I. PRINCIPLE OF LOCK-IN AMPLIFIER

Before reading this handout, I recommend to briefly review the material presented in *Handout 3: Noise in Electrical Circuits*. As it is clear from the discussion of electrical noise, making the bandwidth B of your experimental setup narrower decreases the standard deviation of your random noise signal as $\sqrt{B}$. An unprecedented bandwidth narrowing (as narrow as 0.01 Hz!) can be achieved using a lock-in amplifier, which makes this instrument one of the most frequently used one whenever high precision of measurements is required. It also allow to efficiently filter out the unwanted signals, for example due to coherent interference. Let us consider how the lock-in works.

Suppose we would like to measure a harmonic signal $u(t) = u_0 \cos(\omega_0 t + \phi)$. In addition to this signal, we also have a noise signal $u_n(t)$, thus the total signal at the input of our measuring device (that is the lock-in amplifier) is $u_0 \cos(\omega_0 t + \phi) + u_n(t)$. Inside the lock-in, we mix this signal with a reference harmonic signal $u_{ref} \cos \omega_0 t$. Note that the reference signal must have the same frequency as the signal that we want to measure. By mixing we mean that we produce a signal which is the product of the two mixed signals, that is at the output of our mixer (see Fig. 1) we obtain the signal

$$(u_0 \cos(\omega_0 t+\phi)+u_n(t))\cdot u_{ref} \cos \omega_0 t = \frac{1}{2} u_0 u_{ref} \cos \phi + \frac{1}{2} u_0 u_{ref} \cos(2\omega_0 t+\phi) + u_{ref} u_n(t) \cos \omega_0 t. \tag{1}$$

Note that the first term on the right-hand side is proportional to the amplitude of the signal that we want to measure, and it does not depend on time. Therefore, if we pass the total

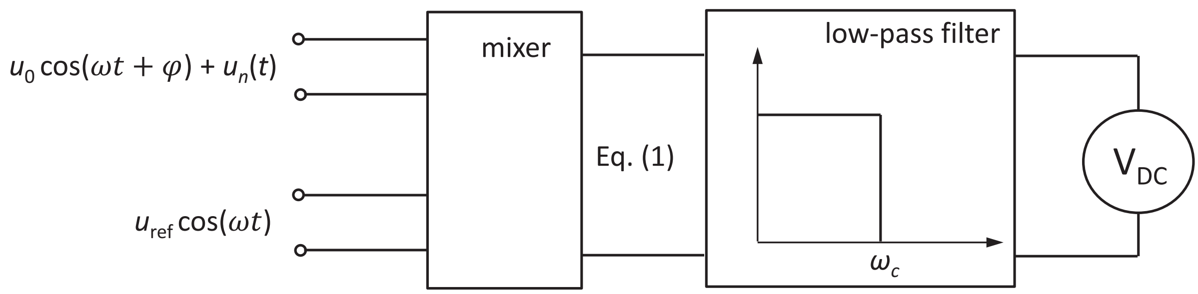

FIG. 1: Simplified diagram of lock-in amplifier for signal measurements.

signal on the right-hand side of the above equation through a low-pass filter (see Fig. 1), the first term will pass unattenuated. The second term on the right-hand side of Eq. (1) has harmonic time dependence at the frequency $2\omega_0$. Therefore, if the cut-off frequency of the low-pass filter is chosen to be sufficiently lower than $2\omega_0$, that is $\omega_c << 2\omega_0$, the second term can be almost completely eliminated.

Finally, let's see what happens to the third term, the noisy term, as we pass it through the low-pass filter. Let us again consider the complex representation of the reference signal, $u_{ref}(t) = (e^{j\omega_0 t} + e^{-j\omega_0 t})/2$, and let us call the noisy signal at the output of the low-pass filter $\tilde{u}_n(t)$. We can write the third term in Eq. (1) as (we omit u_{ref} for the moment)

$$\frac{1}{2}u_n(t)e^{j\omega_0 t} + \frac{1}{2}u_n(t)e^{-j\omega_0 t} = \frac{1}{2}\int_{-\infty}^{\infty} u(\omega-\omega_0)e^{-j\omega t}d\omega + \frac{1}{2}\int_{-\infty}^{\infty} u(\omega+\omega_0)e^{-j\omega t}d\omega, \quad (2)$$

where $u(\omega)$ is the Fourier transform of $u_n(t)$. After passing this signal through the low-pass filter, we obtain

$$\begin{aligned}
\tilde{u}_n(t) &= \frac{1}{2}\int_{-\omega_c}^{\omega} u(\omega-\omega_0)e^{-j\omega t}d\omega + \frac{1}{2}\int_{-\omega}^{\omega} u(\omega+\omega_0)e^{-j\omega t}d\omega = \\
&= \frac{1}{2}\int_{-\omega_c}^{\omega}\left(\int_{-\infty}^{\infty} u_n(t')e^{j\omega_0 t'}e^{-j\omega t'}dt'\right)e^{j\omega t}d\omega + \frac{1}{2}\int_{-\omega_c}^{\omega}\left(\int_{-\infty}^{\infty} u_n(t')e^{-j\omega_0 t'}e^{-j\omega t'}dt'\right)e^{j\omega t}d\omega = \\
&= \frac{1}{2}\int_{-\infty}^{\infty} u_n(t')e^{j\omega_0 t'}\left(\int_{-\omega_c}^{\omega} e^{j\omega(t-t')}d\omega\right)dt' + \frac{1}{2}\int_{-\infty}^{\infty} u_n(t')e^{-j\omega_0 t'}\left(\int_{-\omega_c}^{\omega} e^{j\omega(t-t')}d\omega\right)dt' = \\
&= \frac{1}{2}\int_{-\infty}^{\infty} u_n(t')e^{j\omega_0 t'}\left(\frac{\sin\omega_c(t'-t)}{t'-t}\right)dt' + \frac{1}{2}\int_{-\infty}^{\infty} u_n(t')e^{-j\omega_0 t'}\left(\frac{\sin\omega_c(t'-t)}{t'-t}\right)dt'. \quad (3)
\end{aligned}$$

Note that the function $\frac{\sin\omega_c(t'-t)}{t'-t}$ significantly differs from zero only in the interval $t - \frac{\tau_c}{2} < t' < t + \frac{\tau_c}{2}$, where $\tau_c = 2\pi/\omega_c$ is the correlation time set by the low-pass filter cut-off. Thus we can write the last line approximately as

$$\tilde{u}_n(t) \approx \frac{1}{2\tau_c}\int_{t-\frac{\tau_c}{2}}^{t+\frac{\tau_c}{2}} u_n(t')e^{j\omega_0 t'}dt' + \frac{1}{2\tau_c}\int_{t-\frac{\tau_c}{2}}^{t-\frac{\tau_c}{2}} u_n(t')e^{-j\omega_0 t'}dt' = \frac{1}{\tau_c}\int_{t-\frac{\tau_c}{2}}^{t+\frac{\tau_c}{2}} u_n(t')\cos\omega_0 t' dt'. \quad (4)$$

Thus the effect of low-pass filter is to average the input random noise over the correlation time. The averaging in the above equation is called the *rolling average*. This shows the equivalence between the rolling averaging over the time τ_c and low-pass filtering with the cut-off frequency $\omega_c = 2\pi/\tau_c$.

Finally, we detect the output of the low-pass filter with a DC voltmeter to obtain (by choosing e.g. u_{ref}=2 V)

$$u_X = u_0 \cos\phi + \langle u_n(t) \cos\omega t \rangle, \tag{5}$$

where $\langle ... \rangle$ means rolling average over the correlation time τ_c. This is what should be shown in channel X of your typical lock-in amplifier. Actually, for some reason the manufacture decides to show the effective (RMS) value, that is $u_0/\sqrt{2}$, for the measured signal. Usually, the lock-in repeats the same procedure using a second reference signal of the form $u_{ref} \sin\omega t$, which gives

$$u_Y = u_0 \sin\phi + \langle u_n(t) \sin\omega t \rangle. \tag{6}$$

This is what is shown in channel Y. You can use the readings of channels X and Y to find the amplitude of the signal u_0 and its phase ϕ. Usually, lock-in has an option to show them directly.

So, we can measure both the amplitude and phase of the signal of interest, although our measurements are still affected by the averaged noise (the second term in in the last two equations). Let us calculate the dispersion of this averaged noise. You can calculate it either in a straightforward way as

$$u_\sigma^2 = \overline{\tilde{u}_n^2} = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \tilde{u}^2(t)dt, \tag{7}$$

where $\tilde{u}_n(t)$ is given by Eq. (4). However, it is more instructive to calculate the auto-correlation function of the third term in Eq. (1), pass it through the low-pass filter, and find the dispersion as $u_\sigma^2 = \psi(0)$. The auto-correlation function is

$$\psi(\tau) = \frac{1}{2}\overline{u_n(t)e^{j\omega_0 t}u_n(t+\tau)e^{j\omega_0(t+\tau)}} + \frac{1}{2}\overline{u_n(t)e^{-j\omega_0 t}u_n(t+\tau)e^{-j\omega_0(t+\tau)}}. \tag{8}$$

Doing exactly the same way as in our derivation of Eq. 6 in *Handout 3: Noise in Electrical Circuits*, it is straightforward to obtain

$$\psi(\tau) = \frac{1}{2}\int_{-\infty}^{\infty} S(\omega - \omega_0)e^{j\omega t}d\omega + \frac{1}{2}\int_{-\infty}^{\infty} S(\omega + \omega_0)e^{j\omega t}d\omega. \quad (9)$$

Passing the noise signal with this characteristic through a narrow low-pass filter, and assuming that $S(\omega \pm \omega_0) \approx S(\omega_0)$ (it's easy to figure out that $S(\omega)$ is an even function of ω) within the pass-band of the filter, we find the auto-correlation function at the filter output

$$\psi(\tau) = S(\omega_0)\omega_c \frac{\sin \omega_c \tau}{\omega_c \tau}, \quad (10)$$

and the noise signal dispersion

$$u_\sigma^2 = S(\omega_0)\omega_c. \quad (11)$$

Note that the "real-world" power spectral density function is twice smaller because in the real-world we do not consider negative frequencies. So, the dispersion of the averaged noise in Eqs. (5,6) decreases with the decreasing cut-off frequency of the low-pass filter. By choosing low cut-off frequency (in a typical lock-in you can choose it as low as 0.01 Hz), you can significantly suppress RMS of your noise. The price that you pay is that by decreasing the bandwidth of your lock-in you simultaneously increase the rolling averaging time, which is $\tau_c \propto 1/\omega_c$. Thus, if you start changing the amplitude of you signal u_0, you will also see only averaged amplitude over the time τ_c. This means that you can not change u_0 faster than the rolling averaging time, which is set by your filter cut-off

In a typical lock-in, you can measure RMS of both channels X and Y, that is the signals in Eqs. (5,6), directly by choosing the the 'X Noise' and 'Y noise' buttons. In this case, the

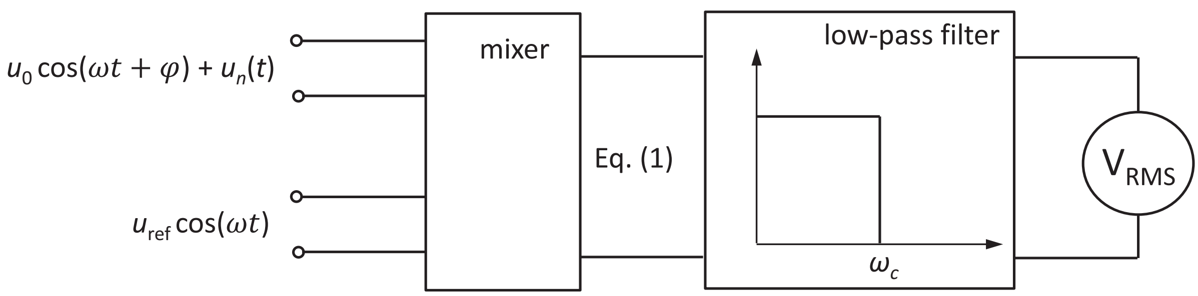

FIG. 2: Simplified diagram of lock-in amplifier for noise measuremnts.

lock-in replaces the DC voltmeter with an RMS voltmeter, see Fig. 2, and both channels will display RMS of the averaged noise given by

$$\sqrt{\overline{\tilde{u}_n^2(t)}} = u_\sigma = \sqrt{S(\omega_0)\omega_c}. \tag{12}$$

Thus, the lock-in allows you to measure the power spectral density $S(\omega)$ of your noise within the frequency range of your lock-in.