^\mu\cdot\deg LL^{alg}/|G_{\mathbb{Z}}|)$: Deligne '74.

Tables for $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$

$$|\{\text{Stokes matrices}\}| = 2^{\mu-1} \cdot |\{\text{Stokes matrices}\}/\{\pm 1\}^\mu|,$$

$$|\{\text{Stokes matrices}\}/\{\pm 1\}^\mu| = \frac{\deg LL^{alg}}{\deg(M^{alg} \to M^{mar}/G_{\mathbb{Z}})}.$$

Sing	$\deg LL^{alg}$	$\deg(M^{alg} \to M^{mar}/G_{\mathbb{Z}})$	$\lvert\{\text{Stokes matrices}\}\rvert$
$\tilde{E}_6$	$2^2 \cdot 3^{11} \cdot 5 \cdot 7$	$6 \cdot 2 \cdot 3 \cdot 3^2 = 326$	$2^7 \cdot 3^7 \cdot 5 \cdot 7$
$\tilde{E}_7$	$2^{18} \cdot 3 \cdot 5^3 \cdot 7$	$6 \cdot 1 \cdot 4 \cdot 2^2 = 96$	$2^{21} \cdot 5^3 \cdot 7$
$\tilde{E}_8$	$2^9 \cdot 3^{10} \cdot 7 \cdot 101$	$6 \cdot 1 \cdot 6 \cdot 1^2 = 36$	$2^{16} \cdot 3^8 \cdot 7 \cdot 101$

$\deg LL^{alg}$ and $\deg(M^{alg} \to M^{mar}/G_{\mathbb{Z}})$: H-Roucairol '07/'18.
$|\{\text{Stokes matrices}\}|$: $\tilde{E}_6$ Kluitmann '83, $\tilde{E}_7$ Kluitmann '87,
$\tilde{E}_8$ H-Roucairol '07/'18.

Interpretation: Torelli result at semisimple points

Choose $t \in M^{mar} - (\mathcal{K}_3^{mar} \cup \mathcal{K}_2^{mar})$.

$\rightsquigarrow$ The TEZP-structure of F_t is a hol vector bundle $H^{osc}|_{\mathbb{C}\times\{t\}}$ with merom conn ∇^{osc} with a semisimple pole of order 2 at 0.

$\rightsquigarrow$ The marked TEZP-structure is equivalent to

$$\begin{cases} (u_1, ..., u_\mu) = \text{(eigenvalues of the pole part)} \\ LD(t) \in \mathcal{B}/\{\pm 1\}^\mu : \text{(the Stokes structure of the conn)} \end{cases}$$

The bijections $\widetilde{LD}^{mar} : \{\text{Stokes regions in } M^{mar}\} \to \mathcal{B}/\{\pm 1\}^\mu$

and $LL^{mar} : \text{(One Stokes region)} \to \mathbb{C}^\mu / \operatorname{Sym}_\mu - \mathcal{D}_{LL}$

together give a global Torelli result for the marked semisimple TEZP-structures above $M^{mar} - (\mathcal{K}_3^{mar} \cup \mathcal{K}_2^{mar})$.