Study of integer resonance crossing in non-scaling FFAGs with an ion trap system

K. Fukushima
K. Moriya, T. Okano, K. Ito, H. Okamoto
- *AdSM, Hiroshima University*

S. L. Sheehy, D. Kelliher, S. Machida, C. Prior
- *STFC, Rutherford Appleton Laboratory*

What is a resonance crossing in nsFFAGs

<u>Non</u>-scaling FFAG(Fixed Field Alternating Gradient) Accelerator

By choosing to ignore the scaling law ($B(r)=B_0(r/r_0)^k$), an accelerator lattice designer may introduce attractive properties such as magnet simplicity, design flexibility and compact orbits.

>> This leads to resonance crossing phenomena during acceleration.

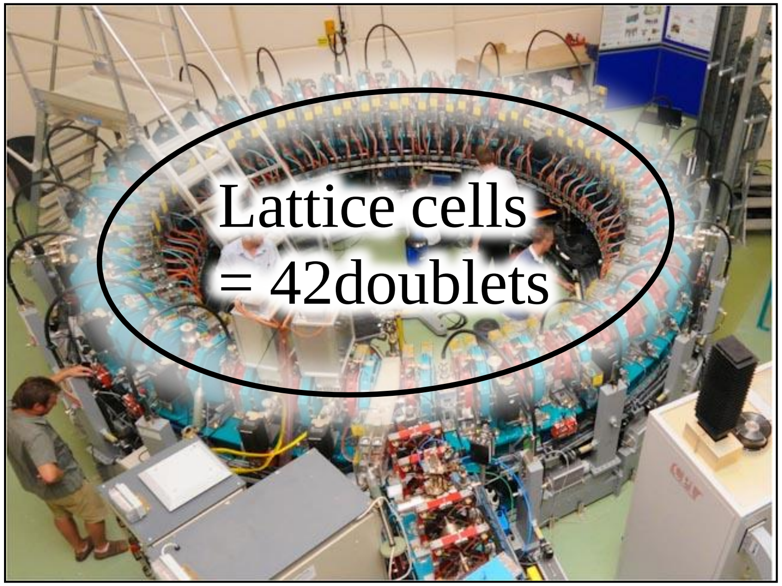
While other accelerators may routinely or experimentally cross low order resonances, linear nsFFAGs including EMMA cross first order integer resonances, in some cases more than ten times per acceleration cycle.

Non-scaling FFAG EMMA accelerator

EMMA@STFC/UK (Daresbury Laboratory)

• The first experimental demonstration of non-scaling FFAG
Complete acceleration from 10 MeV to 20 MeV in less than 20 turns

Electron **M**achine with **M**any **A**pplications



http://www.stfc.ac.uk/home.aspx

Energy range	10 to 20 MeV
Cell type	FD Doublet
Number of cells	42
RF	19 cavities; 1.3 GHz
Cell length	394.481mm
Ring circumference	16.57m

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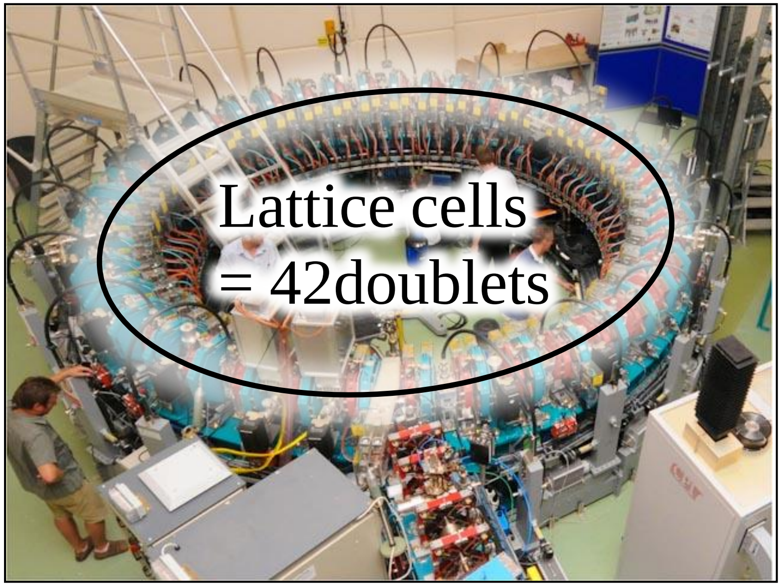

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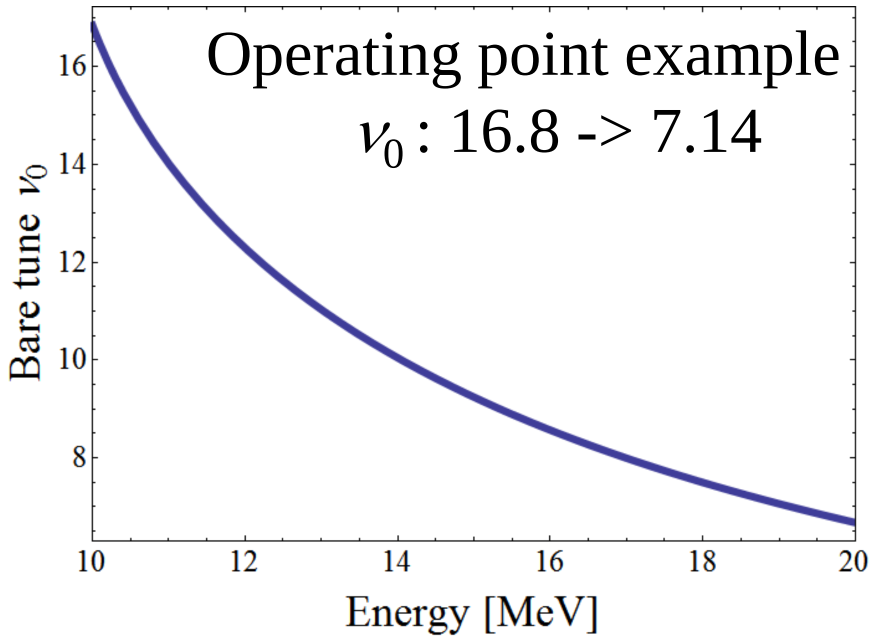

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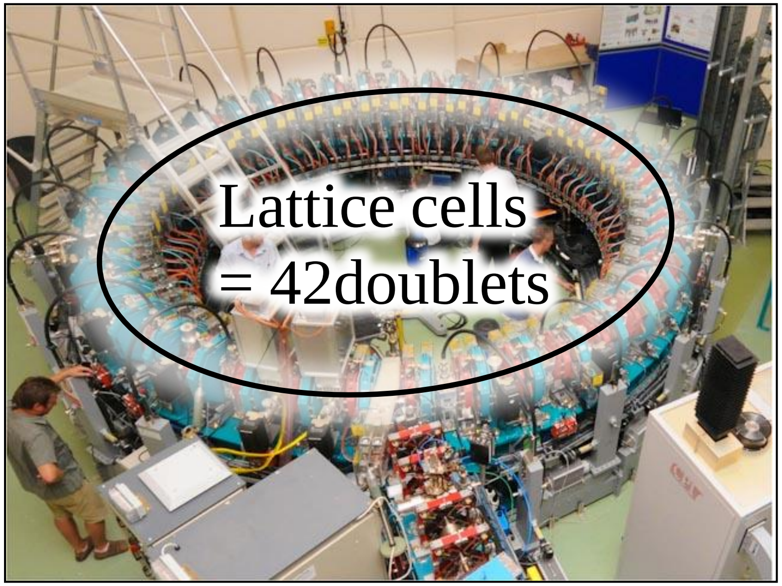
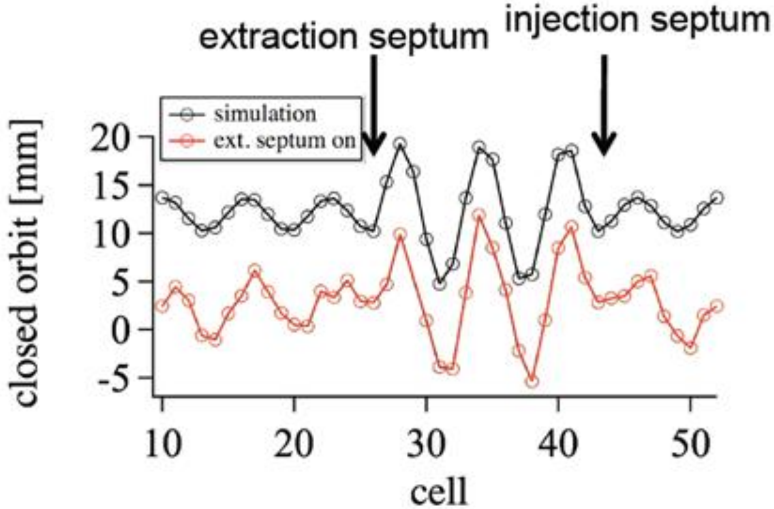
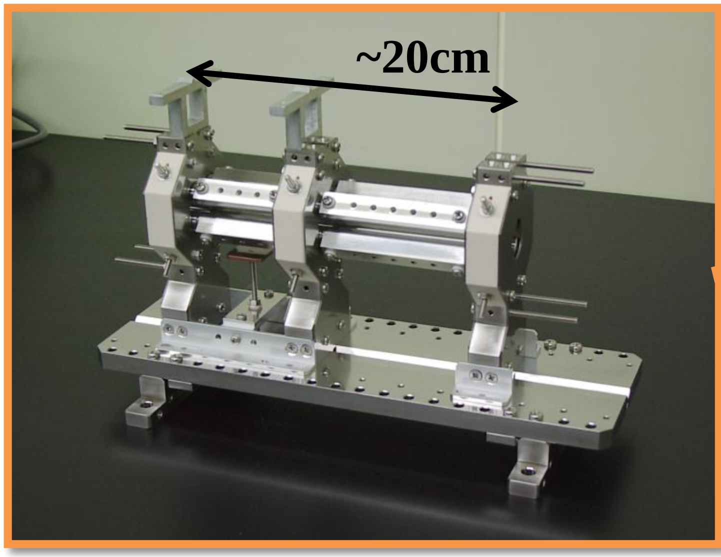


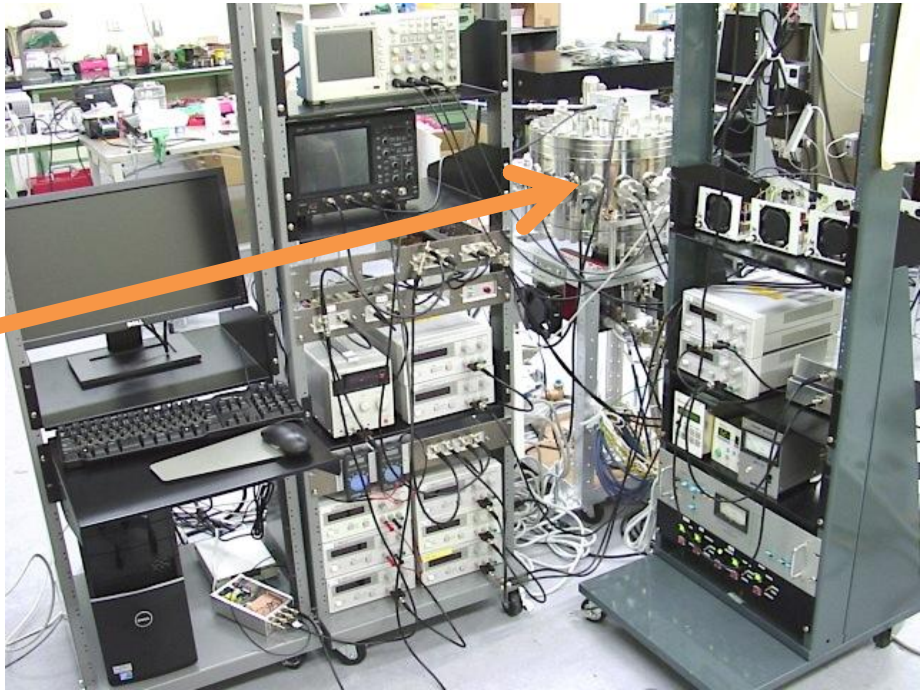
http://www.stfc.ac.uk/home.aspx

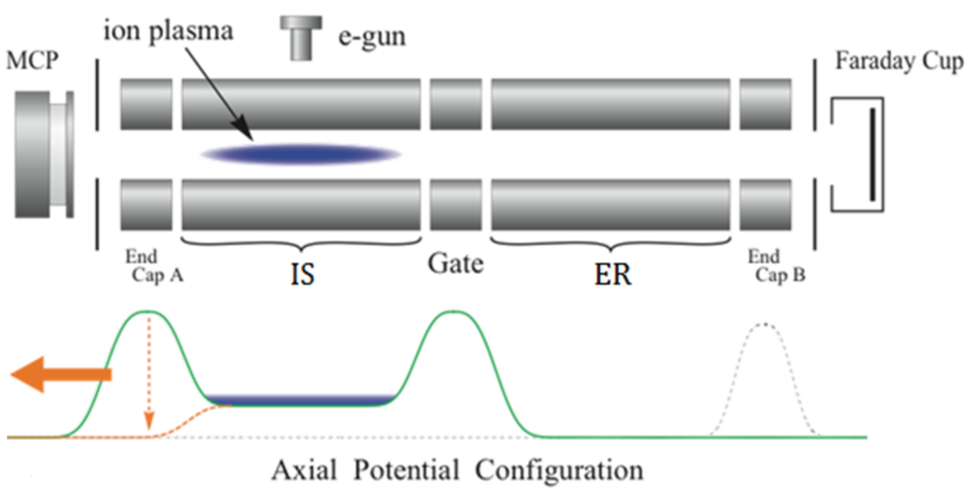
Closed Orbit in EMMA
(equivalent momentum beam)



About S-POD (linear Paul trap) experiments

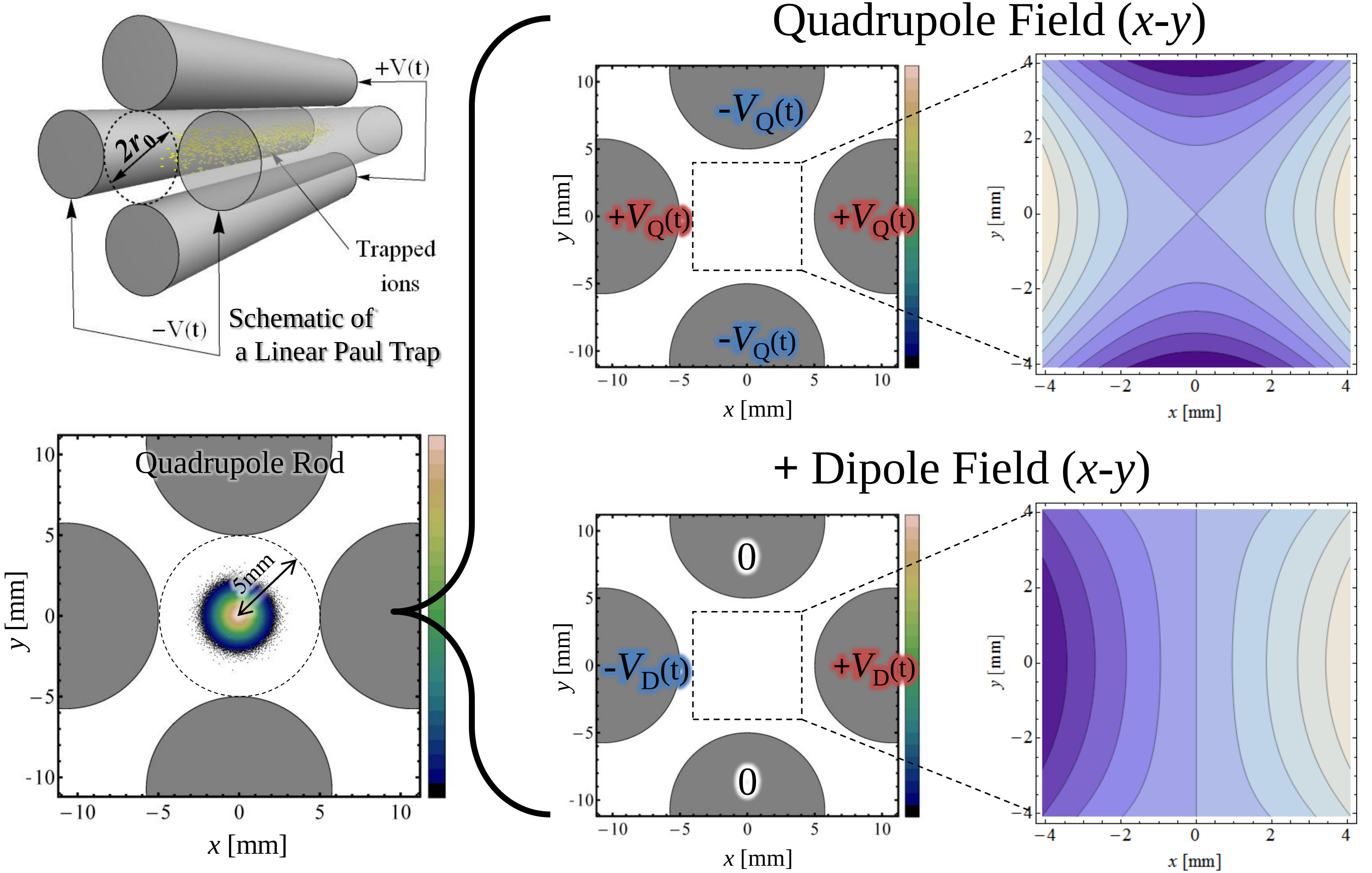






1. Ionize neutral Ar gas with a low-energy electron beam (Section IS).
2. Confine Ar^+ ions typically for 1~10 msec corresponding to $10^3 \sim 10^4$ FODO periods.
3. Switch off the potential wall of MCP side to dump ions for measurement.

Dipole component in S-POD



Correspondence between an accelerator error field $\Delta B/B$ and applied voltage V_D in a LPT

The equation of the transverse ion motion in a LPT with the dipole driving field is

$$\frac{d^2x}{d\tau^2} + K_\mathrm{rf}(\tau)x = -\frac{q}{mc^2 r_0} V_D(\tau)$$

$$\begin{cases} \tau = ct \\ K_\mathrm{rf}(\tau) = 2qV_\mathrm{Q}/mc^2 r_0^2 \\ V_\mathrm{D} = \sum_n w_n \cos(n\theta + \varphi_n) \end{cases}$$

The transverse closed orbit distortion in a circular accelerator due to a dipole error field obeys

$$\frac{d^2 x_\mathrm{COD}}{ds^2} + K_x(s) x_\mathrm{COD} = -\frac{\Delta B}{B\rho}$$

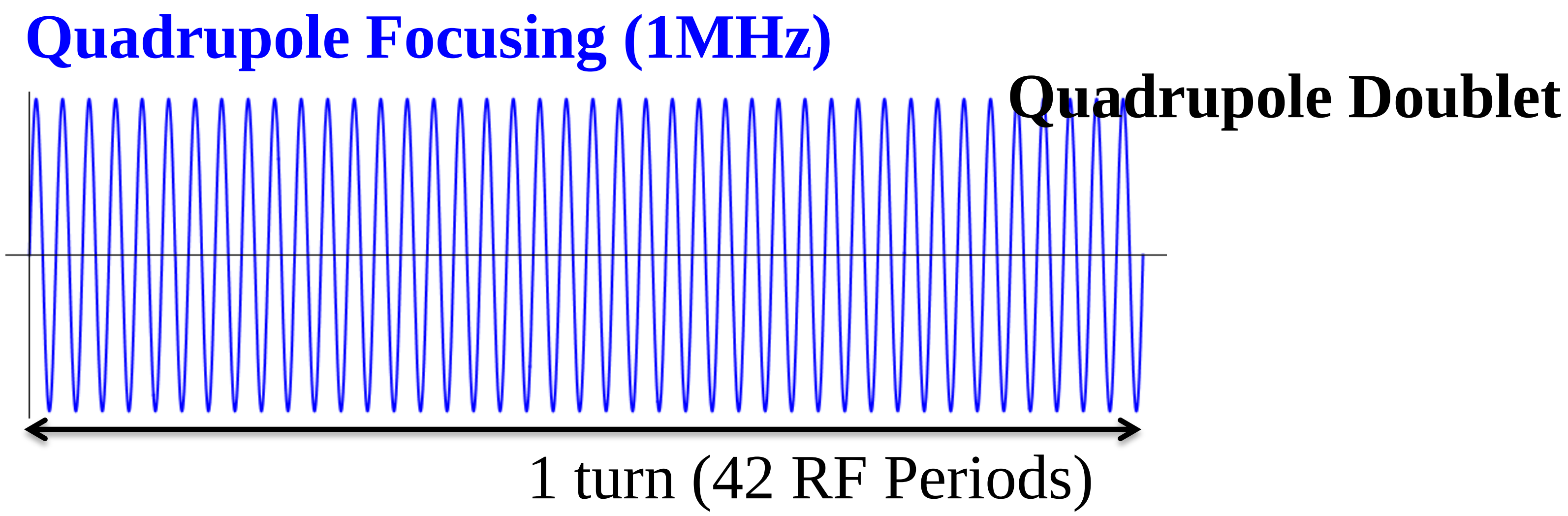
$$\Delta B = \sum_n B_n \cos(n\theta + \varphi_n)$$

Comparison between these equations with the smooth approximation, we get

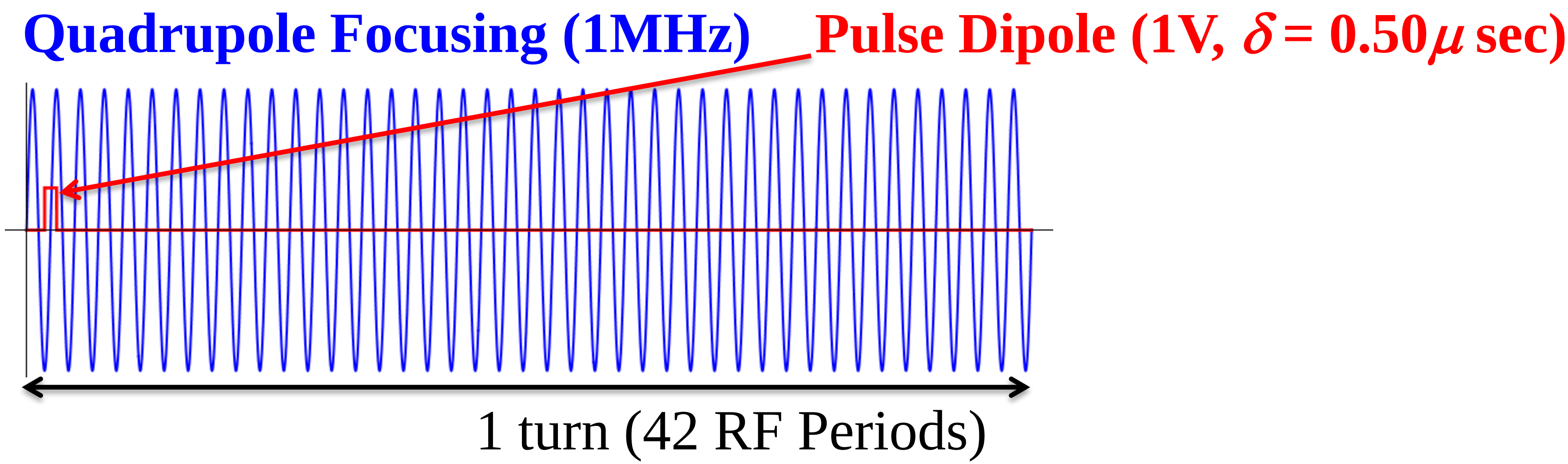
$$V_D \approx \frac{mc^2 r_0}{q} \left(\frac{2\pi R}{N_\mathrm{cell}\lambda} \right)^2 \frac{\Delta B}{B\rho}$$

r_0 : inner radius of the LPT
R : average radius of the ring
λ : wave length

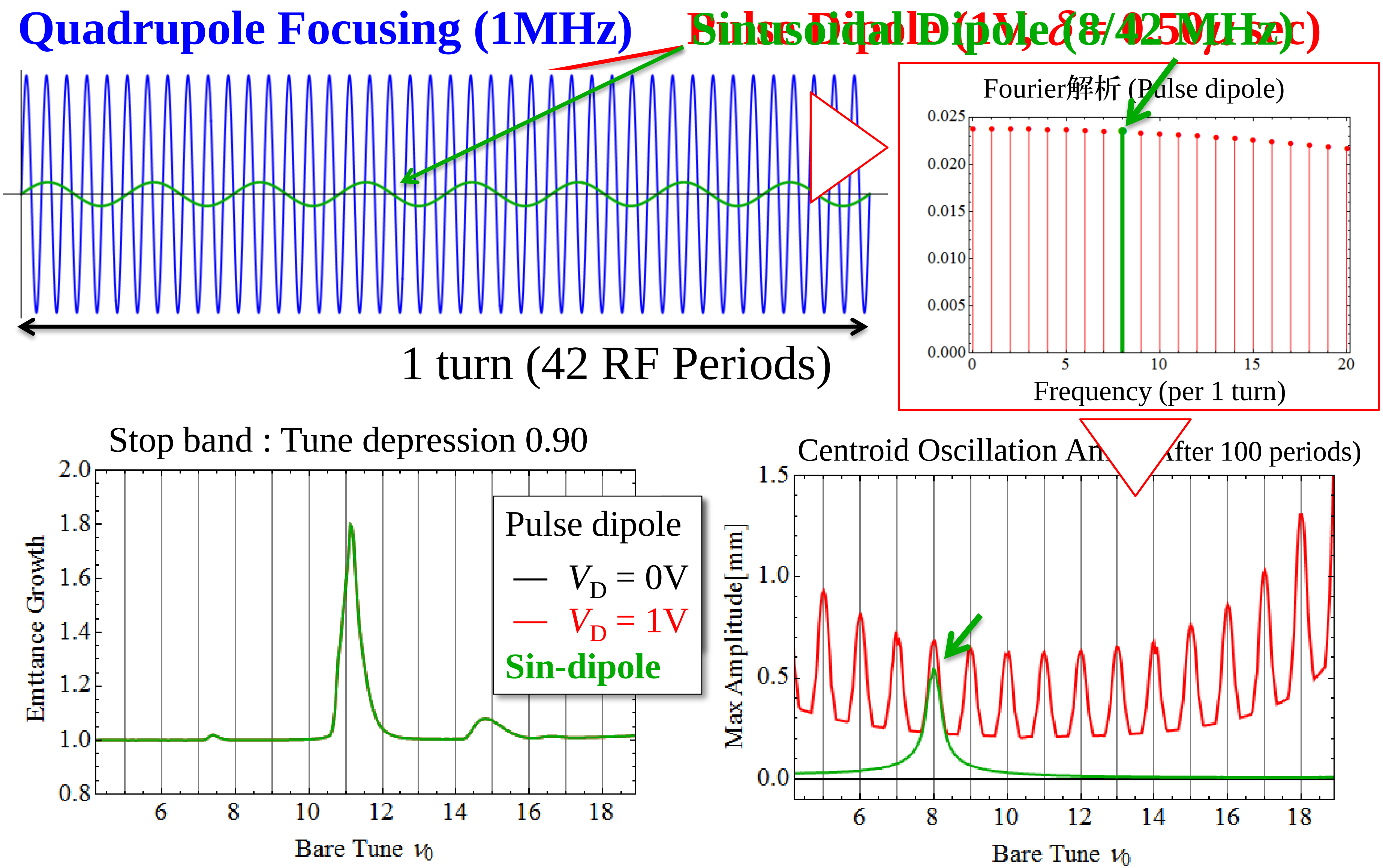
Quadrupole focusing and dipole wave form



Quadrupole focusing and dipole wave form



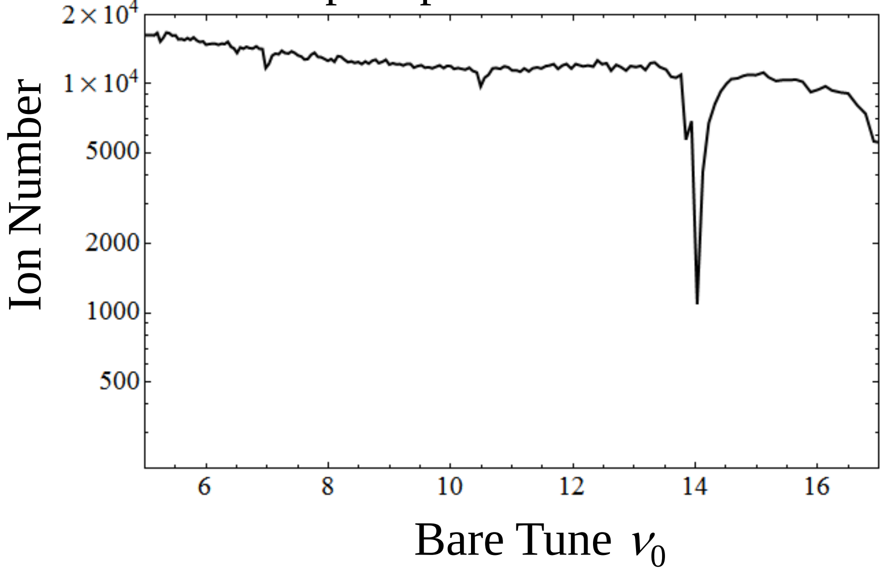
Stop band - fixed bare tune - numerical simulation



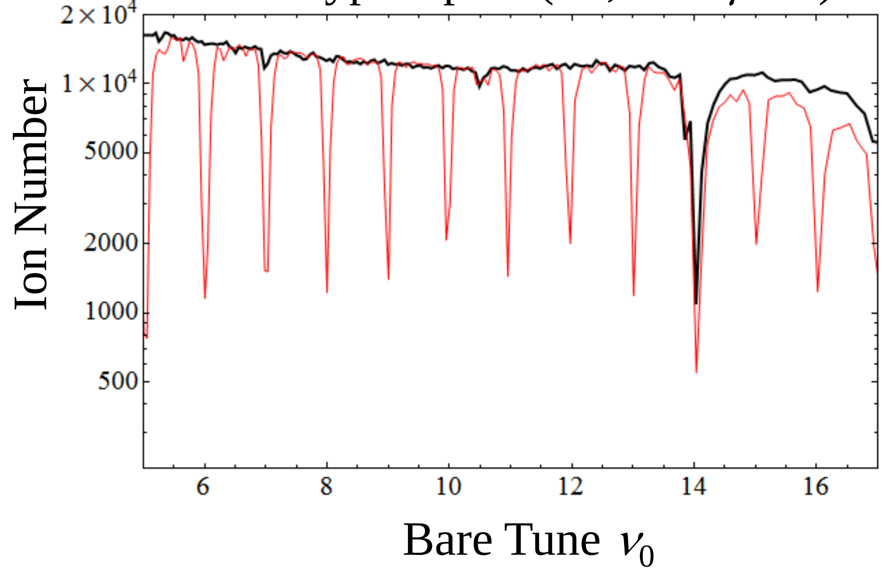
Stop band - experimental result

Resonance stop bands identified by the S-POD system.

No dipole perturbation



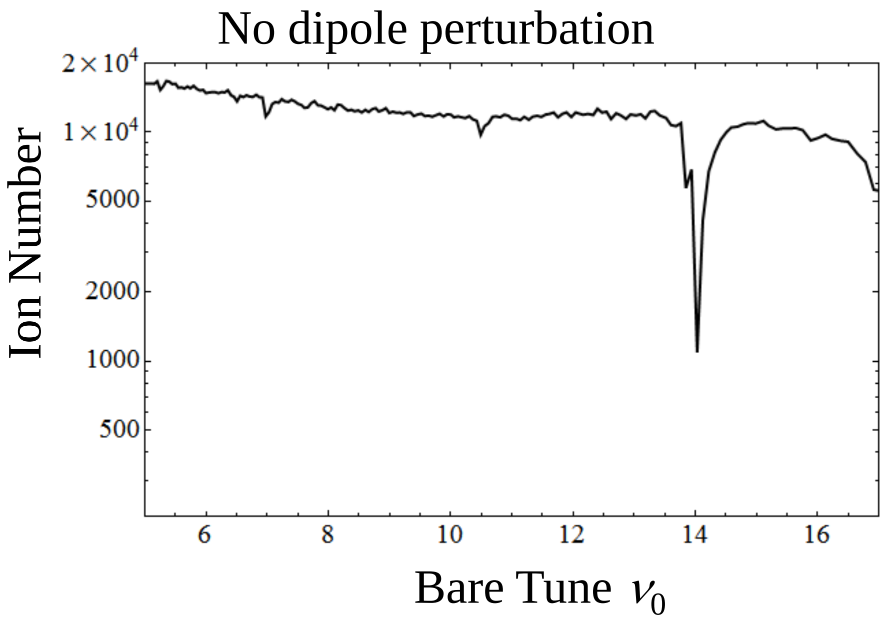
Pulse type dipole (1V, $\delta = 1\mu$sec)

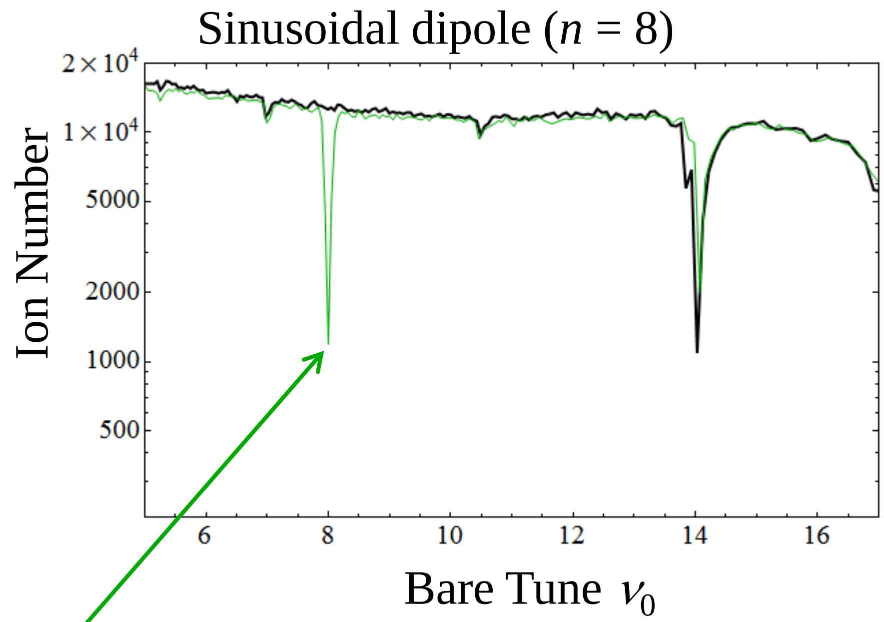


- About ten thousand 40Ar+ ions are initially trapped in the LPT and stored for 10 msec at a certain fixed value of bare tune ν_0.
- The number of ions surviving after the 10-msec storage is measured with a micro-channel plate (MCP) sitting beside the LPT.
- The same experimental procedure was repeated many times, changing ν_0 a small steps over a wide range.

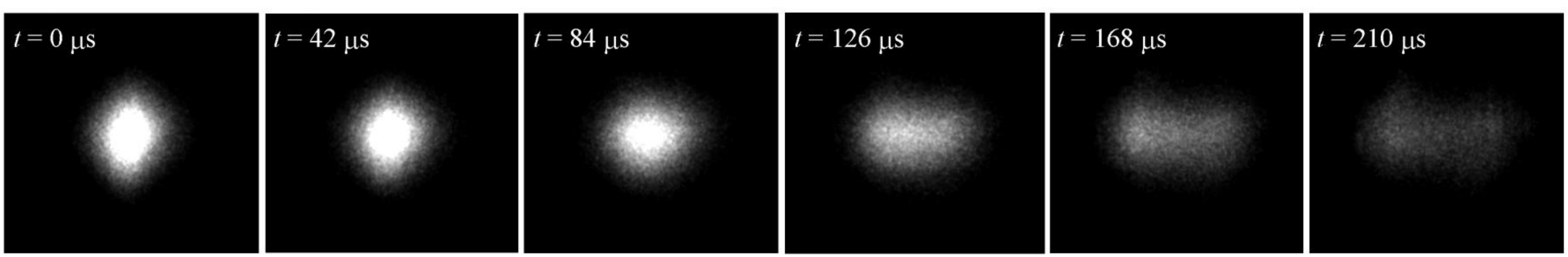
Stop band - experimental result

Resonance stop bands identified by the S-POD system.



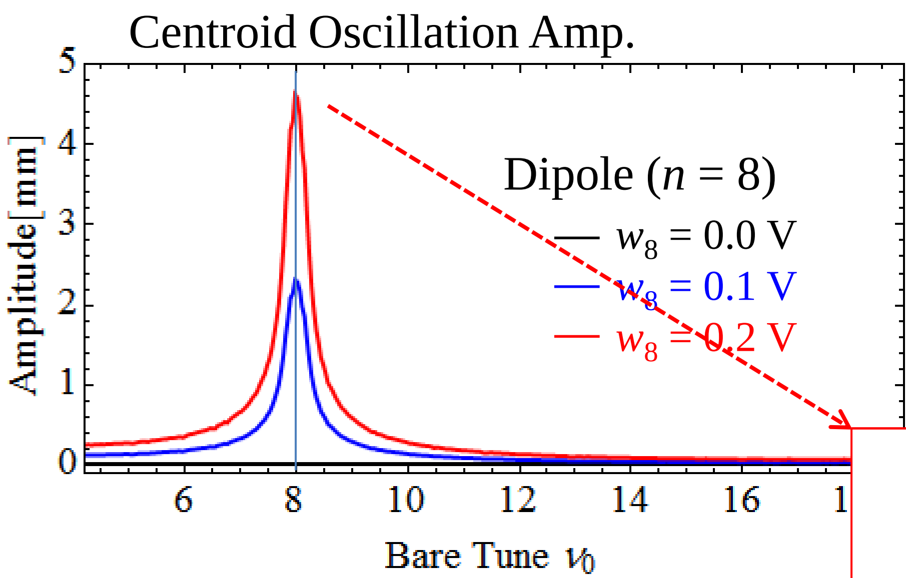


Time evolution of a transverse plasma profile on the phosphor screen measured with a CCD camera.



Time evolution on the integer resonance

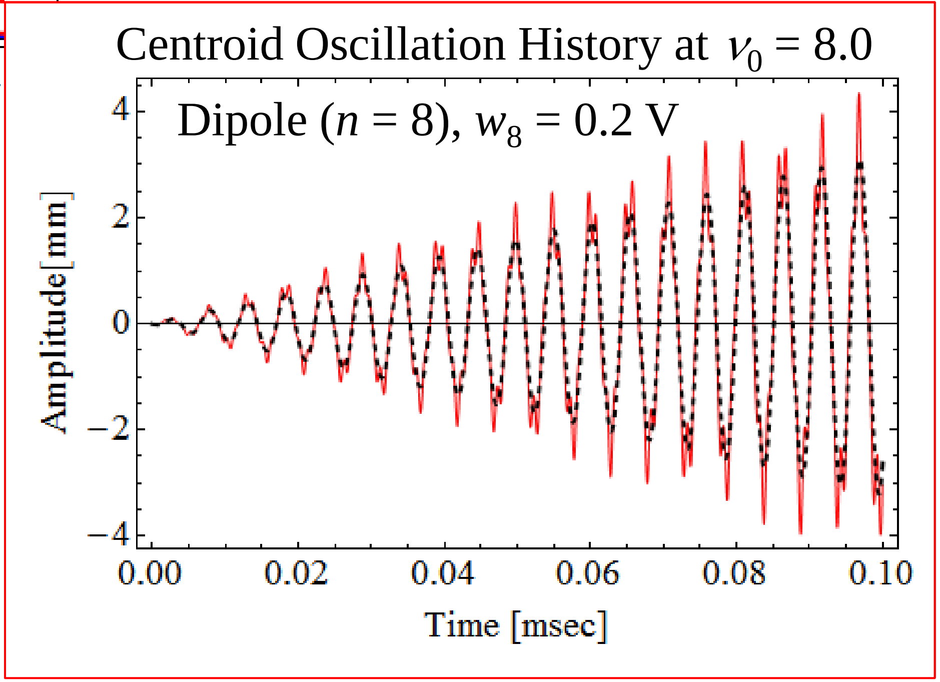
Time evolution on the integer resonance at $\nu_0 = 8$



$$V_D = \sum_n w_n \cos(n\theta + \varphi_n)$$

$$w_8 = \text{const.} \quad w_{n\neq 8} = 0$$

>>In case of the bare tune is on the integer resonance, the centroid amplitude increase linearly.



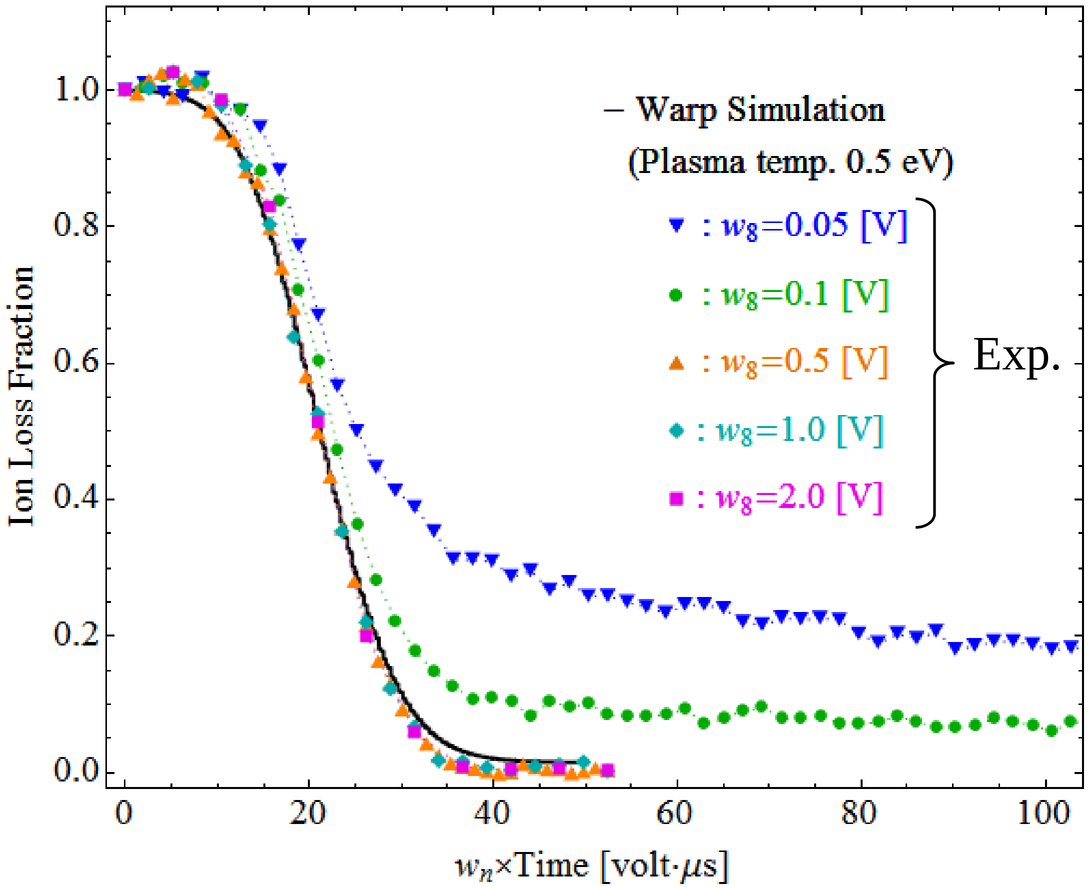
Forced vibration（Smooth approx.)

$$x = x_0 \sin(\omega t + \phi_0) + \frac{q}{m r_0} \frac{w_n}{2\omega} t \cos \omega t$$

$$\omega = 2\pi \nu_0 f_{\mathrm{rf}}$$

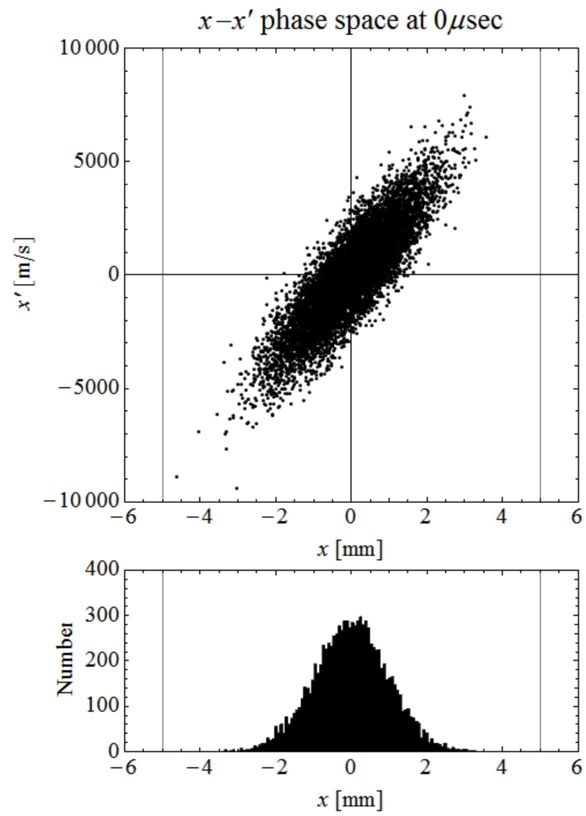
Time evolution on the integer resonance at $\nu_0 = 8$

Time evolution of ion losses



>> Numerical simulation result in which perfectly aligned quadrupole rods have been assumed and the initial ion distribution is Gaussian with a temperature of 0.5 eV.

c.f. Simulation result
(1-dimension, Time evolution on the resonance n = 8, w_8=1.0[V])



Single resonance crossing

Single resonance crossing theory

1) Smooth approximation

Baartman's Eq. (Smooth Approx.) [1]

Amplitude growth ΔA is

$$\Delta A = \frac{\pi}{N_{\text{cell}}\sqrt{u}} \frac{R^2}{B\rho} \frac{B_n}{n}$$

$$\approx \frac{q N_{\text{cell}} \lambda^2}{4\pi m c^2 r_0} \frac{w_n}{n\sqrt{u}}$$

$$= K(n) \frac{w_n}{\sqrt{u}}$$

$$\because K(n) = \frac{q N_{\text{cell}} \lambda^2}{4\pi m c^2 r_0} \frac{1}{n}$$

$$\left\{ \begin{array}{l} \text{crossing speed } u = \dfrac{\Delta\nu}{N_{\text{turn}} \times N_{\text{cell}}} \\[2ex] \Delta\nu = \nu_{\text{cell}}^{ini} - \nu_{\text{cell}}^{fin} \\[2ex] V_D \approx \dfrac{mc^2 r_0}{q} \left(\dfrac{2\pi R}{N_{\text{cell}}\lambda} \right)^2 \dfrac{\Delta B}{B\rho} \end{array} \right.$$

[1]R. Baartman, "Fast Crossing of Betatron Resonances", FFAG workshop 2004, Vancouver, Canada, 2004.

Single resonance crossing theory

2) Guignard's formula

Square root of emittance growth $\Delta\sqrt{\varepsilon}$ is

$$\Delta\sqrt{\varepsilon} = \frac{\pi}{N_{\text{cell}}\sqrt{u}}\frac{2R}{B\rho}\times\left|\frac{1}{2\pi}\int_0^{2\pi}\sqrt{\beta}\,\Delta B e^{in\theta}d\theta\right|$$

$$= \frac{qN_{\text{cell}}\lambda^2}{2\pi mc^2 r_0}\times\left|\frac{1}{2\pi R}\int_0^{2\pi}\sqrt{\frac{2\pi R}{N_{\text{cell}}\lambda}\beta_{\text{rf}}}\,\frac{V_D}{w_n}e^{in\theta}d\theta\right|\cdot\frac{w_n}{\sqrt{u}}$$

$$\because \beta(\theta) = \frac{2\pi R}{N_{\text{cell}}\lambda}\beta_{\text{rf}}(\theta)$$

where, we use the β matched for tune *n*.

$$\Delta A = \Delta\sqrt{\varepsilon}\times\sqrt{\beta^{\max}}$$

$$= \frac{q\lambda\sqrt{\beta_{\text{rf}}^{\max}}}{2\pi mc^2 r_0}\times\left|\int_0^{2\pi}\sqrt{\beta_{\text{rf}}}\,\frac{V_D}{w_n}e^{in\theta}d\theta\right|\cdot\frac{w_n}{\sqrt{u}} = K(n)\frac{w_n}{\sqrt{u}}$$

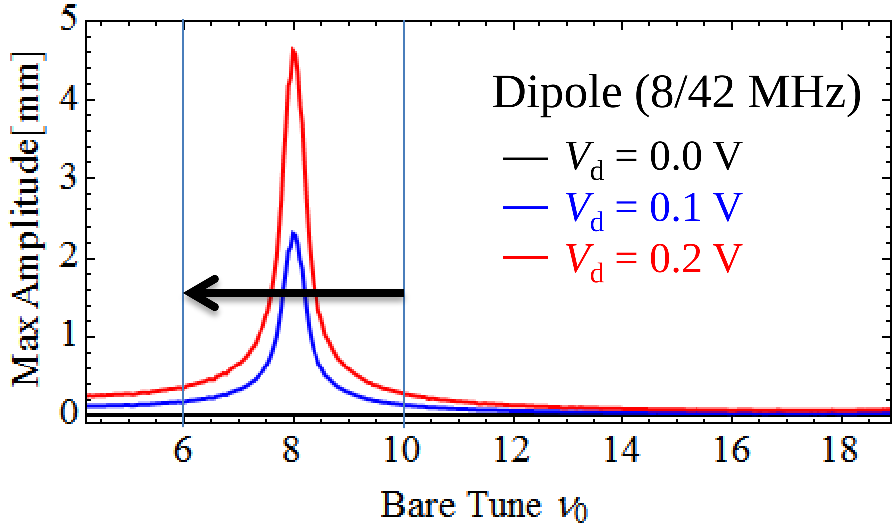
$$\because K(n) = \frac{q\lambda\sqrt{\beta_{\text{rf}}^{\max}}}{2\pi mc^2 r_0}\times\left|\int_0^{2\pi}\sqrt{\beta_{\text{rf}}}\,\frac{V_D}{w_n}e^{in\theta}d\theta\right|$$

c.f. smooth approx.

$$K(n) = \frac{qN_{\text{cell}}\lambda^2}{4\pi mc^2 r_0}\frac{1}{n}$$

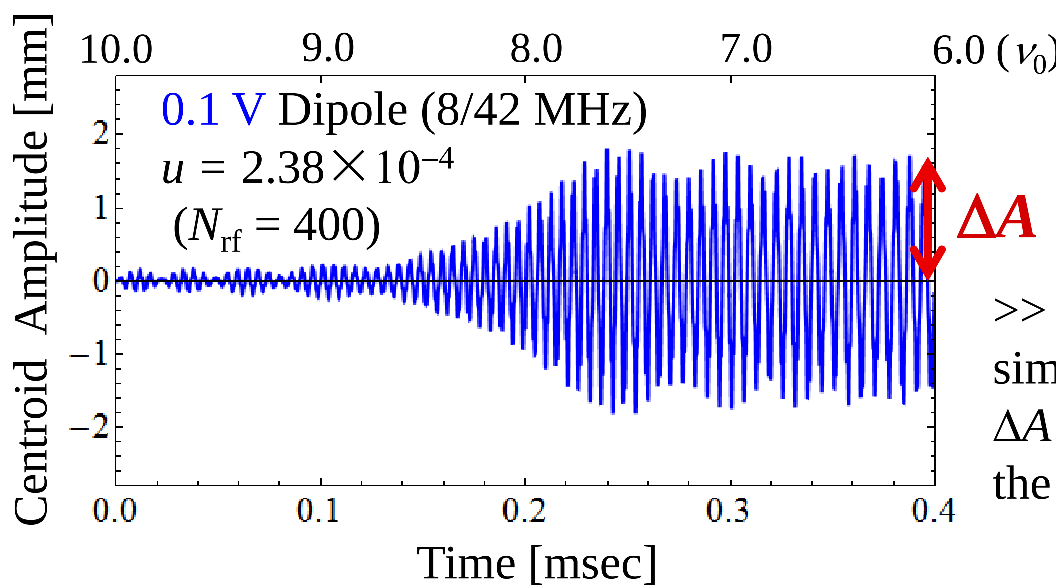
Single resonance crossing simulation

Centroid Oscillation Amp. (After 100 periods)



Sweep range ν_0 :10 -> 6
resonance at $n = 8$

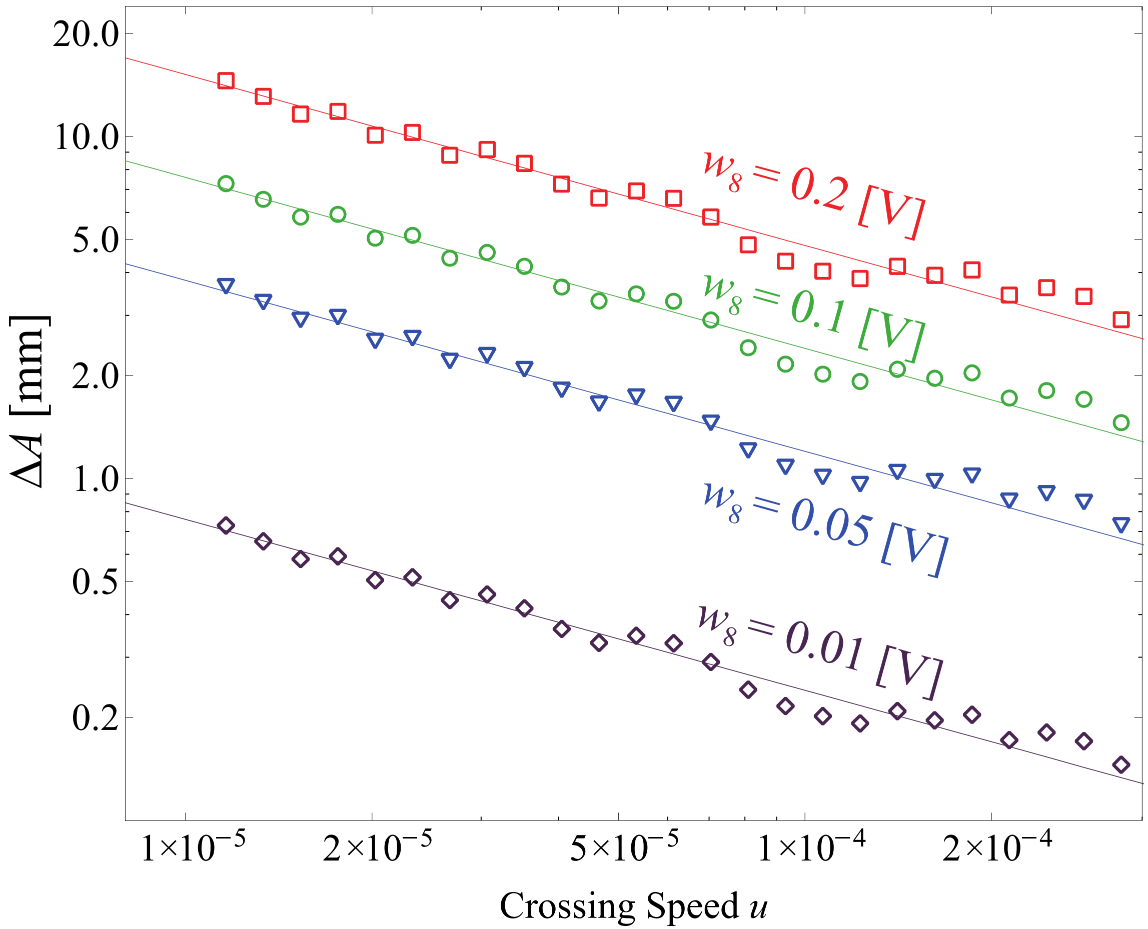
$$\left(\text{crossing speed } u = \frac{\Delta \nu}{N_{\text{turn}} \times N_{\text{cell}}} \right)$$



>> In the numerical simulation, we defined ΔA as the final state of the centroid amplitude.

Single resonance crossing simulation

Amplitude growth ΔA *v.s.* Crossing speed u



Lines :
Guignard's formula

$$\Delta A = K(n)\frac{w_n}{\sqrt{u}}$$

$$\because K(n) = \frac{q\lambda\sqrt{\beta_{\mathrm{rf}}^{\max}}}{2\pi m c^2 r_0} \times \left| \int_0^{2\pi} \sqrt{\beta_{\mathrm{rf}}}\, \frac{V_D}{w_n} e^{in\theta} d\theta \right|$$

>> Numerical simulation results almost agree with Guignard's formula ($K(8) = 0.2955\times10^{-3}$ [m/V]).

Growth factor $K(n)$ on the nth integer-resonance crossing (comparison with the theory and the simulation)

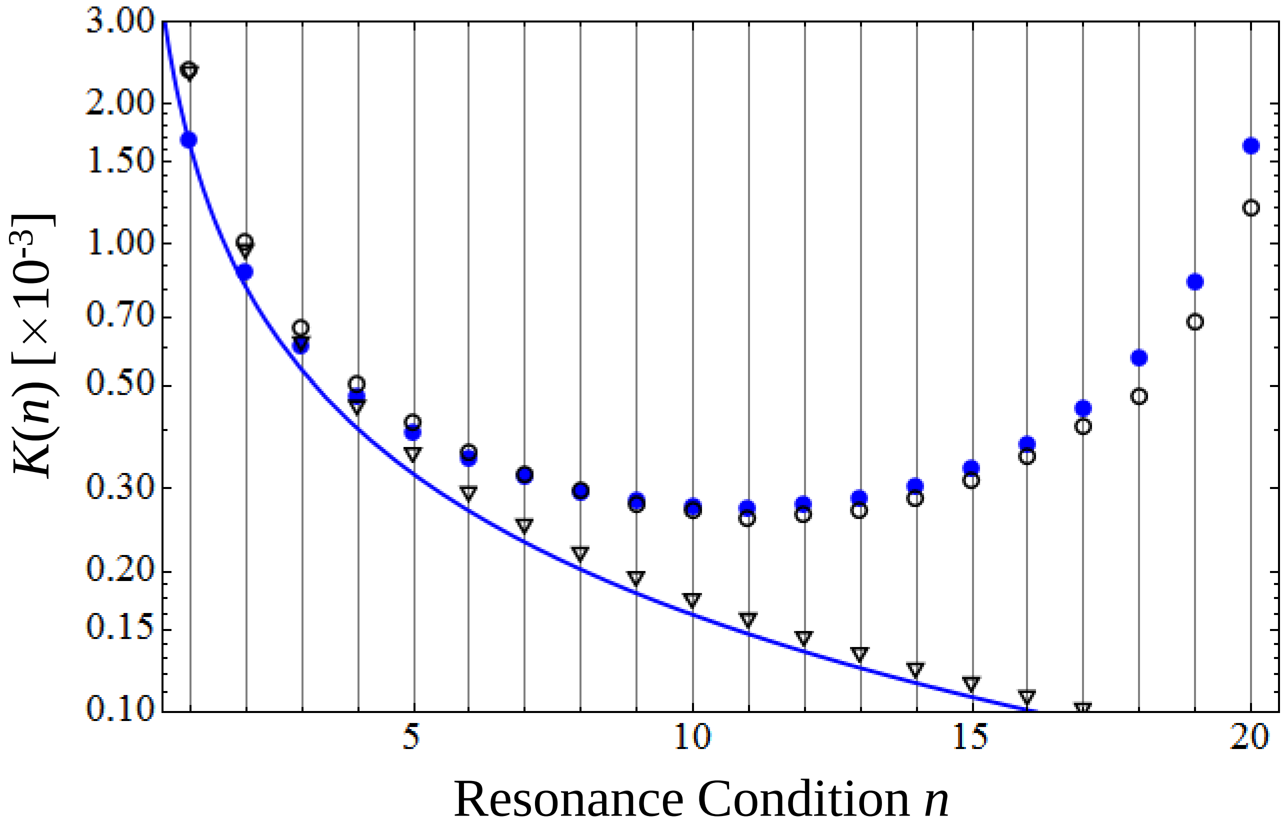
Black symbols : simulation results
empty **circle** : sinusoidal focusing
empty **triangle** : smoothed focusing

Blue **dots** : Guignard's formula

$$K(n)=\frac{q\lambda\sqrt{\beta_{\rm rf}^{\rm max}}}{2\pi mc^2 r_0}\times\left|\int_0^{2\pi}\sqrt{\beta_{\rm rf}}\,\frac{\hat{V}_D}{V_D}e^{in\theta}d\theta\right|$$

Blue **line** : smoothed theory

$$K(n)=\frac{qN_{\rm cell}\lambda^2}{4\pi mc^2 r_0}\frac{1}{n}$$

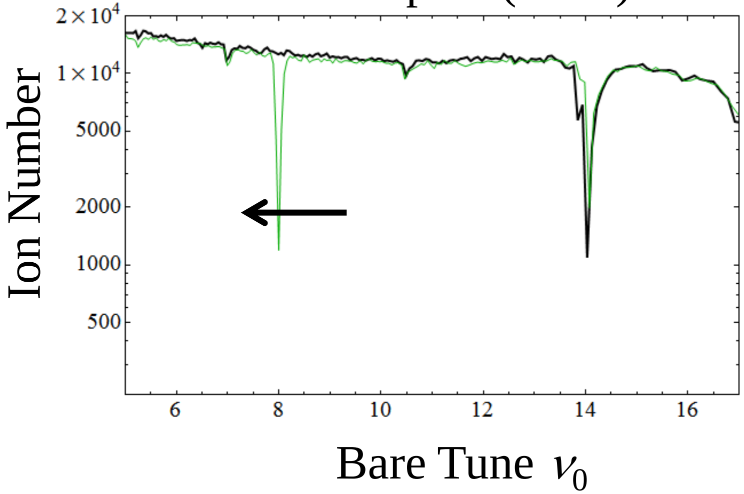


$$\left(\Delta A = K(n)\frac{w_n}{\sqrt{u}}\right)$$

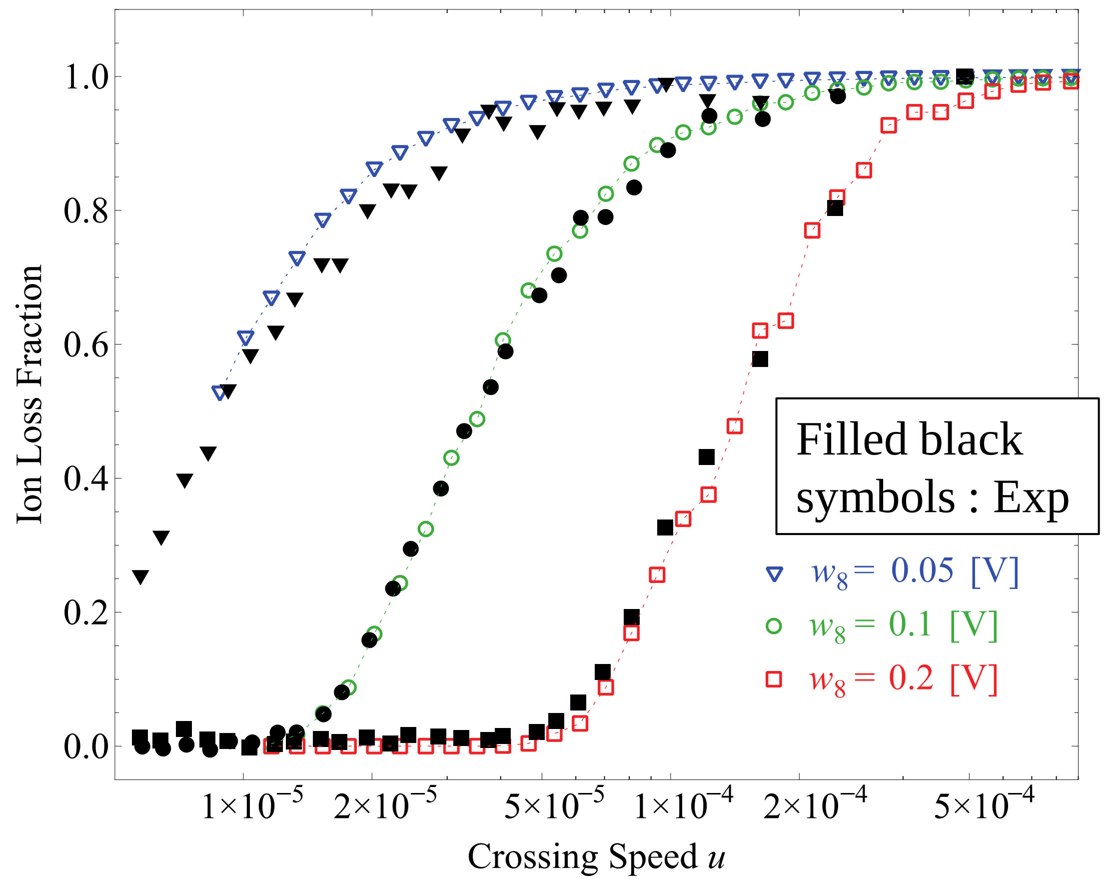
>> The simulation results show the similar behavior with the equation' curve only in the low tune (n) range. This is because the smooth approximation is available in about n < 7.

Single resonance crossing experiments

Sinusoidal dipole (n = 8)



Sweep range ν_0 : 9.5 -> 7.5
resonance at n = 8



>> The numerical data are in good agreement with the experimental observation. However, we note that the experimentally observed ion losses are somewhat fewer than the numerical predictions, which due to nonlinear effects.

Crossing-speed dependence of the perturbation voltage

In the case of the centroid amplitude grow up by the integer resonance, the plasma is scraped from approximately one side.
Therefore, when the plasma centroid amplitude reaches to 5mm, roughly half of the ions are lost because of the inner radius is 5mm in our trap system.

Solid line : Guignard's formula $K(8) = 0.2955 \times 10^{-3}$ [m/V]

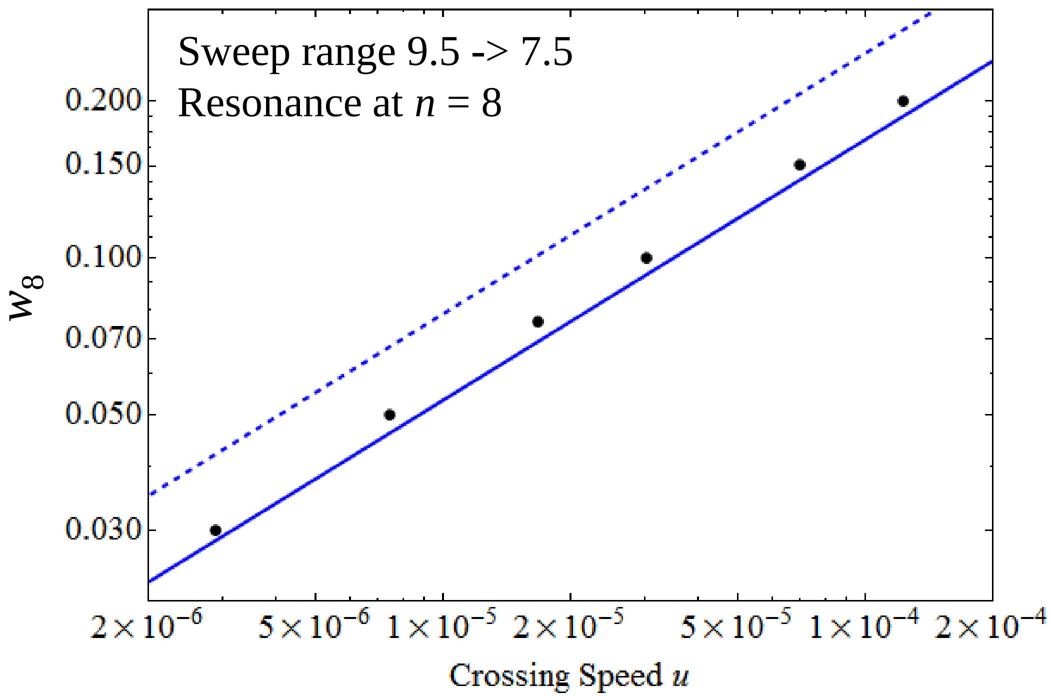
Dashed line : Smoothed theory $K(8) = 0.2018 \times 10^{-3}$ [m/V]

$$\Delta A = K(n)\frac{w_n}{\sqrt{u}}$$

and

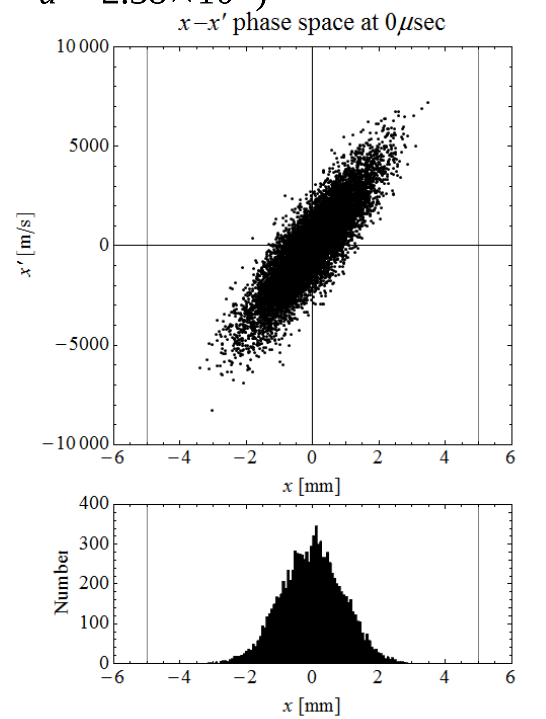
ΔA=5mm

Dots : Experimental result (50% loss condition)



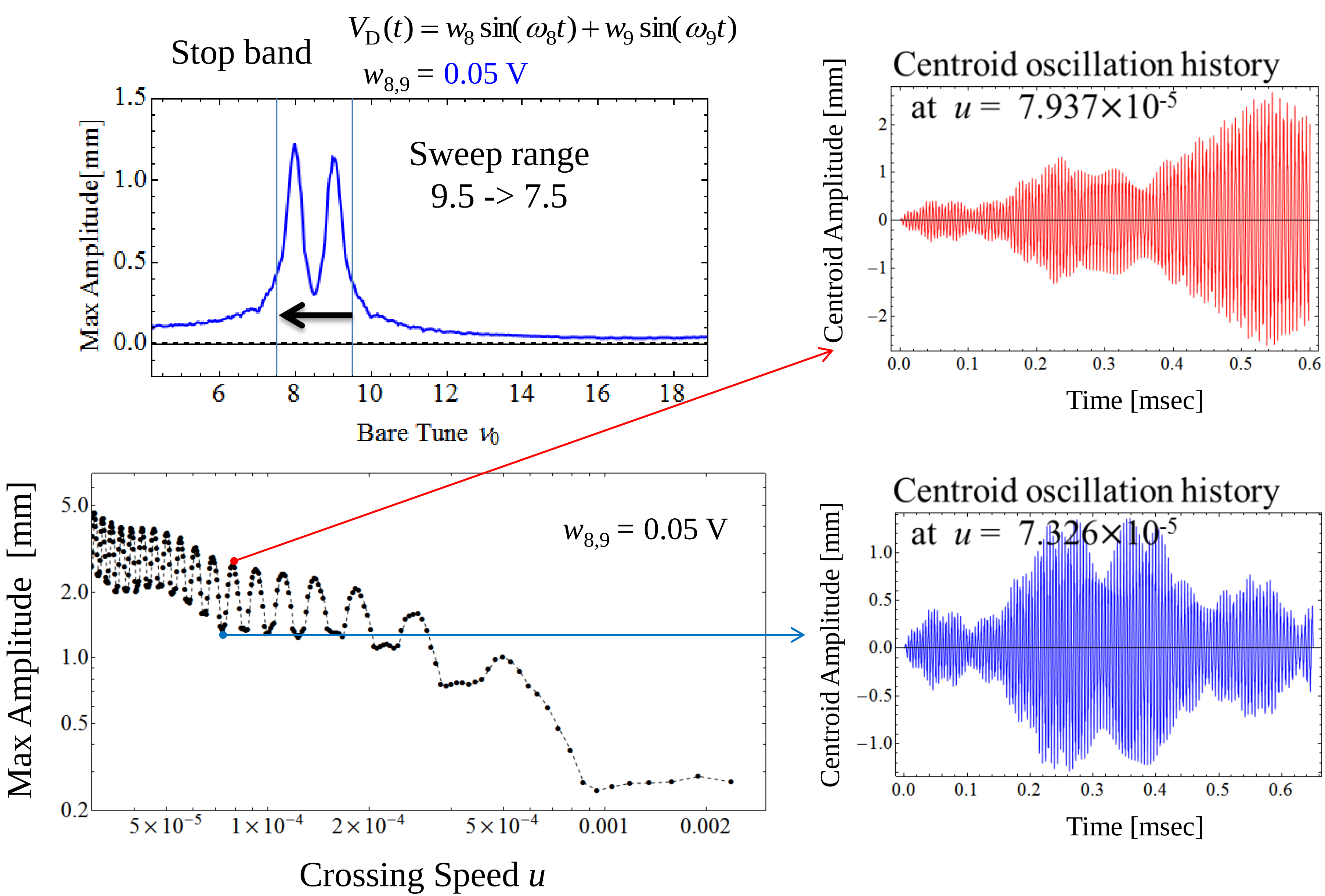
c.f. Simulation result

(1-dimension, resonance crossing at the resonance n = 8, V_D=0.5[V] u = 2.38×10^{-4})



Double resonance crossing

Double resonance crossing simulation (n = 8, 9)



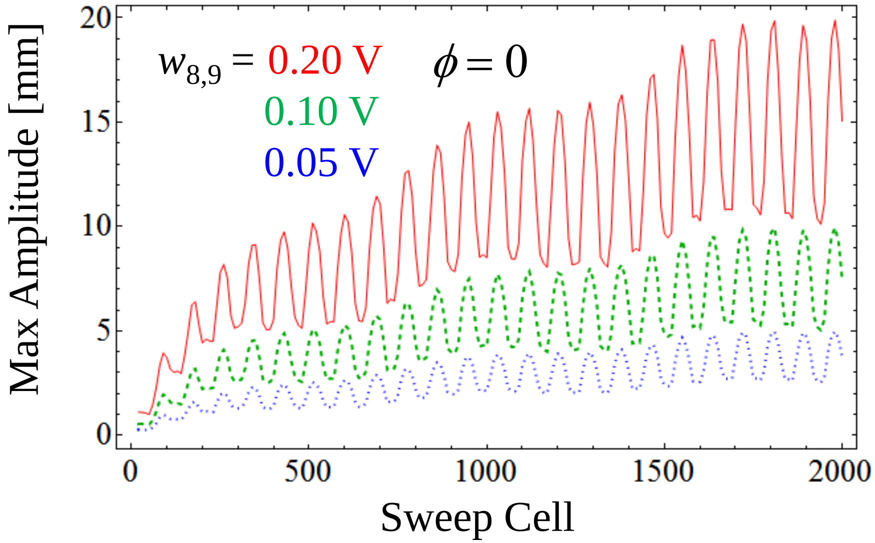
Double resonance crossing simulation (n = 8, 9)

$$V_D(t) = w_8 \sin(\omega_8 t) + w_9 \sin(\omega_9 t - \phi)$$

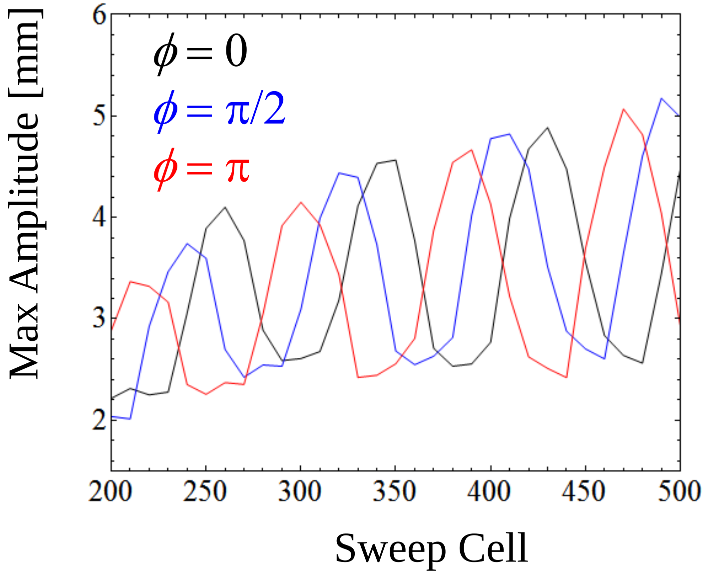
ϕ : relative initial phase

$$\begin{cases} w_8 = w_9 \\ \omega_8 = 2\pi \times (8/42) \times f_{rf} \\ \omega_9 = 2\pi \times (9/42) \times f_{rf} \\ f_{rf} = 1.0 \ [\mathrm{Mhz}] \end{cases}$$

Max amplitude v.s. sweep time

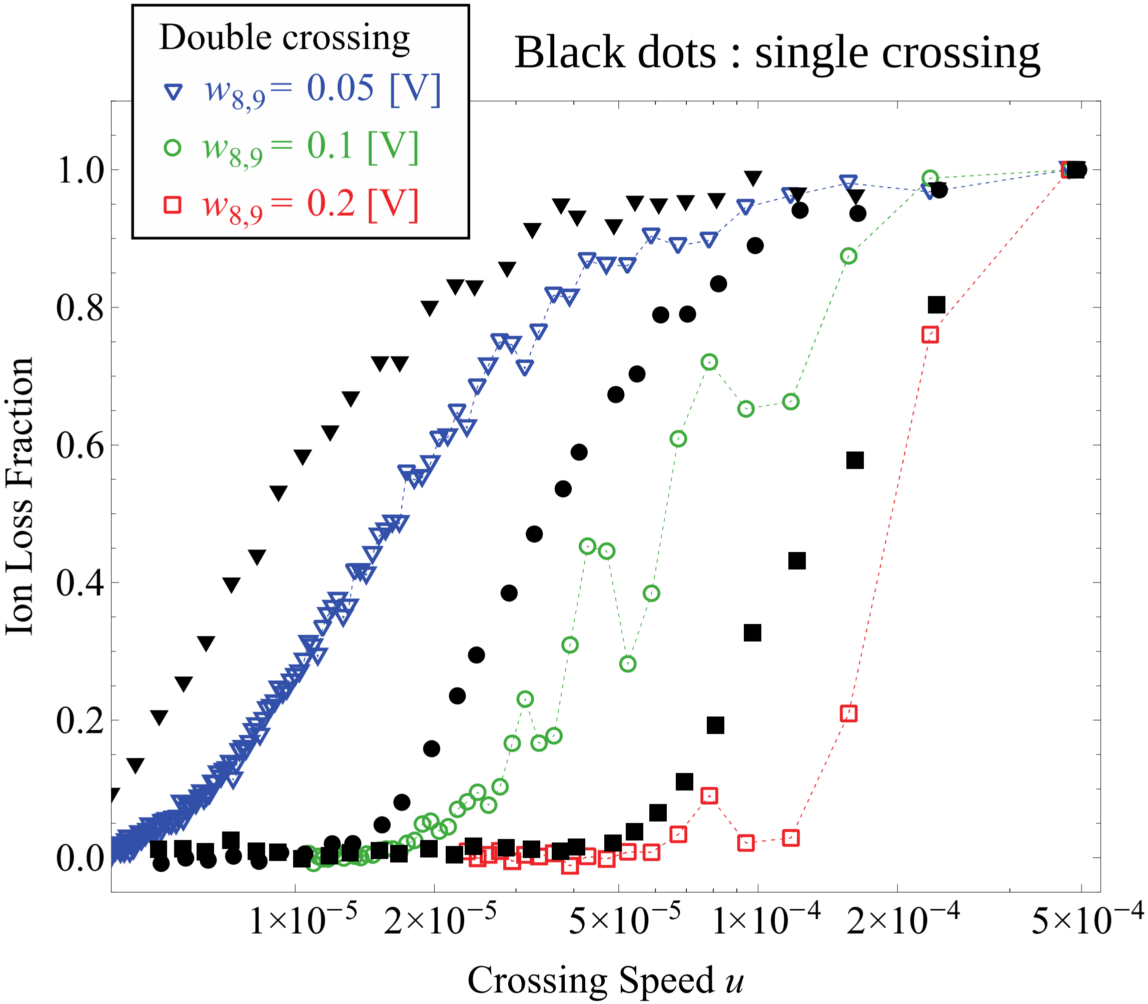


Max amplitude v.s. sweep time
(relative phase dependence)



>>The centroid oscillation is scaled by the dipole voltage w_n. The peaks of max amplitudes move with the relative phase.

Double resonance crossing experiments (n = 8, 9)



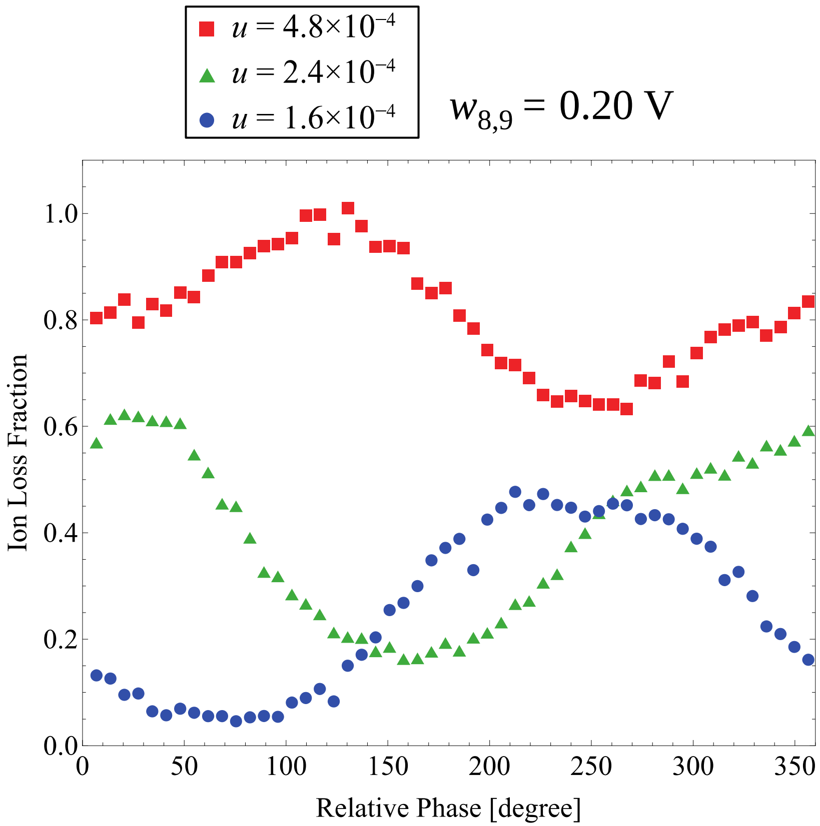
Sweep range ν_0 : 9.5 -> 7.5
resonance at n = 8, 9 and $\phi = 0$

>> The ion-loss rate has been clearly enhanced compared to the case of single resonance crossing (black dots).

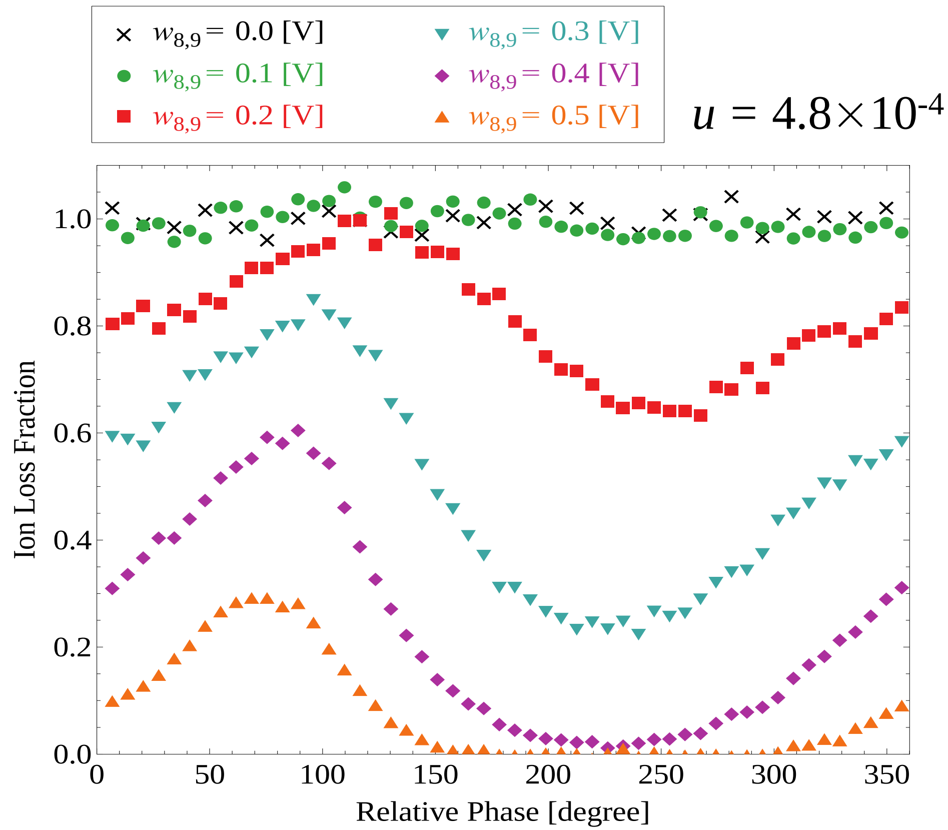
Double resonance crossing experiments (n = 8, 9)

Dependence of ion losses on the relative initial phase ϕ

- fixed dipole strengths, various crossing speed



- fixed crossing speed, various dipole strengths



>> In case the second kick is given at the optimal timing, the ion losses from the second resonance crossing can be suppressed.

Summary

- Theory of integer resonance crossing is verified experimentally.
- In the real situation, the relative betatron phase advance between consecutive integer tunes should be considered. It sometime helps ie. reduces the amplitude growth.
- In reality, nonlinear effects either due to alignment of trap in S-Pod or due to chromaticity and momentum spread in FFAG are inevitable. This introduces a real emittance growth after integer crossing, not only excitation of dipole oscillation. When the decoherence time is short relative to the traversal time from one integer to another, cancellation from consecutive excitations cannot be expected.

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- Etc.

The equation of the transverse ion motion in a LPT with the dipole driving field is

$$\frac{d^2 x}{d\tau^2} + K_{\mathrm{rf}}(\tau)x = -\frac{q}{mc^2 r_0} V_D(\tau) \qquad (1) \qquad \begin{cases} \tau = ct \\ K_{\mathrm{rf}}(\tau) = 2qV_{\mathrm{Q}} / mc^2 r_0^2 \\ V_{\mathrm{D}} = \sum_n w_n \cos(n\theta + \varphi_n) \end{cases}$$

$$\downarrow$$

$$\frac{d^2 \tilde{x}}{d\psi^2} + \nu_0^2 \tilde{x} = -(\nu_0 \beta_{\mathrm{rf}})^{3/2} \frac{q}{mc^2 r_0} V_D(\tau) \qquad (2) \qquad \because \psi = \frac{1}{\nu_0}\int^{\tau} \frac{d\tau}{\beta_{\mathrm{rf}}} \text{ and } \tilde{x} = \frac{x}{\sqrt{\nu_0 \beta_{\mathrm{rf}}}}$$

The transverse closed orbit distortion in a circular accelerator due to a dipole error field obeys

$$\frac{d^2 x_{\mathrm{COD}}}{ds^2} + K_x(s) x_{\mathrm{COD}} = -\frac{\Delta B}{B\rho} \qquad (3) \qquad \because \Delta B = \sum_n B_n \cos(n\theta + \varphi_n)$$

$$\downarrow$$

$$\frac{d^2 \tilde{x}_{\mathrm{COD}}}{d\psi^2} + \nu_0^2 \tilde{x}_{\mathrm{COD}} = -(\nu_0 \beta)^{3/2} \frac{\Delta B}{B\rho} \qquad (4) \qquad \because \psi = \frac{1}{\nu_0}\int^{\tau} \frac{d\tau}{\beta} \text{ and } \tilde{x}_{\mathrm{COD}} = \frac{x_{\mathrm{COD}}}{\sqrt{\nu_0 \beta}}$$

When we use the smooth approximation, we get the matched betatron function of the rf trap

$$\bar{\beta}_{\mathrm{rf}} = N_{\mathrm{cell}}\lambda / 2\pi\nu_0$$

and the averaged value of β is roughly given by

$$\bar{\beta} = R / \nu_0 .$$

Here, we compare Eq.(2) and Eq.(4). The particular solutions of these equations can be written with the smooth approximation as

$$\tilde{x} = \frac{-(\nu_0 \bar{\beta}_{\mathrm{rf}})^{3/2} \dfrac{q}{mc^2 r_0} V_D}{\nu_0^2} = \frac{x}{\sqrt{\nu_0 \bar{\beta}_{\mathrm{rf}}}},$$

$$\tilde{x}_{\mathrm{COD}} = \frac{-(\nu_0 \bar{\beta})^{3/2} \dfrac{\Delta B}{B\rho}}{\nu_0^2} = \frac{x_{\mathrm{COD}}}{\sqrt{\nu_0 \bar{\beta}}}$$

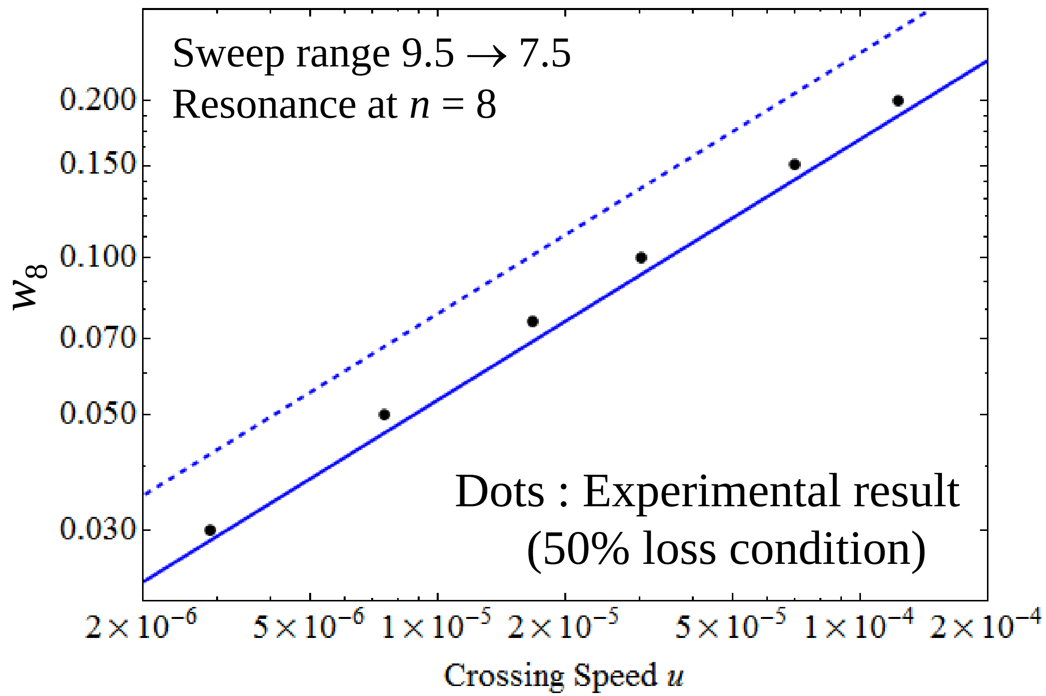
Now, we assume $x \approx x_{\mathrm{COD}}$,

$$V_D \approx \frac{mc^2 r_0}{q} \left(\frac{2\pi R}{N_{\mathrm{cell}}\lambda} \right)^2 \frac{\Delta B}{B\rho}$$

Crossing-speed dependence of the perturbation voltage

In the case of the centroid amplitude grow up by the integer resonance, the plasma is scraped from approximately one side.
Therefore, when the plasma centroid amplitude reaches to 5mm, roughly half of the ions are lost because of the inner radius is 5mm in our trap system.

Crossing-speed dependence of the **critical** perturbation voltage



Solid line : Guignard's formula

$$K(8) = 0.2955 \times 10^{-3}\ [\mathrm{m/V}]$$

Dashed line : Smoothed theory

$$K(8) = 0.2018 \times 10^{-3}\ [\mathrm{m/V}]$$

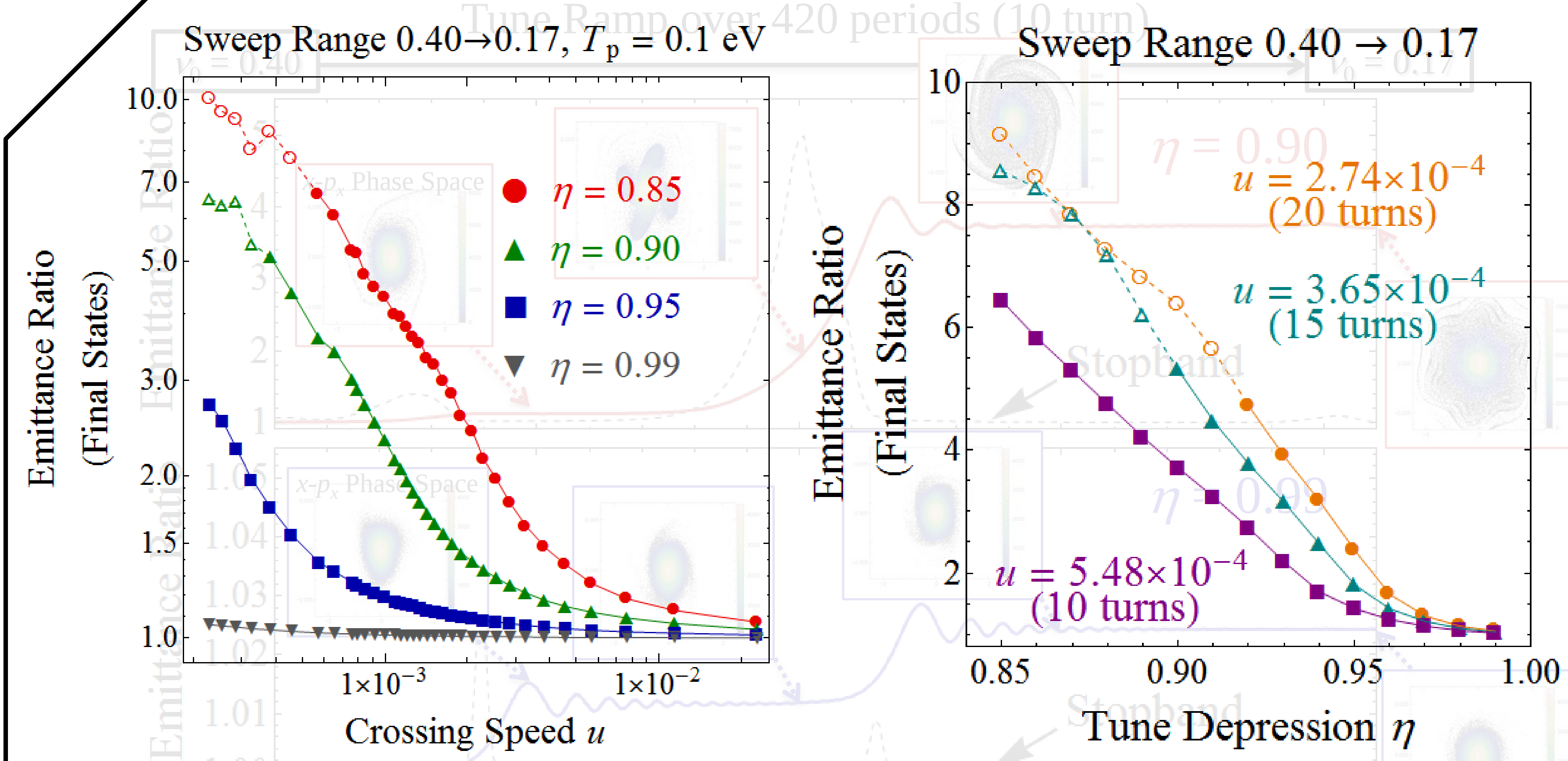
and $\Delta A = 5$ [mm]

: critical amplitude in S-POD

共鳴横断によるビームの不安定性(1)

Beam Instability via Resonance Crossing

共鳴を横断した場合のエミッタンス変化

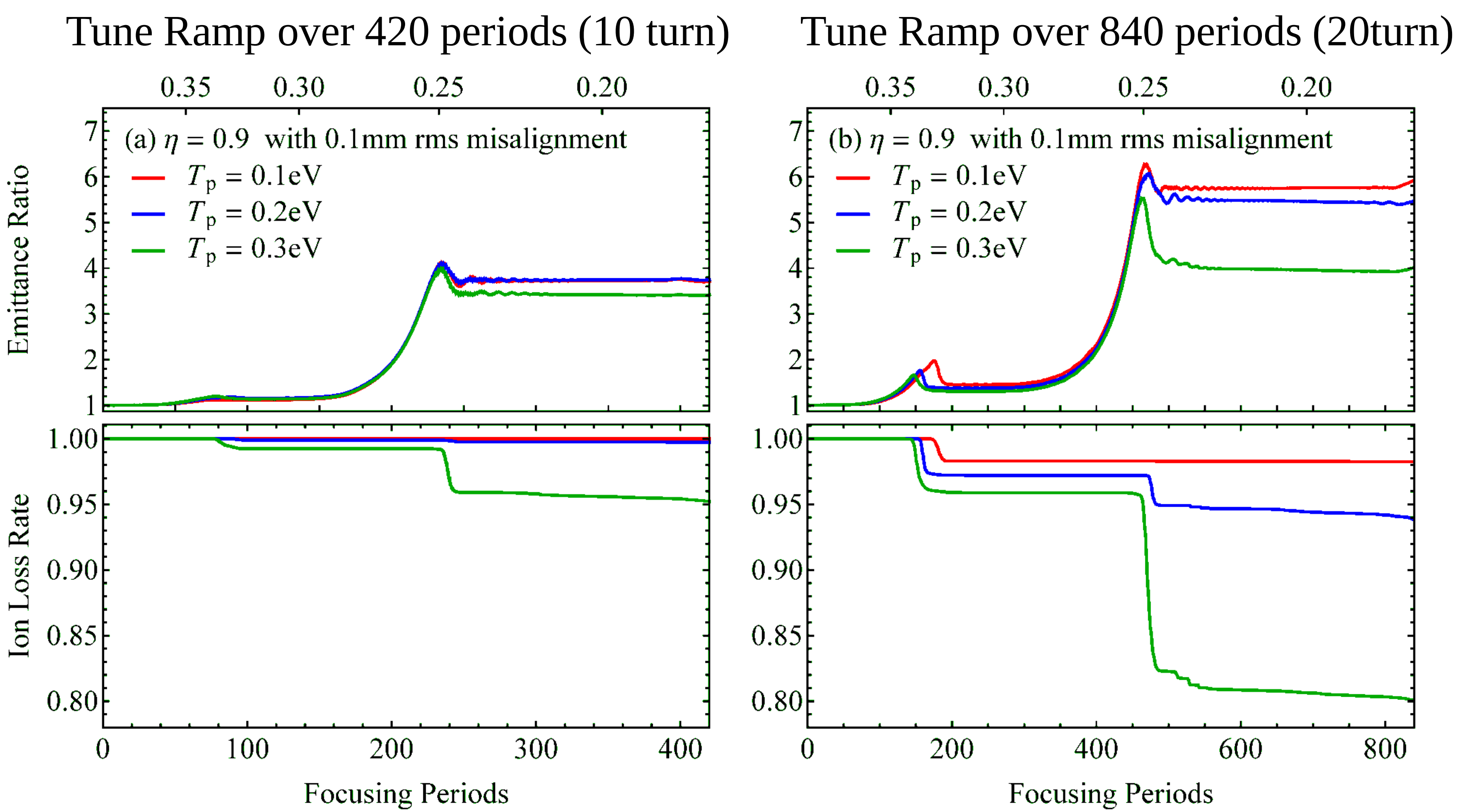


$$\text{Crossing Speed} : u = \frac{\text{(Sweep Range)}}{\text{(Focusing Periods)}}$$

共鳴横断によるビームの不安定性(2)

Beam Instability via Resonance Crossing

共鳴を横断した場合のエミッタンス変化(上)と粒子数変化(下)



加速器中のビームとトラップ中のプラズマ

Beams in a Storage Ring and Plasmas in a Linear Paul Trap

Storage Ring Hamiltonian (2D)

$$H_{\text{2D-ring}} = \frac{p_x{}^2 + p_y{}^2}{2} + \frac{K(s)}{2}(x^2 - y^2) + \frac{q}{p_0\beta_0 c\gamma_0^2}\phi$$

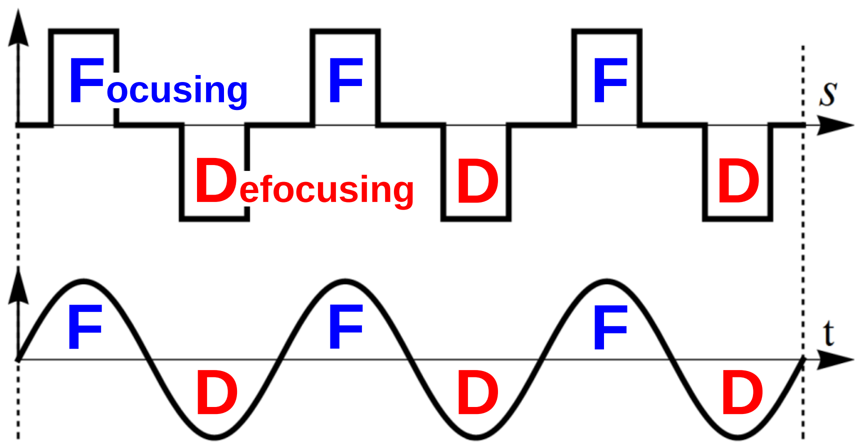
Linear Paul Trap Hamiltonian (2D)

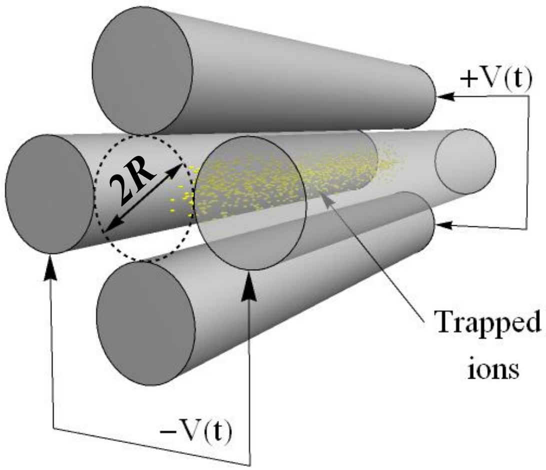
$$H_{\text{2D-trap}} = \frac{p_x{}^2 + p_y{}^2}{2} + \frac{qV(\mathrm{t})}{mc^2R^2}(x^2 - y^2) + \frac{q}{mc^2}\phi_{\mathrm{sc}}$$

+ Vlasov-Poisson Eqs.

FODO Cell
(Storage Ring)

集束正弦波電圧
(L.P.Trap)





Schematic of a Linear Paul Trap

共鳴横断シミュレーションのパラメータ
S-POD parameters to model the EMMA lattice

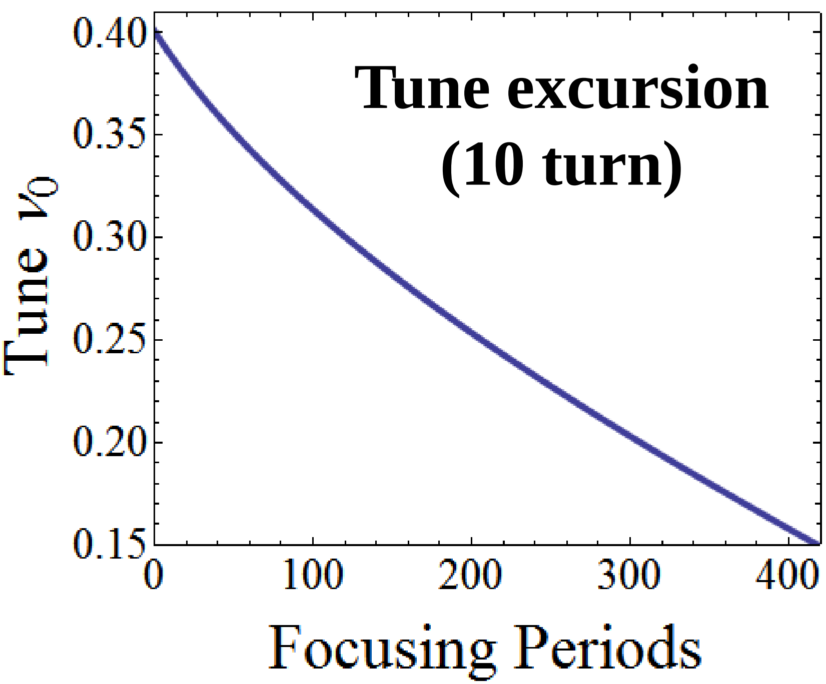
- EMMA parameters (one operating point)

Particle species : Electron
Energy range : 10 to 20 MeV
Number of turns : < 16
Lattice : F/D doublet

Number of cells : 42
Tune range : 16.8 → 7.14
- (per one cell : 0.40 → 0.17)

- S-POD parameters

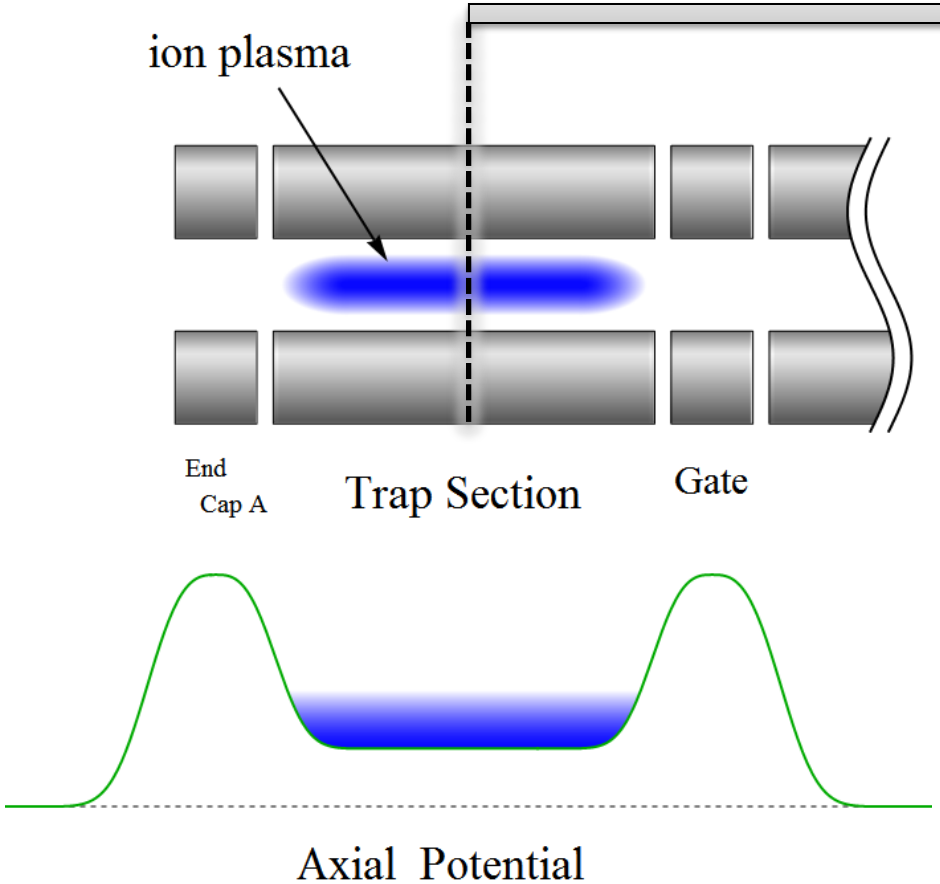
Particle species : Ar^+
Focusing wave : Sinusoidal (1008 kHz)
Focusing periods : < 840 (= 42cell×20 turn)
Tune range
- per one cell : 0.40 → 0.17

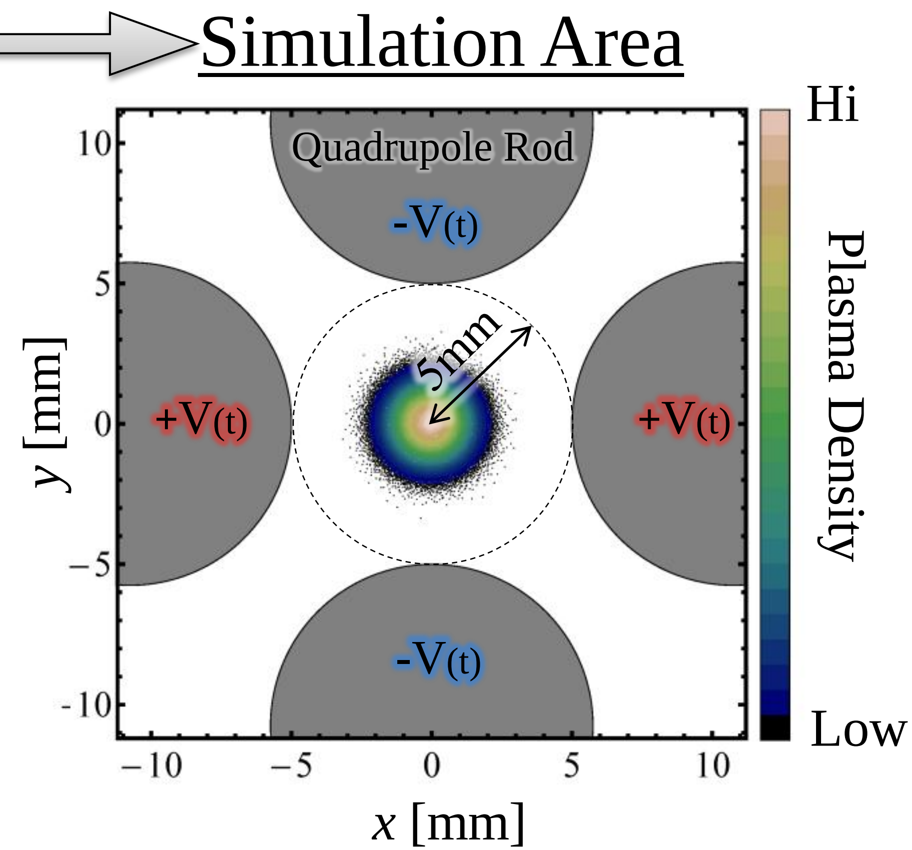


2D-PICシミュレーション

2D Simulation using PIC Code "Warp"

- 実験で用いているトラップの断面を想定し、
  PIC法を用いて空間電荷効果を計算する。

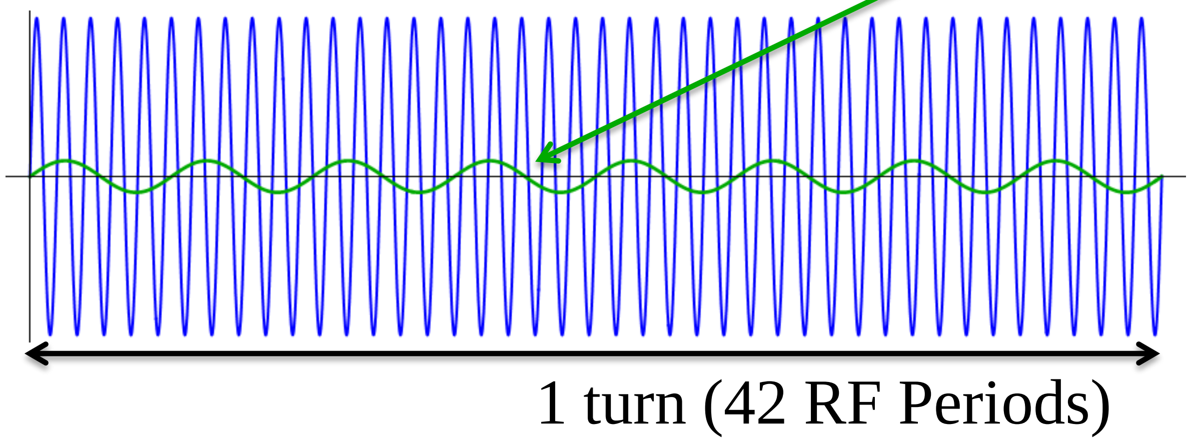


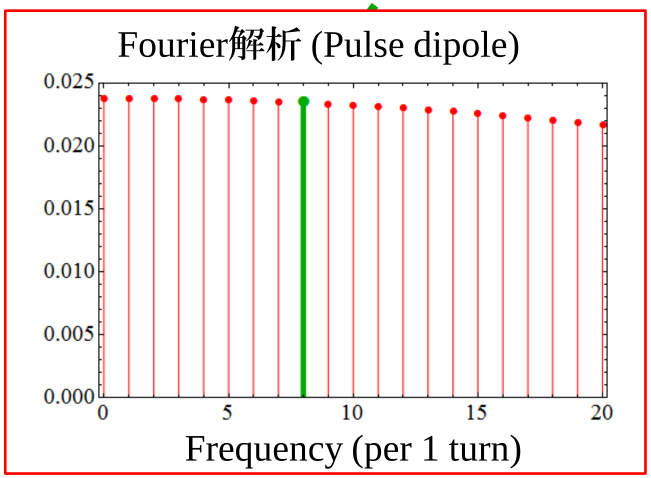


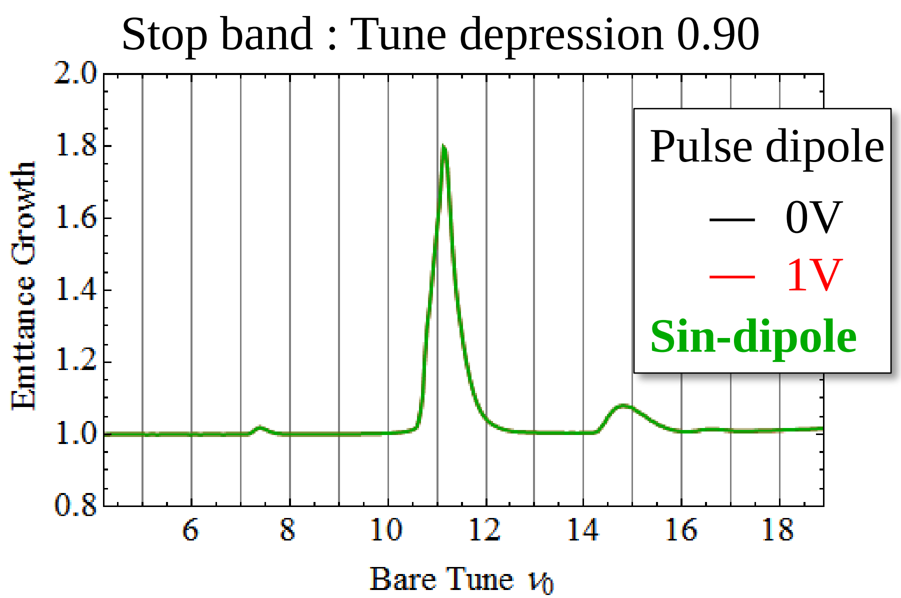
リング中に局在する双極子成分の影響

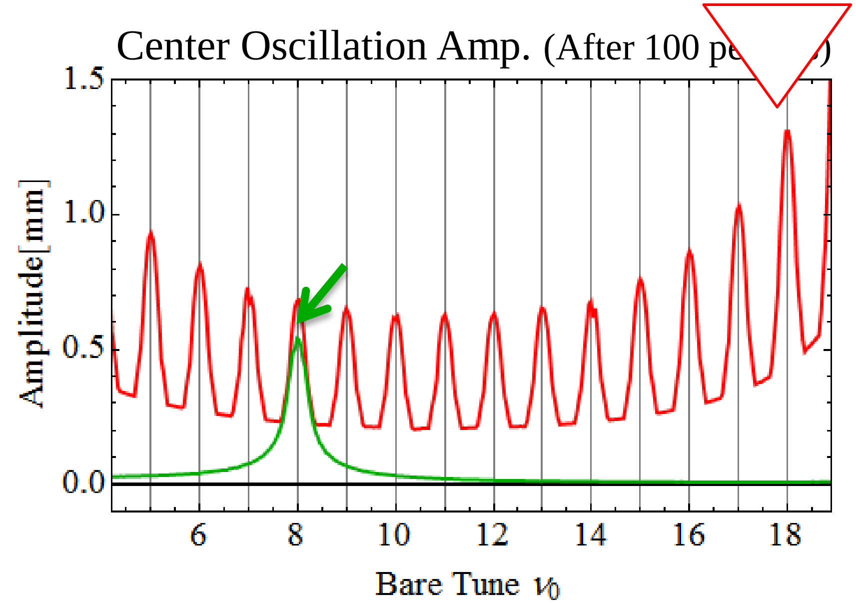
Pulse Type Dipole Component

Quadrupole Focusing (1MHz) **Sinusoidal Dipole (8/42 MHz)**









Single resonance crossing formulas

1) Smooth approximation

Baartman's Eq. (Smooth Approx.) [1]

Amplitude growth ΔA is

$$\Delta A = \frac{\pi}{\sqrt{Q_\tau}} \frac{\overline{R}}{\overline{\overline{B}}} \frac{B_n}{Q}$$

$$= \frac{\pi}{N_{\text{cell}}\sqrt{u}} \frac{R^2}{B\rho} \frac{B_n}{n}$$

$$\approx \frac{q N_{\text{cell}} \lambda^2}{4\pi m c^2 r_0} \frac{w_n}{n\sqrt{u}}$$

$$= K(n) \frac{w_n}{\sqrt{u}}$$

$$Q = N_{\text{cell}} \cdot \nu_0 = n, \quad Q_\tau = \frac{\Delta\nu \cdot N_{\text{cell}}}{N_{\text{turn}}}$$

$$\text{and} \quad \Delta\nu = \nu_{\text{cell}}^{ini} - \nu_{\text{cell}}^{fin}$$

$$\text{crossing speed} \quad u = \frac{\Delta\nu}{N_{\text{turn}} \times N_{\text{cell}}}$$

$$V_D \approx \frac{mc^2 r_0}{q} \left(\frac{2\pi R}{N_{\text{cell}} \lambda} \right)^2 \frac{\Delta B}{B\rho}$$

$$\because K(n) = \frac{q N_{\text{cell}} \lambda^2}{4\pi m c^2 r_0} \frac{1}{n} \quad \text{and} \quad \Delta B = \sum_n B_n \cos(n\theta + \varphi_n)$$

[1]R. Baartman, "Fast Crossing of Betatron Resonances", FFAG workshop 2004, Vancouver, Canada, 2004.

Single resonance crossing formulas

Guignard's formula

Square root of emittance growth is

$$\Delta\sqrt{\varepsilon} = \frac{\pi}{\sqrt{Q_\tau}}\frac{2R}{B\rho}\times\left|\frac{1}{2\pi}\int_0^{2\pi}\sqrt{\beta}\Delta B e^{in\theta}d\theta\right|$$

$$= \frac{\pi}{N_{\mathrm{cell}}\sqrt{u}}\frac{2R}{B\rho}\times\left|\frac{1}{2\pi}\int_0^{2\pi}\sqrt{\beta}\Delta B e^{in\theta}d\theta\right|$$

$$= \frac{qN_{\mathrm{cell}}\lambda^2}{2\pi mc^2 r_0}\times\left|\frac{1}{2\pi R}\int_0^{2\pi}\sqrt{\frac{2\pi R}{N_{\mathrm{cell}}\lambda}\beta_{\mathrm{rf}}}\frac{V_D}{w_n}e^{in\theta}d\theta\right|\cdot\frac{w_n}{\sqrt{u}}$$

$$\because \beta(\theta) = \frac{2\pi R}{N_{\mathrm{cell}}\lambda}\beta_{\mathrm{rf}}(\theta)$$

where, we use the β matched for tune *n*.

$$\Delta A = \Delta\sqrt{\varepsilon}\times\sqrt{\beta^{\max}}$$

$$= \frac{q\lambda\sqrt{\beta_{\mathrm{rf}}^{\max}}}{2\pi mc^2 r_0}\times\left|\int_0^{2\pi}\sqrt{\beta_{\mathrm{rf}}}\frac{V_D}{w_n}e^{in\theta}d\theta\right|\cdot\frac{w_n}{\sqrt{u}} = K(n)\frac{w_n}{\sqrt{u}}$$

$$\because K(n) = \frac{q\lambda\sqrt{\beta_{\mathrm{rf}}^{\max}}}{2\pi mc^2 r_0}\times\left|\int_0^{2\pi}\sqrt{\beta_{\mathrm{rf}}}\frac{V_D}{w_n}e^{in\theta}d\theta\right|$$

c.f. smooth approx.

$$K(n) = \frac{qN_{\mathrm{cell}}\lambda^2}{4\pi mc^2 r_0}\frac{1}{n}$$