NEARLY GORENSTEIN RINGS AND ALMOST GORENSTEIN RINGS AND OF DIMENSION 2 – METHODS USING RESOLUTIONS OF SINGULARITIES

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This is a joint work in progress with Tomohiro Okuma (Yamagata Univ.) and Ken-ichi Yoshida (Nihon Univ.) (partially [6]).

In this talk, we always assume that $(A, \mathfrak{m})$ is an excellent normal local ring of dimension 2, containing an algebraic closed field isomorphic to $A/\mathfrak{m}$.

Let $f : X \to \mathrm{Spec}(A)$ be a resolution of singularities of A and $\mathbb{E} = \bigcup_{i=1}^r E_i$ be the exceptional set of f, where E_i $(1 \le i \le r)$ be the irreducible components of $\mathbb{E}$.

Recently, 2 concepts on Cohen-Macaulay rings which try to determine which Cohen-Macaulay rings are "near" to Gorenstein rings came up.

(1) Nearly Gorenstein Rings (Herzog, Hibi, and Stamate (2019), [5]),
(2) Almost Gorenstein Rings ((Barucci and Fröberg (1997) [2] in dimension 1 and Goto, Takahashi and Taniguchi (Endo) in dimension ≥ 1 [3]),)

We analyze these properties using the methods of resolution of singularities – cohomology of coherent shaves and intersection theory.

1. Nearly Gorenstein Rings

For an A-module M, the Trace ideal of M is defined by

$$\mathrm{Tr}_A(M) = \{f(m) \mid m \in M \text{ and } f \in \mathrm{Hom}_A(M, A)\}$$

In particular, we are interested in Trace ideals of the canonical module K_A os A. Note that $\mathrm{Tr}_A(K_A)$ is always $\mathfrak{m}$-primary ideal in our case and $\mathrm{Tr}_A(K_A) = A$ if and only if A is Gorenstein.

Definition 1.1. ([5]) A is nearly Gorenstein if A is Cohen-Macaulay and if $\mathrm{Tr}_A(K_A) \supset \mathfrak{m}$.

Our main Thorem is;

Theorem 1.2. *Assume that A is a 2-dimensional rational singularity which is not Gorenstein and that X is the minimal resolution of A. Then we can describe $\mathrm{Tr}_A(K_A)$ as*

$$\mathrm{Tr}_A(K_A) = H^0(X, \mathcal{O}_X(-\mathbb{F})),$$

where K_X is the canonical divisor of X and $\mathbb{F}$ is the **minimal** *positive cycle such that $K_X + \mathbb{F}$ is anti-nef. Namely, $(K_X + \mathbb{F})E_i \le 0$ for every E_i.*

In particular, A is nearly Gorenstein if and only if $K_X + \mathbb{Z}_0$ is anti-nef, where Z_0 is the fundamental cycle (minimal positive anti-nef cycle).

Example 1.3. Let A be a rational triple point (multiplicity 3). Then the dual graph of $\mathbb{E}$ is star-shaped ([1]), all E_i are $\mathbb{P}^1$ and there is unique E_i with $E_i^2 = -3$ (otherwise $E_i^2 = -2$). Then A is nearly Gorenstein if and only if (-3) curve is an end curve (intersects to only one E_j). By the classification of rational triple points in [1], rational triple points are classified into 9 families and among them 4 families are nearly Gorenstein.

2. Almost Gorenstein Rings.

Almost Gorenstein rings are defined as follows;

Definition 2.1. ([3]) Let A be a Cohen-Macaulay ring of dimension d and let $\omega \in K_A$ a "general" element of K_A. Then A is almost Gorenstein if $U := K_A/\omega A$ is an Ulrich module of dimension $d-1$, where we say U is an Ulrich module if the multiplicity (as an $d-1$-dimensional A-module) is equal to $\mu(U) = \mathrm{type}(A)-1$, the number of minimal generators of U. (Almost Gorenstein rings of dimension 1 are nearly Gorenstein. But in dimension 2, the situation is very different.)

Our main Theorem is;

Theorem 2.2. *Let $(A, \mathfrak{m})$ be a normal 2-dimensional ring. Then A is almost Gorenstein in the following cases.*
(1) *rational singularities (this is already proved in [3]).*
(2) *elliptic singularities*
(3) $\mathfrak{m}$ *is a p_g-ideal (multiplicity of A is* $\mathrm{embdim}(A)-1$ *and* $\mathfrak{m}^n$ *are integrally closed for all $n \geq 2$).*

The following example is interesting but at the same time, show that the notion of almost Gorenstein ring is very complicated in dimension 2.

Example 2.3. ([4] Chapter 5, §2). Let C be a smooth curve over $k = A/\mathfrak{m}$ (of genus $g \geq 2$) and $P \in C$ be a closed point. We define

$$R(C,P) = \bigoplus_{n\geq 0} H^0(C, \mathcal{O}_C(nP))T^n \subset k(C)[T].$$

and $H_{C,P} := \{n \geq 0 \mid h^0(\mathcal{O}_C(nP)) > h^0(\mathcal{O}_C((n-1)P))\}$ the "Weierstrass semigroup" of $P \in C$. Then $R(C,P)$ (localized at the unique graded maximal ideal) is almost Gorensein if and only if $H_{C,P}$ is almost symmetric; namely, if $2g(H) = \mathrm{F}(H) + \mathrm{type}(H)$, where $g(H) = g(C)$ and $\mathrm{F}(H) = \max\{n \in \mathbb{Z} \mid n \notin H\}$.

References

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