Schedule for OIST/RIMS workshop

Monday, June 10

Morning Session

9:00 AM - 10:30 AM

Speaker: Kenji Matsuki
Affiliation: Purdue University/KUIAS
Title of the talk: Introduction to the Idealistic Filtration Program toward resolution of singularities in positive characteristic I
Abstract of the talk: Look for the abstract and content of the talks by Matsuki at the end of the schedule.

10:45 AM - 12:15 AM

Speaker: Diego Sulca and Orlando Villamayor
Affiliation: Universidad Nacional de Córdoba (Sulca) and Universidad Autónoma de Madrid (Villamayor)
Title of the talk: Multiplicity, blow ups of finte morphisms, and applications to radicial coverings of a regular variety I
Abstract of the talk: Look for the abstract and content of the talks by Sulca and Villamayor at the end of the schedule.

Afternoon Session

2:00 PM - 3:00 PM

Speaker: Takehiko Yasuda
Affiliation: Tohoku University
Title of the talk: The wild McKay correspondence for an arbitrary finite group
Abstract of the talk: I will speak about the wild McKay correspondence for an arbitrary finite group, which I recently proved. After proving the case of cyclic group of prime order, I formulated a conjectural generalization to an arbitrary finite groups. An important step towards this conjecture was to construct the moduli space of torsors over the punctured formal disk for the given finite group. This has been done in my joint work with Fabio Tonini, generalizing an earlier work of Harbater. Based on this work, I have developed the motivic integration theory over wild Deligne-Mumford stacks. The wild McKay correspondence is obtained as an application of this theory.

3:15 PM - 4:15 PM

Speaker: Shunsuke Takagi
Affiliation: University of Tokyo
Title of the talk: On the equivalence of Fano type and globally F-regular type
Abstract of the talk: A projective variety X over an algebraically closed field of characteristic zero is said to be of globally F-regular type if its modulo p reduction X_p is globally F-regular for almost all p. Schwede and Smith asked whether projective varieties of globally F-regular type are of Fano type or not. Gongyo-Takagi and Hwang-Park gave an affirmative answer to the two-dimensional case of their question. In this talk, we will discuss the three-dimensional case. This talk is based on joint work with Paolo Cascini.

4:30 PM - 5:30 PM

Speaker: Herwig Hauser
Affiliation: University of Vienna, Austria
Title of the talk: Surfing on singular curves.
Abstract of the talk: We wish to present a recent approach of Hana Melánová towards a more geometrically inspired resolution process.

Tuesday, June 11

Morning Session

9:00 AM - 10:30 AM

Speaker: Diego Sulca and Orlando Villamayor
Affiliation: Universidad Nacional de Córdoba (Sulca) and Universidad Autónoma de Madrid (Villamayor)
Title of the talk: Multiplicity, blow ups of finte morphisms, and applications to radicial coverings of a regular variety II
Abstract of the talk: Look for the abstract and content of the talks by Sulca and Villamayor at the end of the schedule.

10:45 AM - 12:15 AM

Speaker: Kenji Matsuki
Affiliation: Purdue University/KUIAS
Title of the talk: Introduction to the Idealistic Filtration Program toward resolution of singularities in positive characteristic II
Abstract of the talk: Look for the abstract and content of the talks by Matsuki at the end of the schedule.

Special Lunch Break
Prof. Hikami will take us to a campus tour of OIST.
We will have lunch at Cafe "GURANO", where you can have a buffet type lunch for 500 yen.
Afternoon Session
3:00 PM - 4:00 PM

Speaker: Stefan Perlega
Affiliation: UniCredit Bank Austria AG
Title of the talk: Indefinite increase of the residual order
Abstract of the talk: The subject of the talk are singularities defined over a field of characteristic $p > 0$ by a so-called *purely inseparable equation* of the form

$$z^{p^e} + F(x_1, \ldots, x_n) = 0.$$

These singularities are often studied in the context of resolution of singularities since they exhibit certain pathologies which do not appear over fields of characteristic zero. A common invariant that is associated to these singularities for the purpose of measuring improvement under blowups is the *residual order*. It appears as a natural generalization of the resolution invariant in characteristic zero, although it has much less desirable properties.

It is a well-known fact that the residual order may increase under blowups, even if the center is chosen as a closed point. A result of Moh asserts that the increase of the residual order under a single blowup is limited by p^{e-1}, but it leaves open the question how big this increase may become under longer sequences of blowups.

In this talk, we will discuss an example of a sequence of singularities under point-blowups which is constructed as a *cycle* and can be iterated indefinitely. In this example, the residual order increases during each iteration of the cycle, proving that that there is no limit to how much the invariant may increase under longer sequences of blowups.

4:15 PM - 5:15 PM

Speaker: Daisuke MATSUSHITA
Affiliation: Hokkaido University
Title of the talk: On period loci of a subgroup of the automorphism group of an irreducible symplectic manifold.
Abstract of the talk: Let X be an irreducible symplectic manifold and L a isotropic nef line bundle on X. We also let g be an element of $\mathrm{Aut}(X, L)$. Assume that the order of g is infinite. I will report a geometric nature of the fixed loci of g^n, $(n > 0)$ and give an approach to Abundance conjecture.

Wednesday, June 12

Morning Session

9:00 AM - 10:30 AM

Speaker: Kenji Matsuki
Affiliation: Purdue University/KUIAS
Title of the talk: Introduction to the Idealistic Filtration Program toward resolution of singularities in positive characteristic III
Abstract of the talk: Look for the abstract and content of Matsuki's talks at the end of the schedule.

10:45 AM - 12:15 AM

Speaker: Diego Sulca and Orlando Villamayor
Affiliation: Universidad Nacional de Córdoba (Sulca) and Universidad Autónoma de Madrid (Villamayor)
Title of the talk: Multiplicity, blow ups of finte morphisms, and applications to radicial coverings of a regular variety III
Abstract of the talk: Look for the abstract and content of the talks by Sulca and Villamayor at the end of the schedule.

Afternoon Session

2:00 PM - 3:00 PM

Speaker: Nobuo Hara
Affiliation: Tokyo University of Agriculture and Technology
Title of the talk: Self-dual Frobenius summands on a quintic del Pezzo surface
Abstract of the talk: Given a polarized variety (X, L) in characteristic $p > 0$, we want to know how many and what kind of indecomposable direct summands appear in the direct sum decomposition of the iterated Frobenius direct images $F^e_*(L^n)$, where e, n are non-negative integers. In this talk, I will consider this problem on a quintic del Pezzo surface with anti-canonical polarization, focusing on the case where $F^e_*(L^n)$ is self-dual.

3:15 PM - 4:15 PM

Speaker: Santiago Encinas
Affiliation: Universidad de Valladolid
Title of the talk: Nash multiplicity sequences and Hironaka's order function
Abstract of the talk: When X is a d-dimensional variety defined over a field k of characteristic zero, a constructive resolution of singularities can be achieved by successively lowering the maximum multiplicity via blow ups at smooth equimultiple centers. This is done by stratifying the maximum multiplicity locus of X by means of the so called *resolution functions*. The most important of these functions is what we know as *Hironaka's order function in dimension d*. Actually, this function can be defined for varieties when the base field is perfect; however if the characteristic of k is positive, the function is, in general, too coarse and does not provide enough information so as to define a resolution. It is very natural to ask what the meaning of this function is in this case, and to try to find refinements that could lead, ultimately, to a resolution. In this talk we will show that Hironaka's order function in dimension d can be read in terms of the *Nash multiplicity sequences* introduced by Lejeune-Jalabert. Therefore, the function is intrinsic to the variety and has a geometrical meaning in terms of its space of arcs.

4:30 PM - 5:30 PM

Speaker: Ana Bravo
Affiliation: Universidad Autónoma de Madrid
Title of the talk: Contact loci and Hironaka's order
Abstract of the talk: This talk is a continuation of "Nash multiplicity sequences and Hironaka's order function" by S. Encinas. Here we study contact loci sets of arcs and the behavior of Hironaka's order function defined in constructive Resolution of singularities. We show that this function can be read in terms of the irreducible components of the contact loci sets at a singular point of an algebraic variety. This is joint work with S. Encinas and B. Pascual-Escudero.

Thursday, June 12

Morning Session

9:00 AM - 10:30 AM

Speaker: Diego Sulca and Orlando Villamayor
Affiliation: Universidad Nacional de Córdoba (Sulca) and Universidad Autónoma de Madrid (Villamayor)
Title of the talk: Multiplicity, blow ups of finte morphisms, and applications to radicial coverings of a regular variety IV
Abstract of the talk: Look for the abstract and content of the talks by Sulca and Villamayor at the end of the schedule.

10:45 AM - 12:15 AM

Speaker: Hiraku Kawanoue
Affiliation: Chubu University/RIMS
Title of the talk: Surface resolution via IFP
Abstract of the talk: I will talk about the problem of resolution of singularities of a surface embedded in a nonsingular ambient space W. When $\dim W = 3$, this corresponds to the problem of resolution of singularities of an idealistic filtration in Matsuki's talk on Day 3, where the difficulty is concentrated in the monomial case.

I will present the original (old) invariant in the monomial case in $\dim W = 3$, based upon the work of Benito-Villamayor. The new invariant, which Matsuki talks about on Day 3, is the product of his effort to understand why this old invariant works. (At the beginning we could not see why the old invariant works.) The old one actually behaves better than the new one in $\dim W = 3$, while it seems more difficult to generalize it to the higher dimensional case than the new one.

Afternoon Session

2:00 PM - 3:00 PM

Speaker: Shihoko Ishii
Affiliation: Yau Mathematical Science Center, Tsinghua University/ University of Tokyo
Title of the talk: $\mathbb{R}$-multiideal on a smooth surface in positive characteristic
Abstract of the talk: In the talk, I will show that Mustaţă-Nakamura's conjecture holds for pairs consisting of a smooth surface and a real multiideal over the base field of positive characteristic. As corollaries, we obtain the ascending chain condition of the minimal log discrepancies and of the log canonical thresholds for those pairs. We also obtain finiteness of the set of the minimal log discrepancies of those pairs for a fixed real exponent.

3:15 PM - 4:15 PM

Speaker: Eamon Quinlan-Gallego
Affiliation: University of Michigan and University of Tokyo
Title of the talk: Bernstein-Sato polynomials in positive characteristic
Abstract of the talk: In 2009 Mustaţă defined Bernstein-Sato polynomials in prime characteristic for hypersurface singularities and proved that the roots of these polynomials are related to the F-jumping numbers of the hypersurface. We follow Mustaţă's approach and develop a similar definition for the case of arbitrary ideals. We then show that these polynomials still retain information about the F-jumping numbers. We also generalize previous work of Bitoun to this new setting.

4:30 PM - 5:30 PM

Speaker: Hironobu Maeda
Affiliation: Tokyo University of Agriculture and Technology
Title of the talk: Plücker Coordinates, Gauss Composition and the Principal Genus Theorem – High School Algebra in Arithmetic
Abstract of the talk: In 1801 Gauss gave an algorithm to calculate an integral binary quadratic form whose duplication is a given binary form belonging to the principal genus (article 286 in Disquisitiones Arithmeticae). We show that his method is also valid in the case, where the coefficient domain is the principal ideal domain, in which 2 is either a unit or a prime element whose residue field is perfect and generated by units of the domain.

Friday, June 14

Morning Session

9:00 AM - 10:30 AM

Speaker: Hiraku Kawanoue
Affiliation: Chubu University/RIMS
Title of the talk: IFP with radical saturation
Abstract of the talk: I will talk about a different approach to realize the algorithm for resolution of singularities in the framework of IFP from the one Matsuki talks about. My wish is to include the radical saturation, namely, the integral closure of an idealistic filtration, in the algorithm. I will explain why this is more desirable from a theoretical point of view, and present some possible changes of the framework of the IFP itself so that it would better serve my wish.

10:45 AM - 12:15 AM

Speaker: Diego Sulca and Orlando Villamayor
Affiliation: Universidad Nacional de Córdoba (Sulca) and Universidad Autónoma de Madrid (Villamayor)
Title of the talk: Multiplicity, blow ups of finte morphisms, and applications to radicial coverings of a regular variety V
Abstract of the talk: Look for the abstract and content of the talks by Sulca and Villamayor at the end of the schedule.

Afternoon Session

2:00 PM - 3:00 PM

Speaker: Shinobu Hikami
Affiliation: OIST
Title of the talk: Singularity theory for the negative value p of spin curves
Abstract of the talk: The moduli space of p-spin curves ($p > 0$) is related to minimum ADE singularities. For A_n type singularities, n corresponds to $p-1$ ($p = 2, 3, \cdots$). We extend the value pf p to negative integers ($p < 0$) and discuss the relation to the 2 dimensional quasi homogeneous surface singularities for arbitrary genus.

3:15 PM - 4:15 PM

Speaker: Angelica Benito
Affiliation: Universidad Autónoma de Madrid
Title of the talk: "A semicontinuous invariant of singularities in positive characteristic" (joint work with O. Villamayor).
Abstract of the talk: Fix a hypersurface X embedded in a smooth scheme V over a field k. The multiplicity of X at each point defines a function, $mult_X : X \longrightarrow \mathbb{Z}$, which is upper semicontinouos. Hironaka introduces a refinement of this function: $(mult_X, H-ord_X) : X \longrightarrow \mathbb{Z} \times \mathbb{Q}$. When X is a plane curve, and $k = \overline{k}$ is a field of characteristic zero, the value $H-ord(x)$ at a closed point $x \in X$, where the curve is analytically irreducible, is known as the *first characteristic exponent*. For hypersurfaces of arbitrary dimension Hironaka proves that, if k is of characteristic zero, this refinement of the function $mult_X : X \longrightarrow \mathbb{Z}$ is again upper semicontinuous ($\mathbb{Z} \times \mathbb{Q}$ ordered lexicografically).

The semicontinuity of this important refinement fails to hold if the characteristic of k is positive. In this paper we introduce a natural modification of Hironaka's function, say $\widetilde{H}-ord_X$, making use of invariants specific to the characteristic, so that $(\text{mult}_X, \widetilde{H}-ord_X) : X \longrightarrow \mathbb{Z}\times\mathbb{Q}$ is semicontinuous when k is a perfect field. The new function $\widetilde{H}-Hord_X$ coincides with $H-ord_X$ when evaluated at closed points. In particular we prove that Hironaka's function is upper semi continuous in the closed spectrum also in positive characteristic.

Abstract and schedule for the talks by Sulca and Villamayor

Multiplicity, blow ups of finte morphisms, and applications to radicial coverings of a regular variety

The proposal is to study of the multiplicity as the main invariant to resolve singularities in characteristic zero. This also leads to new questions in arbitrary characteristic which we want to explore.

Basically we will view a singular variety X, at least locally, as finite cover of a regular variety, say $X \to V$; as opposed to viewing the singular variety as a subscheme of a regular variety, say $X \subset V$. This approach enables us to consider some natural non-embedded operations. For example if X is normal and $X' \to X$ is the blow up at a regular center Y followed either by the normalization or by some prescribed finite birational morphism. This happens, for instance, if we blow up at the integral closure of some power of the ideal $I(Y)$.

As for positive characteristic we aim to apply these techniques in the study of some very specific singularities: Let k be a perfect field of characteristic $p > 0$, and fix an irreducible polynomial of the form $Z^{p^e} + f(X_1, \ldots, X_d) \in k[X_1, \ldots, X_d][Z]$. This polynomial defines a hypersurface, say $H \subset \mathbb{A}_k^{d+1}$, and the induced extension of coordinate rings $k[X_1, \ldots, X_d] \subset k[H]$ (ring of functions of H) is finite and purely inseparable. Moreover, the highest possibly multiplicity at points of H is at most p^e. We will analyze these finite extensions in more generality, namely when there is a finite and surjective morphisms $\delta : X \to V$ of k-varieties, where V is regular and the extension of function fields $k(V) \subset k(X)$ is purely inseparable. In this case the multiplicity at points of X is bounded by the degree $p^e := [k(X) : k(V)]$.

Assume that the set $F_{p^e}(X) \subset X$ of points of multiplicity p^e is non-empty. We shall introduce and discuss invariants of singularities $x \in X$ when $x \in F_{p^e}(X)$. We will use these invariants to stratify $F_{p^e}(X)$, and we shall study the behavior of these invariants under blow-ups $X' \to X$ at regular centers Y included in a stratum. The techniques involved in this discussion will make use of differential operators on regular varieties.

On the multiplicity, finite morphisms, and blow-ups of finite morphisms.

0.1. Let X be a variety over a perfect field, and let

$$\mathrm{mult}_X : X \to \mathbb{N}$$

be the function that assigns to each point $x \in X$ the multiplicity of the local ring $\mathcal{O}_{X,x}$. It is well-known that mult_X is upper semi-continuous when $\mathbb{N}$ is given the usual order topology ([22], [8]), whence X is stratified as a finite union of locally closed subsets, namely *the level sets*

$$F_n(X) := \{x \in X : \mathrm{mult}_X(x) = n\}, \quad n \in \mathbb{N}.$$

It is also well-known that X is regular if and only if $\mathrm{mult}_X(x) = 1$ for all $x \in X$ ([22, Thm. 40.1]). When X is not regular and n denotes the maximum value of mult_X, then the level set $F_n(X)$ (the maximal multiplicity locus) is a proper closed subset of X. The following theorem shows that the multiplicity fulfills a fundamental principle for invariants attached to singularities which we call *the fundamental point-wise inequality.*

Theorem 0.2 (Dade, [8]; see also [24])**.** *Let $X \xleftarrow{\pi} X_1$ be a blow up at an irreducible regular center included in $F_n(X)$. Then*

$$\mathrm{mult}_X(\pi(x_1)) \geq \mathrm{mult}_{X_1}(x_1), \qquad \forall x_1 \in X_1.$$

In particular, $\max \mathrm{mult}_X \geq \max \mathrm{mult}_{X_1}$.

We remark that this result remains valid if we replace $X \xleftarrow{\pi} X_1$ by a composition $X \leftarrow X_1 \leftarrow X_1'$, where $X_1 \leftarrow X_1'$ is any finite birational morphism over X_1; see the discussion in Remark 0.5,(d).

0.3. When studying the singularities of an algebraic variety, it is often convenient to view the variety as a finite ramified covering of a regular one. Namely, we consider a variety X together with a finite and surjective morphism $\delta : X \to V$ where V is regular. This approach has been shown to be efficient when studying the multiplicity at singular points of X. For instance, Lipman uses this approach to discuss the multiplicity of complex analytic varieties, and also of algebraic varieties, and the effect of blowing up at equimultiple centers; see [21]. More recently, this idea was taken further in [27] to give an alternative proof of resolution of singularities in characteristic zero by using the multiplicity as main invariant of singularity; see also [4] and [26].

The following theorem, which is valid in arbitrary characteristic, serves as a basis for the study of the multiplicity when a variety is presented as a ramified covering of a regular one. We use the following terminology. Given a finite and surjective morphism of integral schemes $\delta : X \to V$, its *generic rank* is the degree $[K(X) : K(V)]$ of the induced extension of function fields $K(V) \subset K(X)$.

Theorem 0.4 ([27]). *Let $\delta : X \to V$ be a finite and surjective morphism of varieties such that V is regular, and let n denote the generic rank of δ. Then*

$$\mathrm{mult}_X(x) \leq n, \quad \forall x \in X.$$

Moreover, when $F_n(X) := \{x \in X : \mathrm{mult}_X(x) = n\}$ is non-empty, then the following holds.

(1) $F_n(X)$ is closed and each fiber $\delta^{-1}(\delta(x))$, $x \in F_n(X)$, contains only one point. In particular, δ induces a homeomorphism $F_n(X) \equiv \delta(F_n(X))$.

(2) An irreducible subvariety $Y \subset X$ included in $F_n(X)$ is regular if and only if its image $\delta(Y) \subset V$ is also regular. In that case, the blow-up of X at Y and the blow-up of V at $\delta(Y)$ fit into a commutative diagram

(0.4.1)
$$\begin{array}{ccc} X & \longleftarrow & X_1 \\ {\scriptstyle \delta}\downarrow & & \downarrow{\scriptstyle \delta_1} \\ V & \longleftarrow & V_1 \end{array}$$

for a uniquely determined morphism $\delta_1 : X_1 \to V_1$. This morphism is again finite, surjective, and has generic rank n.

Remark 0.5.

a) The proof of this theorem follows essentially from a well-known formula of Zariski that describes the behavior of the multiplicity with respect to finite extensions of rings; see [28, VII, Corollary 1 to Theorem 24].

b) The horizontal arrows in (0.4.1) are blow-ups at regular centers; in particular, V_1 is regular since V is so. We remark that the existence of δ_1 making (0.4.1) commutative is not guaranteed if Y is not included in $F_n(X)$, even if $\delta(Y)$ is also regular.

c) Part (2) of the theorem states that δ_1 has again generic rank n, so an application of the theorem to δ_1 shows that $\mathrm{mult}_{X_1}(x_1) \leq n$ for all $x_1 \in X_1$. In particular, this theorem provides a simple proof of Theorem 0.2 in the case that X is provided with a finite and surjective morphism $\delta : X \to V$ with V regular, such that the generic rank of δ coincides with the maximal multiplicity at points of X.

d) Let $X_1 \leftarrow X_1'$ be a finite birational morphism. An application of the formula of Zariski that was mentioned in a) shows that this morphism satisfies the fundamental point-wise inequality for the multiplicity, whence the same holds true for the composition $X \leftarrow X_1 \leftarrow X_1'$. Notice that if $\delta_1' : X_1' \to V_1$ denotes the composition $X_1' \to X_1 \xrightarrow{\delta_1} V_1$, then δ_1' is a finite and surjective morphism of generic rank n, and the commutativity of (0.4.1) remains valid if we replace $X \leftarrow X_1$ by $X \leftarrow X_1 \leftarrow X_1'$ and δ_1 by δ_1'. As an example, we might take for $X_1 \leftarrow X_1'$ the normalization of X_1. In this case, the composition $X \leftarrow X_1 \leftarrow X_1'$ is called *the normalized blow-up at* Y.

e) We finally mention that if X is any variety over a perfect field, say with maximal multiplicity n, and if $x \in F_n(X)$, then there exists an étale neighborhood $X' \to X$ of x and a finite and surjective morphism $\delta : X' \to V$ of generic rank n such that V is regular; thus, the conclusions (1) and (2) in the Theorem apply for this δ. This observation together with the observation in c) can be used now to give a complete proof of Theorem 0.2.

In view of the previous Theorem, we now introduce the following terminology.

Definition 0.6. Let $\delta : X \to V$ be finite and surjective morphism of varieties, say of generic rank n, such that V is regular. We say that δ is *transversal* if $F_n(X) \neq \emptyset$. In that case, an irreducible regular subvariety $Z \subset V$ included in $\delta(F_n(X))$ will be called a *permissible center for* δ. Given such a center $Z \subset V$, let $Y \subset V$ be the subvariety whose underlying set is $\delta^{-1}(Z)$. This is a regular subvariety by Theorem 0.4. Let $V \leftarrow V_1$ and $X \leftarrow X_1$ denote the blow-ups at Z and $\delta^{-1}(Z)$, respectively, and let $\delta_1 : X_1 \to V_1$ be the unique morphism which makes the diagram (0.4.1) commutative. We call δ_1 the *blow-up of* δ *at* Z.

Notice that if δ_1 is again transversal, then a blow up of δ_1 at a permissible center, say $Z_1 \subset V_1$, will produce a new morphism δ_2. If we keep doing this, after r steps we will obtain a commutative diagram

(0.6.1)
$$\begin{array}{ccccccc} X & \longleftarrow & X_1 & \longleftarrow & X_2 & \cdots\longleftarrow & X_r \\ \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta_1} & & \downarrow{\scriptstyle\delta_2} & & \downarrow{\scriptstyle\delta_r} \\ V & \longleftarrow & V_1 & \longleftarrow & V_2 & \cdots\longleftarrow & V_r \end{array}$$

where the vertical arrows are finite and surjective morphisms of generic rank n, and for each index $i < r$: (a) $V_i \leftarrow V_{i+1}$ is a blow-up of V_i at a permissible center for δ_i, say $Z_i \subset V_i$; (b) $X_i \leftarrow X_{i+1}$ is the blow-up of X_i at $\delta_i^{-1}(Z_i) \subset F_n(X_i)$; (c) δ_{i+1} is the blow-up of δ_i at Z_i. We obtain

$$n = \max \mathrm{mult}_X = \max \mathrm{mult}_{X_1} = \cdots = \max \mathrm{mult}_{X_{r-1}} \geq \max \mathrm{mult}_{X_r}.$$

A natural question is if it is always possible to construct such a sequence so that the last inequality is strict. If the base field has characteristic zero, then the answer is affirmative; in fact, this construction can be done algorithmically. A fundamental result in the construction of such an algorithm is the following Representation Theorem.

Theorem 0.7 ([27])**.** *Assume that $\delta : X \to V$ is a transversal morphism of varieties over a field of characteristic zero with V regular, and let n be the generic rank. Then one can construct a coherent ideal $\mathcal{J}$ on V and positive integer b such that*

(0.7.1)
$$\delta(F_n(X)) = \mathrm{Sing}(\mathcal{J}, b) := \{x \in V : \nu_x(\mathcal{J}) \geq b\},$$

and such that this equality is stable under blow-ups, in the following sense. Let $Z \subset V$ be a permissible center for δ, let $V \leftarrow V_1 \supset H_1$ be the blow-up of V at Z, where H_1 denotes the exceptional hypersurface, and let $\delta_1 : X_1 \to V_1$ denote the blow-up of δ at Z. Then $\mathcal{J}\mathcal{O}_{V_1} = \mathcal{I}(H_1)^b \mathcal{J}_1$ for an $\mathcal{O}_{V_1}$-ideal $\mathcal{J}_1$, and

$$\delta(F_n(X_1)) = \mathrm{Sing}(\mathcal{J}_1, b).$$

Remark 0.8. It is also required that (0.7.1) is preserved after restriction to open subsets, multiplication by the affine line, and after any sequence of transformations made of these three types of operations. This is expressed by saying that $\delta(F_n(X))$ is represented by an idealistic exponent, in the sense of Hironaka. We do not go into details and refer the reader to [27] where the Theorem is formulated using Rees algebras rather than pairs $(\mathcal{J}, b)$. We also remark that, as a consequence of Theorem 0.7, the algorithmic reduction of the multiplicity is a consequence of the constructive resolution of pairs (or of Rees algebras). We finally mention that in positive characteristic, the pair constructed in Theorem 0.7 only gives an inclusion $\delta(F_n(X)) \subseteq \mathrm{Sing}(\mathcal{J}, b)$, which is in general strict. Moreover, there are examples where $\delta(F_n(X))$ is not represented by any idealistic exponent.

0.9. *Closed immersions vs. finite morphisms.* Blowing up singular varieties at regular centers is crucial in Hironaka's approach for resolution of singularities over fields of characteristic zero, in which he considers an embedding $X \subset V$ of the singular variety into a regular variety, and uses the Hilbert-Samuel function as main invariant of singularities. Note that if we blow up X and V simultaneously at a regular center included in X, then there is a natural inclusion $X_1 \subset V_1$ of the corresponding blow-ups. We refer here to [6] where the semi-continuity of the Hilbert-Samuel function is studied in the first chapter (Theorem 1.33), and the behavior under blow-ups (the fundamental point-wise inequality) is carefully treated in the second chapter.

The situation when using the multiplicity as main invariant of singularity is quite different since the techniques here rely on the existence of finite morphisms $X \to V$ and transformations of finite morphisms. Note that the replacement of X by its normalization $\tilde{X}$ is not compatible with embeddings $X \subset V$, for example the normalization of a hypersurface might not be a hyperpersurface. However, normalization is compatible with finite morphisms $X \to V$ since the composition $\tilde{X} \to X \to V$ remains finite. Moreover, it follows from the observations in Remark 0.5, d), that one can produce diagrams like (0.6.1) so that each morphism in the top row is a normalized blow-up at a regular center included in the set of points of multiplicity n, and each of them satisfies the fundamental point-wise inequality for the multiplicity. This is a peculiar property of the multiplicity, namely that the point-wise inequality also holds for blow ups followed by finite birational morphisms such as the normalization. These will show up in our discussion although we shall restrict our attention on a very specific case in characteristic $p > 0$.

0.10. *The case of radicial morphisms.* The Representation Theorem 0.7 is still valid in the case of characteristic $p > 0$ if n is not divisible by p. On the other hand, if $p|n$, say $n = p^e m$ with $(p, m) = 1$, then we can still attach to $\delta : X \to V$ a pair $(\mathcal{J}, b)$ and a morphism of schemes $\delta' : X' \to V$ of generic rank p^e such that $\delta(F_n(X)) = \mathrm{Sing}(\mathcal{J}, b) \cap \delta'(F_{p^e}(X'))$, and such that this description is preserved under any sequence of morphisms that are either blow-ups at permissible centers, restriction to open subsets or multiplications by the affine line. The next step would be to consider the case when $n = p^e$. We shall focus here in a very special case. The case in which δ is radicial.

For our next discussion, p denotes a fixed prime, and q a power of p. For a ring B of characteristic p we denote B^p the image of the Frobenius endomorphism $F : B \to B$. If B is a domain, we denote $B^{1/p} = \{x \in L : x^p \in B\}$, where L is an algebraic closure of the fraction field of B. Similar notation is used when p replaced by q. This notation extends also to sheaves of rings on schemes. Finally, for a scheme V, when we use the expressions $\mathcal{O}_V$-ideal, $\mathcal{O}_V$-module and $\mathcal{O}_V$-algebra we are assuming in advance that they are respectively quasi-coherent $\mathcal{O}_V$-ideals, modules and algebras.

We now introduce the class of morphisms we shall deal with.

Definition 0.11. We define $\mathscr{C}_q$ as the class of all finite and surjective morphisms $\delta : X \to V$ of integral schemes of characteristic p where V is regular and the following holds.

(i) V is F-finite, that is, every affine ring S of V is finite over its subring S^p (e.g. a variety over a field k of characteristic p such that $[k : k^p]$ is finite).

(ii) If S is an affine ring of V and B is the corresponding affine ring of X (that is, $\mathrm{Spec}(B) = \delta^{-1}(\mathrm{Spec}(S))$), then $B^q \subseteq S$. Here, we are identifying S with a subring of B.

We also denote by $\mathscr{C}_q(V)$ the subclass of $\mathscr{C}_q$ consisting of those morphisms $\delta : X \to V$ for a fixed V.

The following is a list of observations where we also introduce some notation.

(1) Any $\delta : X \to V$ in $\mathscr{C}_q$ is in particular affine, and its underlying continuous map is a homeomorphism. Hence, we can think of $\mathcal{O}_X$ as a sheaf on V which includes $\mathcal{O}_V$ as a subsheaf. Condition (ii) in Definition 0.11 can be therefore written as $\mathcal{O}_X^q \subseteq \mathcal{O}_V$.

(2) Given $\delta : X \to V$ in $\mathscr{C}_q$, the induced extension of function fields, say $K(V) \subset K(X)$, is finite and purely inseparable; in particular, the generic rank of δ is a power of p, say p^t. Theorem 0.4, which was formulated for varieties, have been actually proved in [27] under more general hypothesis, and can be applied to our morphism δ too. In particular, the multiplicity along points of X is at most p^t. For convenience of notation, we shall write $F_\delta(X) \subset X$ instead of $F_{p^t}(X)$ (= the set of points of multiplicity equal to the generic rank of δ). As in the case of varieties, we say that δ is *transversal* if $F_\delta(X) \neq \emptyset$.

(3) $\mathscr{C}_q(V)$ can be viewed as a full subcategory of the category of finite V-schemes. Since the latter category is (anti)-equivalent to the category of coherent $\mathcal{O}_V$-algebras, condition (ii) in Definition 0.11 implies that $\mathscr{C}_q(V)$ is (anti)-equivalent to the category of $\mathcal{O}_V$-subalgebras of $\mathcal{O}_V^{1/q}$. One easily checks that the only morphisms in the latter category are given by inclusions. Finally, by an application of Frobenius, we obtain:

Lemma 0.12. *By assigning to each $\delta \in \mathscr{C}_q(V)$ the $\mathcal{O}_V^q$-subalgebra $\mathscr{B}_\delta := \mathcal{O}_X^q \subseteq \mathcal{O}_V$, we obtain an equivalence between $\mathscr{C}_q(V)$ and the collection of coherent $\mathcal{O}_V^q$-subalgebras of $\mathcal{O}_V$. In addition, given $\delta, \delta' \in \mathscr{C}_q(V)$, there exists a V-morphism $X \to X'$ if and only if $\mathscr{B}_{\delta'} \subseteq \mathscr{B}_\delta$, and in that case there is only one such V-morphism.*

Example: Let $V = \mathbb{A}^d_k = \mathrm{Spec}(k[X_1, \dots, X_d])$, where k is a field of characteristic p such that $[k : k^p]$ is finite. This is an F-finite regular scheme. We obtain a morphism $\delta : X \to V$ in $\mathscr{C}_q(V)$ by setting $X = \mathrm{Spec}(k[X_1, \dots, X_d][Z]/\langle Z^q + f(X_1, \dots, X_d)\rangle)$, where $f(X_1, \dots, X_d) \notin (k[X_1, \dots, X_d])^p$. According to our previous discussion, we shall associate to δ the $(k[X_1, \dots, X_d])^q$-subalgebra of $k[X_1, \dots, X_d]$ generated by $f(X_1, \dots, X_d)$.

0.13. Fix a transversal morphism $\delta : X \to V$ in the class $\mathscr{C}_q(V)$. Our objective is to introduce invariants of singularities for points in $F_\delta(X)$. Lemma 0.12 with the aid that δ is a homeomorphism enable us to replace $F_\delta(X)$ by $\delta(F_\delta(X)) \subset V$, and δ by the $\mathcal{O}_V^q$-subalgebra $\mathscr{B}_\delta := \mathcal{O}_X^q$. Since V is F-finite and regular, the module of differentials Ω^1_V is locally free of finite rank, whence the $\mathcal{O}_V$-module Diff^i_V of differential operators on V of order $\leq i$ is also locally free of finite rank. We denote by $\mathit{Diff}^i_{V,+} \subset \mathit{Diff}^i_V$ the $\mathcal{O}_V$-submodule consisting of those operators D such that $D(1) = 0$. By evaluation, we obtain $\mathcal{O}_V$-ideals

$$\mathit{Diff}^1_{V,+}(\mathscr{B}_\delta) \subseteq \ldots \subseteq \mathit{Diff}^{q-1}_{V,+}(\mathscr{B}_\delta).$$

This list of $q-1$ ideals will play a predominant role in our investigation of invariants of singularities that refine the multiplicity. The following theorem shows an instance of this assertion, namely, it describes $\delta(F_\delta(X))$ as the support of $\mathit{Diff}^{q-1}_{V,+}(\mathscr{B}_\delta)$. It will also illustrate how the notions of permissible centers and blow-ups can be expressed entirely in terms of $\mathcal{O}^q_V$-subalgebras and transformations of these algebras. For completeness, we include part of Theorem 0.4.

Theorem 0.14. *Fix a transversal morphism $\delta : X \to V$ in the class $\mathscr{C}_q$, and set $\mathscr{B}_\delta := \mathcal{O}^q_X \subset \mathcal{O}_V$.*

(1) $F_\delta(X)$ is closed, and its homeomorphic image in V can be expressed as

$$\delta(F_\delta(X)) = \mathcal{V}(\mathit{Diff}^{q-1}_{V,+}(\mathscr{B}_\delta)).$$

(2) Let $Y \subset X$ be a closed irreducible subscheme, and let $Z \subset V$ denote the closed subscheme of V whose sheaf of ideals is $\mathcal{I}(Y) \cap \mathcal{O}_V$ (whence $Z = \delta(Y)$ as sets). Then following assertions are equivalent:

(a) Y is regular and it is included in $F_\delta(X)$.

(b) Z is regular and it is included in $\mathcal{V}(\mathit{Diff}^{q-1}_{V,+}(\mathscr{B}_\delta))$.

(3) If the equivalent conditions in (2) are satisfied, then the blow-up of X at Y and the blow-up of V at Z fit into a commutative diagram

(0.14.1)
$$\begin{array}{ccc} Y \subset X & \longleftarrow & X_1 \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta_1} \\ Z \subset V & \longleftarrow & V_1 \end{array} ,$$

for a uniquely determined morphism δ_1, which is also in the class $\mathscr{C}_q$.

(4) The $\mathcal{O}^q_{V_1}$-algebra $\mathscr{B}_{\delta_1} := \mathcal{O}^q_{X_1} \subseteq \mathcal{O}_{V_1}$ associated with δ_1 coincides with the $\mathcal{O}^q_{V_1}$-subalgebra of $\mathcal{O}_{V_1}$ generated by the conductor $(\mathscr{B}_\delta \mathcal{O}^q_{V_1} : \mathcal{L}_1^{(q)}) \subseteq \mathcal{O}_{V_1}$. Here, $\mathscr{B}_\delta \mathcal{O}^q_{V_1}$ denotes the $\mathcal{O}^q_{V_1}$-subalgebra of $\mathcal{O}_{V_1}$ generated by the sections of $\mathscr{B}_\delta$ when these are viewed as sections on the blow-up V_1, $\mathcal{L}_1$ denotes the exceptional ideal of the blow-up $V \leftarrow V_1$, and $\mathcal{L}_1^{(q)}$ denotes the $\mathcal{O}^q_V$-ideal $\mathcal{L}_1 \cap \mathcal{O}^q_V \subset \mathcal{O}^q_V$.

We introduce the following definition before going into some remarks about this theorem.

Definition 0.15. For a pair $(V, \mathscr{B})$ with $\mathscr{B}$ an $\mathcal{O}^q_V$-subalgebra, we define $\operatorname{Sing}(V, \mathscr{B}) := \mathcal{V}(\mathit{Diff}^{q-1}_{V,+}(\mathscr{B}))$. A regular subscheme $Z \subset V$ included in $\operatorname{Sing}(V, \mathscr{B})$ is called a permissible center for $(V, \mathscr{B})$. If $V \leftarrow V_1$ is the blow-up of V at Z and $\mathcal{L}_1$ is the exceptional ideal, then we call the $\mathcal{O}^q_{V_1}$-subalgebra $\mathcal{O}^q_{V_1}[(\mathscr{B}\mathcal{O}^q_{V_1} : \mathcal{L}_1^{(q)})]$ the 1-transform of $\mathscr{B}$ by the blow-up.

Remark 0.16. Lemma 0.12 and Theorem 0.14 enable us to work entirely within the setting of pairs $(V, \mathscr{B})$, where V is an irreducible F-finite regular scheme and $\mathscr{B}$ is an $\mathcal{O}^q_V$-subalgebra. In fact, if $\mathscr{B} = \mathscr{B}_\delta$ for a morphism $\delta : X \to V$, then part (1) of Theorem 0.14 describes $\delta(F_\delta(X))$ as the closed set $\operatorname{Sing}(V, \mathscr{B})$, part (2) says that a closed subscheme $Z \subset V$ is permissible for δ in the sense of Definition 0.6 if and only if Z is permissible for $(V, \mathscr{B})$ in the sense of Definition 0.15, and finally (4) states that the blow-up of δ at Z is the morphism associated with the 1-transform of $\mathscr{B}$. In summary, we can replace diagrams like (0.14.1) by diagrams of the form

(0.16.1)
$$\begin{array}{ccc} V & \longleftarrow & V_1 \\ \mathscr{B} & & \mathscr{B}_1 \end{array}$$

Similarly, an iteration of transformations like (0.6.1) corresponds to a sequence

(0.16.2)
$$\begin{array}{ccccccc} V & \xleftarrow{\pi_1} & V_1 & \xleftarrow{\pi_2} & \dots & \xleftarrow{\pi_r} & V_r \\ \mathscr{B} & & \mathscr{B}_1 & & \dots & & \mathscr{B}_r \end{array}$$

where $V_i \leftarrow V_{i+1}$ is the blow-up at a regular center, say $Z_i \subset V_i$, that is included in $\operatorname{Sing}(V_i, \mathscr{B}_i)$, and $\mathscr{B}_{i+1}$ is the 1-transform of $\mathscr{B}_i$.

On the definition of invariants of singularities.

We have associated with a pair $(V, \mathscr{B})$ a closed set $\operatorname{Sing}(V, \mathscr{B}) := \mathcal{V}(\mathit{Diff}^{\,q-1}_{V,+}(\mathscr{B})) \subset V$ which, if not empty, is the maximal multiplicity locus of the V-scheme defined by $\mathscr{B}^{1/q}$. Our aim is to refine the multiplicity by defining upper semi-continuous functions on V that could lead to a stratification of $\operatorname{Sing}(V, \mathscr{B})$. A possible way to achieve this is by making use of the following list of ideals:

$$\mathcal{G}(\mathscr{B}) = (\mathit{Diff}^{\,1}_{V,+}(\mathscr{B}), \ldots, \mathit{Diff}^{\,q-1}_{V,+}(\mathscr{B})) \tag{0.16.3}$$

This is in fact an example of the following definition.

Definition 0.17. A *q-differential collection of ideals* is a sequence of $\mathcal{O}_V$-ideals $\mathcal{G} = (\mathcal{I}_1, \ldots, \mathcal{I}_{q-1})$ with the property that $\mathit{Diff}^{\,i}_V(\mathcal{I}_j) \subseteq \mathcal{I}_{i+j}$ whenever $i + j < q$. Given such a sequence and a point $x \in V$, we set

$$\eta_x(\mathcal{G}) := \min\{\nu_x(\mathcal{I}_i) + i : 1 \le i \le q-1\}$$

where $\nu_x(\mathcal{I}_i)$ denotes, as usual, the order of $\mathcal{I}_i$ at x.

The assignation $x \mapsto \eta_x(\mathcal{G})$ defines an upper semi-continuous function $\eta(\mathcal{G}) : V \to \mathbb{N}$ when $\mathbb{N}$ is given the usual order. The following theorem establishes the fundamental point-wise inequality for $\eta(\mathcal{G})$ under an appropriate definition of transformation of q-differential collections. This theorem will be used in the construction of upper semi-continuous functions for pairs $(V, \mathscr{B})$ (Theorem 0.19), and also for pairs $(V, \mathscr{B})$ with a normal crossing divisor (Theorem 0.22).

Theorem 0.18. *Let $\mathcal{G}$ be a q-differential collection on V, let $Z \subset V$ be a regular center included in the maximum locus of $\eta(\mathcal{G})$, say $\eta_x(\mathcal{G}) = aq + b$ for all $x \in Z$ (with $a, b \in \mathbb{N}_0$ and $0 \le b < q$), let $V \xleftarrow{\pi} V_1$ be the blow-up at Z, and let $\mathcal{L}$ denote the exceptional ideal. Then the collection*

$$\mathcal{G}_1 := ((\mathcal{I}_1\mathcal{O}_{V_1} : \mathcal{L}^{qa}), \ldots, (\mathcal{I}_{q-1}\mathcal{O}_{V_1} : \mathcal{L}^{qa})) \tag{0.18.1}$$

is also q-differential, and we have the following point-wise inequality:

$$\eta_{\pi(x_1)}(\mathcal{G}) \ge \eta_{x_1}(\mathcal{G}_1), \quad \forall x_1 \in V_1. \tag{0.18.2}$$

We call the collection (0.18.1) the a-transform of $\mathcal{G}$ by the blow-up. The first part of the following theorem shows that the function η for q-differential collections of the form $\mathcal{G}(\mathscr{B})$, with $\mathscr{B} \subset \mathcal{O}_V$ an $\mathcal{O}_V^q$-subalgebra, can be used to stratify the closed set $\operatorname{Sing}(V, \mathscr{B})$.

Theorem 0.19. *For an $\mathcal{O}_V^q$-subalgebra $\mathscr{B} \subseteq \mathcal{O}_V$, the following holds.*

(1) $\operatorname{Sing}(V, \mathscr{B}) = \{x \in V : \eta_x(\mathcal{G}(\mathscr{B})) \ge q\}$.

(2) Assume that $\operatorname{Sing}(V, \mathscr{B})$ is non-empty, and write the maximum value of $\eta(\mathcal{G}(\mathscr{B}))$ in the form $aq + b$ with $0 \le b < q$. Then a regular center $Z \subset V$ included in the maximum locus defines a sequence

$$\begin{array}{ccccccc} V & \xleftarrow{\ \pi\ } & V_1 & \xleftarrow{\ \cong\ } & \ldots & \xleftarrow{\ \cong\ } & V_1 \\ \mathscr{B} & & \mathscr{B}_1 & & \ldots & & \mathscr{B}_a \end{array} \tag{0.19.1}$$

where

(a) $V \leftarrow V_1$ is the blow-up at Z and $\mathscr{B}_1$ is the 1-transform of $\mathscr{B}$ by π.

(b) For each index $1 \le i < a$, the exceptional hypersurface of π, say $H_1 \subset V_1$, is included in $\operatorname{Sing}(V_1, \mathscr{B}_i)$, the isomorphism $V_i \xleftarrow{\cong} V_{i+1}$ is the blow-up of V_i at H_1, and $\mathscr{B}_{i+1}$ is the 1-transform of $\mathscr{B}_i$.

(c) $H_1 \not\subset \operatorname{Sing}(V_1, \mathscr{B}_a)$,

(3) If $(\mathcal{G}(\mathscr{B}))_1$ denotes the a-transform of $\mathcal{G}(\mathscr{B})$, then there is a component-wise inclusion

$$\mathcal{G}(\mathscr{B}_a) \subseteq (\mathcal{G}(\mathscr{B}))_1. \tag{0.19.2}$$

In particular, $\eta_{x_1}(\mathcal{G}(\mathscr{B}_a)) \ge \eta_{x_1}((\mathcal{G}(\mathscr{B}))_1)$ for all $x_1 \in V_1$, whence

$$\{x_1 \in V_1 : \eta_{x_1}((\mathcal{G}(\mathscr{B}))_1) \ge q\} \subset \operatorname{Sing}(V_1, \mathscr{B}_1). \tag{0.19.3}$$

We call $\mathscr{B}_a$ *the a-transform of* $\mathscr{B}$ by the blow-up. Property (1) in the Theorem asserts that if $\delta : X \to V$ is the V-scheme defined by $\mathscr{B}^{1/q}$, and if $\delta_1 : X_1 \to V_1$ is the blow-up of δ at Z, then we can still blow up δ_1 exactly $a - 1$ times at H_1 (obtaining as a result the V-scheme $\delta_a : X_a \to V_1$ defined by $\mathscr{B}_a^{1/q}$).

Remark 0.20. The inclusion (0.19.2) is in general strict, and therefore we cannot make use of Theorem 0.18 to deduce a point-wise inequality $\eta_{\pi(x_1)}(\mathcal{G}(\mathscr{B})) \geq \eta_{x_1}(\mathcal{G}(\mathscr{B}_a))$, $\forall x_1 \in V_1$, which would be desirable. In fact, examples show that this inequality does not hold in general. Nevertheless, we shall see in the lectures that a sequence of permissible transformations of q-differential collections starting from $\mathcal{G}(\mathscr{B})$, in a way that we will specify, induces a sequence of permissible transformation of $\mathscr{B}$, hence also a sequence of blow-ups as (0.6.1).

0.21. When making successive monoidal transformations it is natural to define invariants by making use of the hypersurfaces introduced in the previous blow-ups. So this time we consider 4-uples $(V, \mathscr{B}, \Lambda, \mathcal{L})$ where V is an irreducible F-finite regular scheme, $\mathscr{B} \subset \mathcal{O}_V$ is an $\mathcal{O}_V^q$-algebra, Λ is a finite collection of invertible ideals with normal crossings, and $\mathcal{L}$ is an invertible ideal included in $\mathcal{I}$ for each $\mathcal{I} \in \Lambda$. For example, we could take for Λ the empty collection and $\mathcal{L} = \mathcal{O}_V$. We attach to this 4-tuple the collection

$$\mathcal{G}(\mathscr{B}, \Lambda, \mathcal{L}) := ((\mathit{Diff}^{1}_{V,\Lambda,+}(\mathscr{B}) : \mathcal{L}^1), (\mathit{Diff}^{2}_{V,\Lambda,+}(\mathscr{B}) : \mathcal{L}^2), \ldots, (\mathit{Diff}^{q-1}_{V,\Lambda,+}(\mathscr{B}) : \mathcal{L}^{q-1}))$$

where $\mathit{Diff}^{i}_{V,\Lambda,+} \subseteq \mathit{Diff}^{i}_{V,+}$ denotes the submodule of those differential operators that are logarithmic with respect to each $\mathcal{I} \in \Lambda$.

Theorem 0.22. *Within the previous setting, the following holds.*

(1) $\mathcal{G}(\mathscr{B}, \Lambda, \mathcal{L})$ *is a q-differential collection, and there is a component-wise inclusion* $\mathcal{G}(\mathscr{B}) \subseteq \mathcal{G}(\mathscr{B}, \Lambda, \mathcal{L})$. *In particular,* $\eta_x(\mathcal{G}(\mathscr{B})) \geq \eta_x(\mathcal{G}(\mathscr{B}, \Lambda, \mathcal{L}))$, $\forall x \in V$, *whence*

$$\{x \in V : \eta_x(\mathcal{G}(\mathscr{B}, \Lambda, \mathcal{L})) \geq q\} \subset \operatorname{Sing}(V, \mathscr{B}).$$

(2) Let Z be a regular center included in the maximum locus of the function $\eta(\mathcal{G}(\mathscr{B}, \Lambda, \mathcal{L}))$, say $\eta_x(\mathcal{G}(\mathscr{B}, \Lambda, \mathcal{L})) = aq + b$ for all $x \in Z$ $(a, b \in \mathbb{N}_0, 0 \leq b < q)$, and suppose that $\{\mathcal{I}(Z)\} \cup \Lambda$ has normal crossings. Let $V \xleftarrow{\pi} V_1$ be the blow-up at Z and let $H_1 \subset V_1$ be the exceptional hypersurface. We set:

(a) $\mathscr{B}_1 :=$ the a-transform of $\mathscr{B}$ by the blow-up.

(b) $\Lambda_1 :=$ the collection of the strict transforms of the ideals $\mathcal{I} \in \Lambda$ plus the exceptional ideal $\mathcal{I}(H_1)$.

(c) $\mathcal{L}_1 := (\mathcal{L}\mathcal{O}_{V_1})\mathcal{I}(H_1)$.

Then Λ_1 is a collection of invertible ideals with normal crossings, $\mathcal{L}_1$ is included in $\mathcal{I}_1$ for each $\mathcal{I}_1 \in \Lambda_1$, and there is an inequality

$$\eta_{\pi(x_1)}(\mathcal{G}(\mathscr{B}, \Lambda, \mathcal{L})) \geq \eta_{x_1}(\mathcal{G}(\mathscr{B}_1, \Lambda_1, \mathcal{L}_1)), \quad \forall x_1 \in V_1.$$

Remark 0.23. As for the fundamental point-wise inequality of our functions η in these theorems, it is worth bearing in mind a classical result in algebraic geometry. Let V be a smooth variety over a field k and let $\mathcal{J}$ be an $\mathcal{O}_V$-ideal. It is well-known that the function $x \mapsto \nu_x(\mathcal{J})$, is upper semi-continuous (see [9, Chap. 2] or [11, Chap. 3]). Let $Z \subset V$ be a smooth subvariety included in the maximum locus of $\nu(\mathcal{J})$, say $\nu_x(\mathcal{J}) = b$ for all $x \in Z$. Let $V \leftarrow V_1$ be the blow up of V at Z, and let H_1 denote the exceptional hypersurface. Then V_1 is smooth, and there is a factorization $\mathcal{J}\mathcal{O}_{V_1} = \mathcal{I}(H_1)^b \mathcal{J}_1$ for some $\mathcal{O}_{V_1}$-ideal $\mathcal{J}_1$ which does not vanish along H_1. Then we have

Theorem 0.24. $\nu_{\pi(x_1)}(\mathcal{J}) \geq \nu_{x_1}(\mathcal{J}_1)$ *for all* $x_1 \in V_1$.

0.25. As for the relation with published works, let us mention [10], where Giraud studies Jung conditions for finite radicial coverings of regular varieties in positive characteristic. The outcome of [5] is a first breakthrough in the resolution of singularities of radicial extensions in positive characteristic by means of blow-ups at regular centers. More recently, in [7] Cossart and Piltant prove resolution of singularities for arbitrary three-dimensional schemes of positive characteristic. These are proofs that introduce suitable invariants, which fulfill the point-wise inequality for blow ups at regular centers. Other invariants with point-wise inequality arise in the work of Kawanoue and Matsuki in [16] and [17]. Also in this line, and related with this exposition, is the joint work of the second author with Benito in [2] (see also [3]). All these cited papers also make use either of differential operators or of logarithmic differential operators.

We thank C. Abad, A. Benito, A. Bravo, and S. Encinas, for stimulating discussion on these questions.

Expected schedule. The first three lectures will be devoted to constructive resolution of singularities in characteristic zero, and we shall also present some questions that arises in positive characteristic.

Lecture 1: On the function mult_X as an invariant of singularities and Theorem 0.2.
Lecture 2: Blow-ups of finite morphisms and Theorem 0.4.
Lecture 3: Theorem 0.7 and a proof of Resolution of singularities in characteristic zero.

The last two lectures will be devoted to radicial morphisms of the form introduced in Definition 0.11. According to Remark 0.16, we can focus on $\mathcal{O}_V^q$-subalgebras of $\mathcal{O}_V$, where V is an F-finite regular scheme. Our discussion will be formulated more generally in terms of $\mathcal{O}_V^q$-submodules of $\mathcal{O}_V$. In the context of radicial morphisms, modules and transformations of modules will resemble in a natural way the roll played by ideals and transformations of ideals in the study of the singular locus of a variety.

Lecture 4: p-basis and differential operators techniques applied to the study of R^q-submodules and transformations by blow-ups (where R is an F-finite regular local ring).
Lecture 5: The definition of invariants of singularities using differential operators, and also logarithmic differential operators. Theorems 0.18, 0.19 and 0.22.

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Abstract and schedule for the talks by Matsuki

Matsuki will be giving the first 3 lectures using the time slots in the morning given to Matsuki and Prof. Kawanoue, while Prof. Kawanoue will be giving the last 2 lectures. Please note that Prof. Villamayor will be delivering 5 lectures in the other time slots in the morning.

I will describe the plan of each lecture for the first 3 days under the uniform title mentioned above. Please note that Prof. Kawanoue may be speaking under a different title, and that he will describe his own plan of lectures.

Day 1. IN CHARACTERISTIC ZERO

Part 1. **Resolution of singularities of a curve embedded in a nonsingular surface**: As a warm-up, we start the lecture with the discussion of resolution of singularities of a curve embedded in a nonsingular surface. This is a classical subject, which you can find, e.g., in the textbook "Algebraic Geometry I Complex Projective Varieties" by Mumford. Mumford attributes his proof to Hironaka. Even though the tools are elementary, the proof contains almost all the essential ideas toward the general solution in higher dimensional case in characteristic zero.

The choice of the center for blow up, the singular points of the curve (and its succesive strict transforms), is obvious in this lower dimensional case.

So the only issue is to show that the process terminates after finitely many steps. For this purpose, we introduce two invariants, which should measure the improvements of the singularities effectively after each blow up. The first invariant "μ" is the order of the defining equation of the curve. But the invariant μ by itself is not good enough to measure the improvements effectively. Thus we introduce the second invariant "ν", which is the order of the coefficient ideal on a **hypersurface of maximal contact**. We observe that the first invariant μ never increases after each blow up, and that, when the first invariant μ stays the same, the second invariant ν strictly decreases, finishing the proof.

We note that Mumford "cheats" at one point. Mumford does not prove that the invariant ν is independent of the choice of a hypersurface of maximal contact. (Actually his definition of the invariant ν is the minimum of all such choices, which is good enough to show the termination of the process. Of course Hironaka shows its independence by the so-called Hironaka's trick. But Mumford did not want to introduce Hironaka's trick, I imagine, on which he would have to spend many pages of his book.) We show this independence following Włodarczyk's idea of homogenization.

In the discussion above, we already see the important role played by the differential operators in the problem of resolution of singularities. This is a vision first provided by Giraud.

Part 2. **Resolution of singularities of a basic object**: We observe that the problem of embedded resolution of singularities in higher dimensional case is reduced to the problem of resolution of singularities of a **basic object** $(W, (\mathcal{I}, b), E)$, consisting of a nonsingular variety W, the pair of an ideal $\mathcal{I}$ and a positive integer $b \in \mathbb{Z}_{>0}$, and a simple normal crossing divisor E: Starting with $(W, (\mathcal{I}, b), E)$ with the singular locus $\mathrm{Sing}(\mathcal{I}, b) := \{P \in W; \mathrm{ord}_P(\mathcal{I}) \geq b\} \neq \emptyset$, construct a sequence of transformations

$$(W, (\mathcal{I}, b), E) = (W_0, (\mathcal{I}_0, b), E_0) \leftarrow (W_1, (\mathcal{I}_1, b), E_1) \leftarrow \cdots \leftarrow (W_l, (\mathcal{I}_l, b), E_l)$$

such that $\mathrm{Sing}(\mathcal{I}_l, b) = \emptyset$. (The notion of a basic object with its terminology is due to Villamayor, following the original notion of an idealistic exponent by Hironaka.)

The basic strategy is to construct a hypersurface of maximal contact H, the coefficient ideal $\mathcal{J}$ at level $c \in \mathbb{Z}_{>0}$, and a bounday divisor $B = E|_H$, such that the problem of resolution of singularities for $(W, (\mathcal{I}, b), E)$ is reduced to the one for $(H, (\mathcal{J}, c), B)$ and hence that we can use the induction on dimension.
Of course the basic strategy as stated is too naive, and there are some obstructions.

> Obstruction 1. The order may strictly increase after a transformation, even though our ultimate goal is to reduce the order of the ideal below the specified level b.
> Obstruction 2. A hypersurface of maximal contact does not exist (at $p \in \mathrm{Sing}(\mathcal{I}, b) \subset W$ where $\mathrm{ord}_P(\mathcal{I}) > b$).
> Obstruction 3. Even when a hypersurface of maximal contact H exists, it may not be transversal to the boundary divisor E, and as a consequence $E|_H$ may not be a simple normal crossing divisor on H.

Obstruction 4. A hypersurface of maximal contact exists only locally. So the globalization issue remains to be resolved: Do the processes of resolution constructed locally patch together to provide a glocal process of resolution ?

Hironaka's way of overcoming these obstructions is sublimated into a constructive and explicit algorithm by Villamayor described below:

Solution to Obstructions 1,2,3:

(a) We introduce the pair of invariants $(\text{w-ord}, s)$ where "w-ord" represents the weak-order and where "s" represents the number of irreducible components in the *old* part E_{old} of the boundary divisor E.

(b) We construct the modification $(W, (m(\mathcal{I}), m(b)), m(E) = E \setminus E_{old})$ such that
• $\text{Sing}(m(\mathcal{I}), m(b)) = \text{Sing}(\mathcal{I}, b) \cap \text{Maximum Locus of } (\text{w-ord}, s)$,
• A hypersurface of maximal contact H together with the coefficient ideal $\mathcal{J}$ at level c for the modification exists, $m(E)$ is transversal to H, and hence $B = m(E)|_H$ is a simple normal crossing divisor on H.

(c) Resolution of singularities for $(H, (\mathcal{J}, c), B)$, which can be done by induction on dimension, induces resolution of singularities for the modification, which in turn induces a sequence of transformations for $(W, (\mathcal{I}, b), E)$, where at the end of the sequence the invariant "w-ord" strictly decreases.

(d) Repeat the procedure described in (c), until we reach the stage where w-ord $= 0$, the monomial case.

(e) Construct a sequence of resolution of singularities for a basic object in the monomial case by some easy combinatorial method.

Solution to Obstruction 4: Overcome the globalization issue by Hironaka's trick or Włodarczyk's homogenization.

This settles the problem of resolution of singularities of a basic object, and hence the problem of embedded resolution of singularities in general. **End of Day 1**

Day 2. IN POSITIVE CHARACTERISTIC

Part 1. **Resolution of singularities of a curve embedded in a nonsingular surface**: Here in positive characteristic, we discuss the above subject from a view point of the **Idealistic Filtration Program**, our approach (with Kawanoue) toward resolution of singularities in positive characteristic.

Again the choice of the center for blow up is obvious, and we choose the center to be the singular points of the curve (and its successive strict transforms).

As in characteristic zero, the only issue is to show that the process terminates after finitely many steps. For this purpose, we introduce two invariants, which should measure the improvements of the singularities effectively after each blow up. The first invariant "μ" is the order of the defining equation of the curve as in characteristic zero. Our second invariant $(\sigma, \widetilde{\nu})$ in positive characteristic is similar in spirit to, yet quite different on its face from, the second invariant ν in characteristic zero, consisting of the two factors σ and $\widetilde{\nu}$. Recall that, in order to define the second invariant ν in characteristic zero, the existence of a smooth hypersurface of maximal contact was crucial. In positive characteristic, a smooth hypersurface of maximal contact does not exist in general. Therefore, instead, we consider the collection of *possibly singular* hypersurfaces of maximal contact, called the **Leading Generator System**. The first factor σ indicates at what levels these hypersufaces of maximal contact live, while the second factor $\widetilde{\nu}$ is the order of the differential saturation (of the idealistic filtration generated by the defining equation of the curve at level μ) modulo the L.G.S. Taking the differential saturation corresponds to taking the (homogenized) coefficient ideal in characteristic zero, and computing the order modulo the L.G.S. corresponds to computing the order on the hypersurface of maximal contact in characteristic zero. We observe that the first invariant μ never increases after each blow up, and that, when the first invariant μ stays the same, the second invariant $(\sigma, \widetilde{\nu})$ strictly decreases, finishing the proof.

The proof for the fact that the invariant $(\sigma, \widetilde{\nu})$ is independent of the choice of the L.G.S. follows a similar path to the one in characteristic zero provided by Włodarczyk's homogenization. In fact, chronologically Kawanoue's idea of taking the differential saturation appeared slightly earlier than Włodarczyk's idea of homogenization, while they share the common spirit mathematically.

<u>Part 2.</u> **From the notion of a basic object to that of an idealistic filtration**: In the second part of Day 2, we explain the notion of an **idealistic filtration**, which generalizes and extends in a natural way the notion of a basic object. It is conceived by Kawanoue as the framework for the I.F.P. toward resolution of singularities in positive characteristic. However, it is also valid and theoretically simplifies the algorithm in characteristic zero.

I.F. Classically, in a basic object $(W, (\mathcal{I}, b), E)$, we consider the pair $(\mathcal{I}, b)$ consisting of a single ideal $\mathcal{I}$ and a fixed level $b \in \mathbb{Z}_{>0}$. In an idealistic filtration $(W, \mathcal{R}, E)$, we consider the collection labled by the levels $n \in \mathbb{Z}_{\geq 0}$

$$\mathcal{R} = \oplus_{n \in \mathbb{Z}_{\geq 0}} (\mathcal{I}_n, n),$$

which has the natural structure of graded $\mathcal{O}_W$-algebra. We require that $\mathcal{R}$ is finitely generated as an $\mathcal{O}_W$-algebra, and that the ideals form a filtration, i.e.,

$$\mathcal{O}_W = \mathcal{I}_0 \subset \mathcal{I}_1 \subset \mathcal{I}_2 \subset \cdots \subset \mathcal{I}_i \subset \mathcal{I}_{i+1} \subset \cdots.$$

By abuse of language, we also call this graded algebra $\mathcal{R}$ an idealistic filtration. It is closely related to the notion of a Rees algebra used by the Spanish group for the framework of their approach. The only difference is that they do not require that the ideals form a filtration, but we do. For example, when the Rees algebra has a set of (local) generators $\{(f_s, n_s)\}_{s \in S}$, it is generated by the set as an $\mathcal{O}_W$-algebra. But when we say the same set generates the idealistic filtration, it is generated by $\{(f_s, n_s), (f_s, n_s - 1), \cdots, (f_s, 1)\}_{s \in S}$ as an $\mathcal{O}_W$-algebra. Despite this small technical difference, the notions of a Rees algbra and that of an idealistic filtration play essentially the same role in the problem of resolution of singularities.

Kawanoue also defines the differential saturation$\mathcal{D}(\mathcal{R})$ of $\mathcal{R}$, by adding the elements obtained by applying the differential opeartors, i.e., $\mathcal{D}(\mathcal{R})$ is generated as an idealistic filtartion by the the set

$$\{(\delta f, \max\{0, n - \deg \delta\}); (f, n) \in \mathcal{R}, \text{ while } \delta \text{ a diff. op. of } \deg \delta\}$$

L.G.S. In an attempt to find a substitute in positive characteristic for a smooth hypersurface of maximal contact in characteristic zero, Kawanoue considers the following Leading Algebra (at a point $P \in \mathrm{Sing}(\mathcal{R}) \subset W$), consisting literally of the leading terms of the elements in the idealistic filtration at the point

$$\mathbb{L}_P(\mathcal{R}) = \oplus_{n \in \mathbb{Z}_{\geq 0}} \{f \bmod \mathfrak{m}_P^{n+1}; (f, n) \in \mathcal{R}_P\} \subset \oplus_{n \in \mathbb{Z}_{\geq 0}} \mathfrak{m}_P^n / \mathfrak{m}_P^{n+1}$$

and proves the crucial lemma regarding the structure of the leading algebra:

Lemma. There exists a regular system of parameters $(x_1, \ldots, x_t, x_{t+1}, \ldots, x_d)$ ($d = \dim W$) and $0 \leq e_1 \leq \cdots \leq e_t$ such that

$$\mathbb{L}_P(\mathcal{R}) = k[x_1^{p^{e_1}}, \ldots, x_t^{p^{e_t}}] \subset k(x_1, \ldots, x_t, x_{t+1}, \ldots, x_d] = \oplus_{n \in \mathbb{Z}_{\geq 0}} \mathfrak{m}_P^n / \mathfrak{m}_P^{n+1},$$

where $p = \mathrm{char}(k)$ is the characteristic of the field.

Now by definition, there exists a set $\mathbb{H} = \{(h_\alpha, p^{e_\alpha})\}_{\alpha=1}^t$ such that $h_\alpha = x_\alpha^{p^{e_\alpha}} \bmod \mathfrak{m}_P^{p^{e_\alpha}+1}$. We call such a set a Leading Generator System, saying that the h_α's define a collection of *possibly singular* hypersurfaces of maximal contact. Note that, in characteristic zero, we observe that $\mathbb{L}_P(\mathcal{R}) = k[x_1, \ldots, x_t] \subset k(x_1, \ldots, x_t, x_{t+1}, \ldots, x_d]$, all the elements of the L.G.S. atre at level 1, i.e., $\mathbb{H} = \{(h_\alpha, 1)\}_{\alpha=1}^t$, and hence that h_α's define the classical *smooth* hypersurfaces of maximal contact.

Invariant σ. We think that the lower the levels of the elements in the L.G.S. the better the L.G.S. is. Accordingly, we define the invariant σ, the indicator of how good the L.G.S. is, as the sequence $\{a_n\}_{n=0}^\infty$ defined by $a_n = d - \#\{\alpha; e_\alpha \leq n\}$.

Invariant $\widetilde{\mu}$. Given $f \in \mathcal{O}_{W,P}$ (or more generally $f \in \widehat{\mathcal{O}_{W,P}}$, there is a unique power series expansion of the following form with resspect to the given L.G.S. $\mathbb{H}$

$$f = \sum c_{f,B} H^B$$

where $B = (b_1, b_2, \ldots, b_t)$, $H^B = h_1^{b_1} h_2^{b_2} \cdots h_t^{b_t}$, and where $c_{f,B}$ is of the form

$$c_{f,B} = \sum \gamma_{f,B,K} x_1^{k_1} x_2^{k_2} \cdots x_t^{k_t}$$

where $K = (k_1, k_2, \ldots, k_t)$ with $0 \leq k_\alpha \leq p^{e_\alpha} - 1$ and $\gamma_{f,B,K} \in k[[x_{t+1}, \ldots, x_d]]$.

We define the order of f module $\mathbb{H}$ to be the order of the "constant term $c_{\mathbb{O}}$ of the expansion, i.e., $\mathrm{ord}_P(f|_{\mathbb{H}}) = \mathrm{ord}_P(c_{\mathbb{O}})$.

We define the order $\mu_P(\mathcal{R})$ of the idealistic filtration $\mathcal{R}$ at P modulo $\mathbb{H}$ by the formula

$$\mu_p(\mathcal{R}) = \min\left\{\frac{\mathrm{ord}_P(f|_{\mathbb{H}})}{n}; (f, n) \in \mathcal{R}_P, n \neq 0\right\}.$$

The weak-order $\widetilde{\mu}_P(\mathcal{R})$ is defined by subtracting the "appropriate" amount of the order associated to the boundary divisor E from $\mu_P(\mathcal{R})$. **End of Day 2**

Day 3. IN POSITIVE CHARACTERISTIC

Part 1. **Resolution of singularities of an idealistic filtration**: The problem of embedded resolution of singularities (and that of a basic object) in positive characteristic (as well as in characteristic zero) is reduced to the problem of resolution of singularities of an idealistic filtration $(W, \mathcal{R}, E)$: construct a sequence of transformations

$$(W, \mathcal{R}, E) = (W_0, \mathcal{R}_0, E_0) \leftarrow (W_1, \mathcal{R}_1, E_1) \leftarrow \cdots \leftarrow (W_l, \mathcal{R}_l, E_l)$$

such that $\mathrm{Sing}(\mathcal{R}_l) = \emptyset$. Note that the singular locus $\mathrm{Sing}(\mathcal{R})$ of an idealistic filtration is defined to be

$$\mathrm{Sing}(\mathcal{R}) = \{P \in W; \mathrm{ord}_P(f) \geq n \quad \forall (f, n) \in \mathcal{R}_P\}$$

The basic strategy to construct such a sequence is to carry out an induction on the invariant σ via L.G.S. in positive characteristic, instead of carrying out an induction on dimension via a smooth hypersurface of maximal contact in characteristic zero, meanwhile following a parallel path in positive characteristic to the solutions of Hironaka to the obstructions in characteristic zero explained on Day 1:

(a) We introduce the triplet of invariants $(\sigma, \widetilde{\mu}, s)$ where we already discussed the invariants σ and $\widetilde{\mu}$ and where "s" represents the number of irreducible components in the *aged* part E_{aged} of the boundary divisor E.
(b) We construct the modification $(W, m(\mathcal{R}), m(E) = E \setminus E_{aged})$ such that
  - $\mathrm{Sing}(m(\mathcal{R})) \supset \mathrm{Sing}(\mathcal{R}) \cap$ Maximum Locus of $(\sigma, \widetilde{\mu}, s)$,
  - $\sigma(\mathcal{R}) > \sigma(m(\mathcal{R}))$
(c) Resolution of singularities for $(W, m(\mathcal{R}), m(E))$, which can be done by induction on the invariant σ, induces a sequence of transformations for $(W, \mathcal{R}, E)$, where in the sequence the invariant $(\sigma, \widetilde{\mu}, s)$ strictly decreases.
(d) Repeat the procedure described in (c), until we reach the stage where $(\sigma, \widetilde{\mu}, s) = (\sigma, 0, 0)$, the monomial case.
(e) Construct a sequence of resolution of singularities for an idealistic filtration in the monomial case.

Are we done ? Not quite ! Unlike in characteristic zero, the problem of resolution of singularities of an idealistic filtration in the monomial case is SUPER HARD !

Note on the globalization issue: Since homogenization is already incorporated in the form of differential saturation, one might think the globalization is automatic in the I.F.P. when we try to go through the local processes described in (a) through (e). Not quite ! There is an issue unique to the I.F.P. (the issue occurs only when the value of the invariant $\widetilde{mu}$ is equal to 1). The issue is rather a technical one than the one which would cause an essential horror like the Moh-Hauser Jumping phenomena. So we ignore the globalization issue of the I.F.P. in this lecture series.

Part 2. **Difficulty in the momomial case**: Why is the problem of resolution of singularities in the monomial case in the I.F.P. in positve characteristic so much more difficult than the monomial caes in the classical setting in characteristic zero ?

Both in characteristic zero and in positve characteristic, we deal with a monomial of the defining ideals of the components of a simple normal crossing divisor on a variety. However, the variety in characteristic zero is nonsingular, since it is the intersection of the successive smooth hypersurfaces of maximal contact trasversal to each other. Therefore, the analysis is easy. On the other hand, the variety in positive characteristic is (possibly) very singular, since it is the intersection of the successive (possibly) singular hypersufaces of maximal contact defined by the elements of an L.G.S. This is where the difficulty lies. (Note that the meaning of a "simple normal crossing" divisor on a singular variety has to be also clarified in the latter.)

In the following, we describe our strategy (locally at $P \in W$) to achieve resolution of singularities of an idealistic filtration in the monomial case, following the original philosophy of Villamayor.

Inductive scheme in terms of the invariant τ: There is an inductive scheme on the invariant $0 \leq \tau = \#\mathbb{H} \leq d = \dim W$, i.e., the number of the defining equation of the variety of our concern.

Case $\tau = 0$: This case is the same as the classical one, easy.

Case $\tau = 1$: This caes is the most difficult one.

Case $\tau = j > 1$: This case is reduced to the one where $\tau = j - 1$ and $\dim = d - 1$.

Case $\tau = d$: This case does not happen.

Therefore, the analysis is reduced to the case where $\tau = 1$, i.e., whete there is only one defining equation. This confirms a folklore that the most essential case in the problem of resolution of singularities is the one of a hypersurface singularities in our framework.

Analysis of the case where $\tau = 1$:

In this case, by definition, we have only one element $\mathbb{H} = \{(h, p^e)\}$ in the L.G.S., where h is of the following form via Weierstrass Preparation Theorem

$$h = x_1^{p^e} + a_1 x_1^{p^e-1} + a_2 x_1^{p^e-2} + \cdots + a_{p^e-1} x_1 + a_{p^e}$$

with $a_i \in k[[x_2, \ldots, x_d]]$ and $\mathrm{ord}_P(a_i) > i$ for $i = 2, \ldots, d$.

We observe that the coefficients a_i for $i = 2, \ldots, p^e - 1$ are "well-controlled", and hence we can pretend that h is of the form $h = x_1^{p^e} + a_{p^e}$. This confirms another folklore that the most essential case in the problem of resolution of singularities in positive characteristic is the case of purely inseparable extensions in our framework.

○ **Cleaning**

We focus our attention on the analysis of a_{p^e}. We soon realize that the basic invaraint attached to a_{p^e}, e.g. the order, is highly dependent of the choice of the coordinate variables. For example, if $h = x_1^{p^e} + a_{p^e}$ and if the initial term $\mathrm{In}(a_{p^e})$ is a p^e-th power, then setting $x_1' = x_1 + \{\mathrm{In}(a_{p^e})\}^{1/p^e}$, we have $h = (x_1')^{p^e} + a_{p^e}'$ with $\mathrm{ord}_P(a_e') > \mathrm{ord}_P(a_e)$. Actually, there is a process, similar to the one demonstarated in this example, to kill the dependency. The process is called "cleaning".

○ **From the usual monomial case to the tight monomial case**

We have the monomial (M, a) at level a appearing in the definition of the monomial case, and the monomial M' at level p^e, which divides a_{p^e} as much as possible. By normalizing, we obtain the usual monomial $M_{\text{usual}} = M^{1/a}$ and the tight monomial $M_{\text{tight}} = (M')^{1/p^e}$.

We set

$$\mathrm{inv}_{\mathrm{MON},\heartsuit}(P) := \min\left\{\frac{\mathrm{ord}_P(a_{p^e})}{p^e} - \mathrm{ord}_P(M_{\text{tight}}), \mathrm{ord}_P(M_{\text{usual}}) - \mathrm{ord}_P(M_{\text{tight}})\right\}.$$

We say we are in the tight monomial case when $\mathrm{inv}_{\mathrm{MON},\heartsuit}(P) = 0$, following Benito-Villamayor. We observe that, when we are in the tight monomial case, resolution of singularities can be achieved easily like the classical case in characeristic zero. So our final task is, starting from the (usual) monomial case with $\mathrm{inv}_{\mathrm{MON},\heartsuit}(P) > 0$, to find a way to reach the tight monomial case with $\mathrm{inv}_{\mathrm{MON},\heartsuit}(P) = 0$.

○ **Induction on dimension**

For the purpose of fulfilling the final task, we look at the refined invariant

$$\begin{aligned} &\mathrm{inv}_{\mathrm{MON},\spadesuit}(P) \\ &\quad := \min\{\rho_D(P) - \mathrm{ord}_P(M_{\text{tight}}); D \text{bad component of } E, \\ &\qquad\qquad \mathrm{ord}_P(M_{\text{usual}}) - \mathrm{ord}_P(M_{\text{tight}})\} \\ &\quad \geq \mathrm{inv}_{\mathrm{MON},\heartsuit}(P) \geq 0. \end{aligned}$$

There are two advantages to go from $\heartsuit$ to $\spadesuit$. First is that, since $\spadesuit$ is looking at the edge of the Newton polygon, the comparison of $\spadesuit$ before and after blow up is easier than $\heartsuit$. Second is that by looking at the components D, we cn use the induction dimension. Actually we apply I.F.P. in one dimension less to D in order to determine the center.

○ **Overcome the Moh-Hauser Jumping phenomenon**

If the invariant $\mathrm{inv}_{\mathrm{MON},\spadesuit}(P)$ strictly decreases after blowing up our choice of the center (determined by the induction on dimension), we would have been a happy camper. While it sometimes stays the same (which is not so bad, as we can overcome this by introducing some supplementary invariants), it can strictly increase from time to time (which is VERY, VERY bad, and could destroy our whole strategy). The strict increase of the invariant, caused by the operation of cleaning, is called "the Moh-Hauser Jumping phenomenon".

In fact, the big portion of Kawanoue's motivation to develop the I.F.P. stems from the desire to come up with an algorithm which avoids "the Moh-Hauser Jumping phenomenon" caused by cleaning. So it is as though we started the journey escaping from this ghost, only to face the ghost again at the end.

Up to $\dim W \leq 3$, we can eliminate the ghost by showing the **eventual decrease** of the invariant $\mathrm{inv}_{\mathrm{MON},\spadesuit}$.

The case $\dim W = 4$ is the intensive focus of our current research ! **End of Day 3**