I. BASICS OF MODULATION TECHNIQUES

Below I present the most common types of signal modulation: the pulse, amplitude and frequency modulations. I also discuss situations when they help to increase sensitivity of your measurements by using them in conjunction with the lock-in amplifier.

A. Pulse modulation (PM)

As we saw previously, the lock-in technique provides an excellent method to measure the amplitude (and the phase) of a harmonic signal $u(t) = u_0 \cos(\omega t + \phi)$ with an unprecedented sensitivity. However, very often we need to measure a small signal with an arbitrary time dependence, for example a signal $u(t)$ slowly varying in time, see Fig. 1a. In addition, very often we have a larger background noise that can obfuscate our signal, see Fig. 1b. In this case, we still can use the lock-in technique by employing the following technique. Let us multiply our slowly varying signal $u(t)$ by the so called *square wave*, that is a rectangular waveform of unit amplitude and period T_0, see Fig. 2.

Exercise
Write the Fourier transform of the square wave shown in Fig. 2.

Technically, it is rather easy to multiply the measured signal by a square wave. For example,

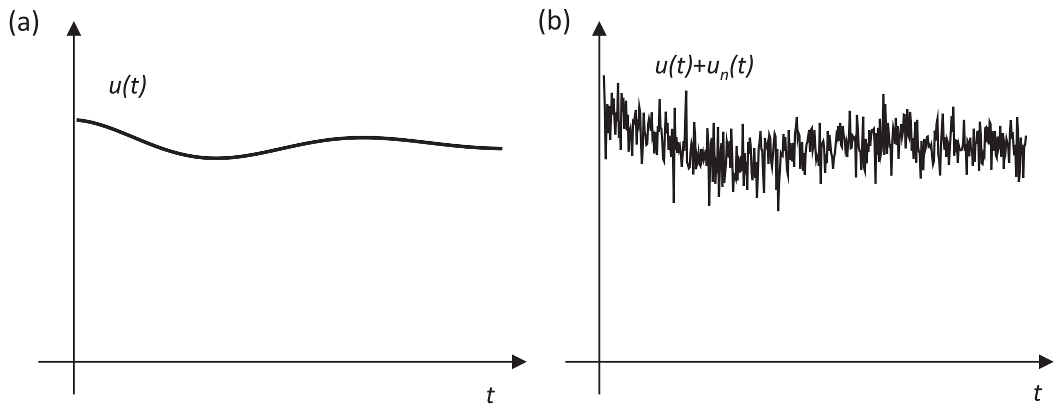

FIG. 1: Slowly time-varying signal (a) and slowly time-varying signal with random noise (b).

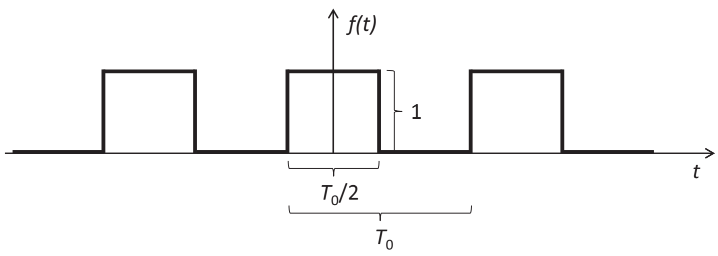

FIG. 2: Rectangular wave-form of period T.

you can simply switch the source of your signal periodically on and off. Or you can put a switch into the transmission line (a wire, cable, or optical fiber) that delivers the signal. Another example is to put a chopper to block a laser beam if you want to measure its intensity. Anyway, after the multiplication, the signal that you measure will look like in Fig. 3(a). Note that different sources of noise are usually unaffected by the square wave. So, the total signal at your measurement device will look like in Fig. 3b.

This procedure of multiplying the measured signal by a square wave is called *pulse modulation* (PM), and the resultant signal in Fig. 3a is called pulse modulated (or very often just *pulse modulated*). Let us see what happen if we use a lock-in amplifier to measure this signal. For this, we will use the reference signal for the lock-in amplifier, $u_{ref} = \cos(\omega_0 t)$, at the frequency $\omega_0 = 2\pi/T_0$, where T_0 is the period of the square wave. Please review the Section H of Handout 4 for the theory of lock-in.

Using the Fourier expansion of the square wave, we can write

$$u(t) \cdot f(t) + u_n(t) = u(t)\left(\frac{1}{2} + \frac{2}{\pi}\cos(\omega_0 t) + \frac{2}{3\pi}\cos(3\omega_0 t) + ...\right) + u_n(t). \tag{1}$$

It is clear from the theory of lock-in amplifier (see Sec. H, Handout 4) that the outputs of the two-phase lock-in amplifier read

$$u_X = \frac{2}{\pi}\langle u(t)\rangle \cos\phi + \langle u_n(t)\cos\omega_0 t\rangle, \quad u_Y = \frac{2}{\pi}\langle u(t)\rangle \sin\phi + \langle u_n(t)\cos\omega_0 t\rangle, \tag{2}$$

where $\langle ... \rangle$ means the rolling average over the lock-in low pass filter time τ_c, ϕ is the phase difference between the square wave and lock-in reference signal, and the averaged noise has the RMS value $u_\sigma = \sqrt{S(\omega_0)\omega_c}$, where $\omega_c = 2\pi/\tau_c$. Thus, we can extract the amplitude of the measured signal, which turns out to be

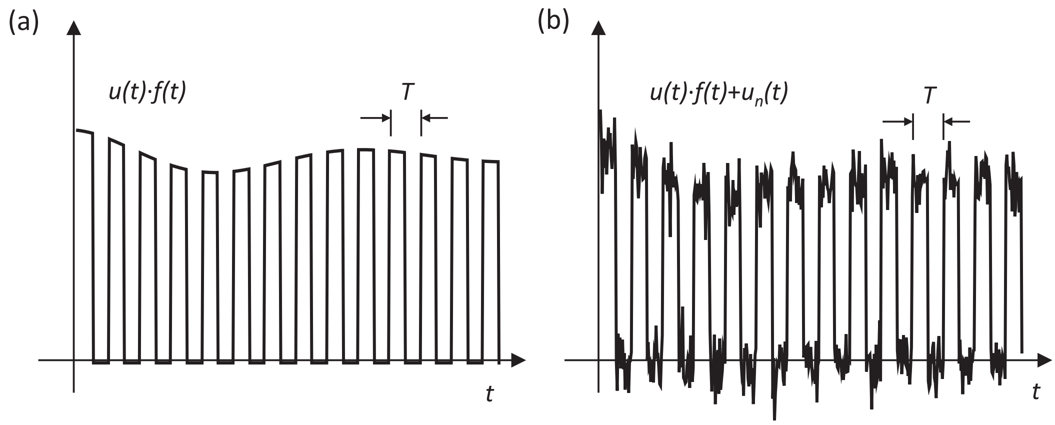

FIG. 3: Pulse modulated signal (a) and pulse modulated signal with noise (b).

$$u_{measure}(t) = \frac{2}{\pi}\langle u(t)\rangle. \tag{3}$$

We have to notice two things. First, the measured signal amplitude is reduced by a factor $2/\pi \approx 0.64$. Obviously, this is because the lock-in amplifier filters out all higher-order harmonics of ω_0 (and a DC component) in Eq. (1). Second, and this is very important, at the lock-in output you measure the rolling average of the slowly varying signal $u(t)$ over the time τ_c. Therefore, if you want to preserve the information about how fast the measured signal changes with time, the lock-in time τ_c should be much shorter than the time over which the measured signal varies significantly. *There is always a compromise between, on the one hand, choosing the lock-in time τ_c long enough to reduce your measurement bandwidth $\omega_c = 2\pi/\tau_c$, and, on the other hand, choosing it short enough in order not to average out the time-varying measured signal.*

Obviously, using the same technique you can measure a DC signal V which is constant in time. In this case you don't need to worry about τ_c and can choose it as long as needed to average out your noise. Another important example is when you measure a harmonic signal $u_0 \cos(\Omega t)$, where $\Omega/2\pi$ is very small, for example in a range of let's say $0.1 - 1$ Hz. You can try to measure this harmonic signal directly using the lock-in amplifier with the reference signal at the frequency Ω. However, as we saw earlier the measurements at such low frequencies are severely affected by the $1/f$ noise. Therefore, it is much better to use PM at the frequency $\omega_0/2\pi$ above 1 kHz where the $1/f$ noise becomes insignificant. Importantly, you need to make sure that the lock-in time τ_c should be much shorter than $2\pi/\Omega$. This is another important point. *Signal modulation together with the lock-in technique allows you to shift the frequency range of your measurements to higher frequencies, where* $1/f$ *noise is insignificant.*

B. Amplitude modulation (AM)

Above we considered the situation where we need to measure a DC or slowly varying signal. Let's now consider another situation where we need to measure a harmonic signal $u(t) = u_0 \cos(\omega t)$ at frequency $\omega/2\pi$, which is much higher than the range of our lock-in amplifier (the upper limit of a typical lock-in is 100 kHz). For example, we might need to measure a 10 MHz signal from the Nuclear Magnetic Resonance (NMR), or a 100 GHz signal from Electron Spin Resonance (ESR). Very often, we use a detector whose output is a DC voltage signal proportional to the square of the amplitude (or the power) of the detected high-frequency signal, see Fig. 4ab. Typical examples of such detectors are radio-frequency rectifying diode detectors, or optical p-n junction photo-detectors. Anyway, when the amplitude of the high-frequency signal is small (we say that the *signal level* is low), the output of the detector is also small (it can be as small as a few micro-Volts), and it can be severely affected by noise. Obviously, in this case it is desirable to use the lock-in amplifier to measure the detector output.

In order to employ lock-in detection, we modify our high-frequency signal according to the equation

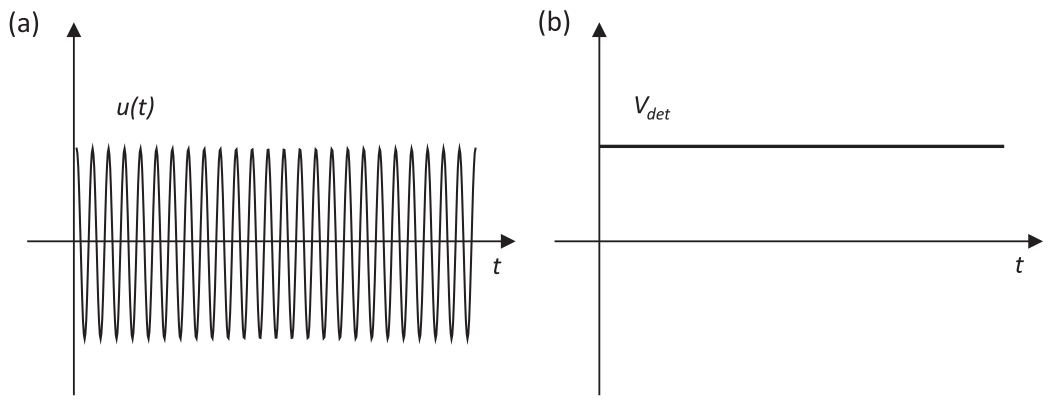

FIG. 4: High-frequency signal (a) and detector response to this signal (b).

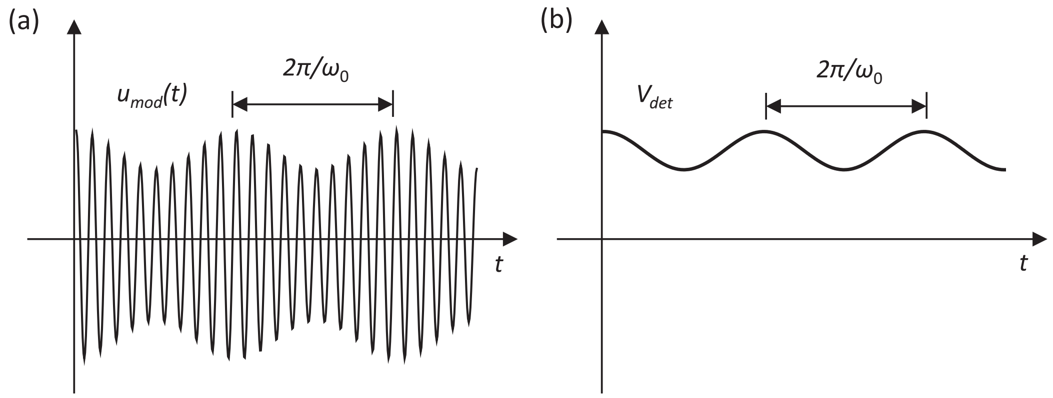

FIG. 5: Amplitude-modulated signal (a) and detector response to this signal (b).

$$u_0 \cos(\omega t) \rightarrow u_{mod}(t) = u_0 \left[1 + m \cos(\omega_0 t)\right] \cos(\omega t). \quad (4)$$

Here m, $0 < m < 1$, is called the *modulation depth*, and the modulation frequency ω_0 is typically much less than ω, $\omega_0 << \omega$. This type of modulation is called the *amplitude modulation* (AM), and the signal on the right-hand side of (4) is called the amplitude-modulated signal.

Exercise
Write the Fourier series expansion of the amplitude-modulated signal.

The amplitude-modulated signal is shown in Fig. 5a. Technically, the amplitude modulation can be done by multiplying (or as we say *mixing*) the high-frequency and low-frequency voltage signals using elements with non-linear current-voltage characteristics, e.g. semiconductor diodes. The response of a detector to this signal will look like in Fig. 5b (note that the response time of the detector must be much shorter than $2\pi/\omega_0$, which is usually the case). Obviously, the detector response is a harmonic signal at frequency $\omega_0/2\pi$, and it can be detected by the lock-in amplifier with the reference signal at the modulation frequency $\omega_0/2\pi$. Thus, in order to employ lock-in detection we typically want to modulate our high-frequency signal at the frequency up to 100 kHz (the upper frequency limit of a typical lock-in).

In principle, a high-frequency signal can be amplitude-modulated by an arbitrary slowly time-varying function $f(t)$, such that $|f(t)| < 1$, accordind to

$$u_{mod}(t) = u_0 \left[1 + f(t)\right] \cos(\omega t). \tag{5}$$

This form of amplitude modulation has wide applications in the *radio-communication*. For example, the modulating function $f(t)$ could be a music or human speech signal from a microphone. Usually, the frequency spectrum of such signals, that is signals that can be heard by the human ear, lie in the range from hundred to a few thousands Hz. These frequencies can not be easily transmitted as radio-waves to large distances. Instead, we would use a high-frequency signal in the range 10^5-10^8 Hz, amplitude-modulated it with our speech signal $f(t)$ according to (5), transmit the modulated high-frequency signal to large distance, and *demodulate* it (that is remove the high-frequency component by a detector, see Fig. 5b) at the receiver. This simple procedure forms the basis for the radio-communication.

Finally, note that one can consider the pulse modulation described in Section A as a particular example of the amplitude modulation. Technically, we should call it the *pulse-amplitude modulation*. There are other types of pulse modulation mainly used in the *digital electronics*, which is beyond this course.

C. Frequency modulation (FM)

Instead of modifying the amplitude of a measured high-frequency signal, as is done in Eq. (4), one can modify in a similar way the frequency of the signal according to

$$u_0 \cos(\omega t) \to u_{mod}(t) = u_0 \cos(\left[\omega + \Delta\omega \cos(\omega_0 t)\right] t). \tag{6}$$

This is called the *frequency modulation* (FM). Here $\Delta\omega$ is called the *amplitude* or *deviation* of FM, and typically $\Delta\omega << \omega$. Note that the amplitude of FM could be either much smaller than ω_0 (*narrow-band FM*) or larger than ω_0 (*wide-band FM*).

Exercise

Write the Fourier series expansion of the frequency-modulated signal. You can use the following series expansion

$$e^{j\beta \sin(\omega_0 t)} = \sum_{n=-\infty}^{+\infty} J_n(\beta) e^{jn\omega_0 t}, \tag{7}$$

where $J_n(\beta)$ is the n-order Bessel function of argument β.

Obviously, the amplitude of the frequency modulated signal $u_{mod}(t)$ in Eq. (6) does not vary in time. Therefore, the response of a rectifying detector also will not vary in time, see Fig. 4. In this case, what is the point of using FM? It is helpful to use FM instead of AM when the frequency of the measured high-frequency signal varies with some experimental parameters, in particular when there is a strong dependence of the frequency on such parameters. The typical example is a measurement of resonance. Suppose we pass a harmonic signal $u_{in}(t) = u_0 \cos(\omega t)$ through a resonant circuit (e.g. see Fig. 7a in Handout 3), and vary the cyclic frequency ω around the resonance frequency ω_{res} of the resonant circuit. Then, the output signal will be in the form

$$u_{out}(t) = u_0 A(\omega) \cos(\omega t), \tag{8}$$

where $A(\omega)$ is the resonant response function, which in general is complex.

Now, instead of using $u_0 \cos(\omega t)$, let us use a frequency-modulated signal from Eq. (6) as the input signal. Then, the output signal becomes

$$u_{out}(t) = u_0 A(\omega + \Delta\omega \cos(\omega_0 t)) \cos\left([\omega + \Delta\omega \cos(\omega_0 t)]\, t\right), \tag{9}$$

and the detector response reads

$$u_{det} \propto |A(\omega + \Delta\omega \cos(\omega_0 t))|^2 \approx |A(\omega)|^2 + \mathrm{Re}\left(A \frac{dA}{d\omega}\right) \Delta\omega \cos(\omega_0 t), \tag{10}$$

where we used the Taylor expansion because typically $\Delta << \omega$. Note that the resultant signal at the detector output is harmonic at the frequency ω_0. Thus, it can be sensitively measured by the lock-in amplifier with the reference signal at the frequency ω_0. Also note that the amplitude of the measured signal will be proportional to the *derivative* of the response $A(\omega)$. This is very typical for FM technique.

As AM technique, FM is widely used in the radio-communication.