I. BASICS OF AC ELECTRICAL CIRCUITS

A. Basic Elements of AC Circuits

The basic elements of AC circuits are resistors, capacitors, and inductors. These elements can be mathematically described by the relation between the time-dependent voltage $u(t)$ across the element and the electrical current $i(t)$ flowing through the elements. In order to distinguish from DC voltage V and DC current I, we will use lower-case letters for the time dependent voltages and currents. Thus, for a resistor of resistance R, capacitor of capacitance C, and inductor of inductance L we write

$$u = Ri, \quad u = \frac{1}{C}\int idt, \quad u = L\frac{di}{dt}. \tag{1}$$

Typically, we consider the time-dependent voltage signals of the form $u(t) = u_0\cos(\omega t)$, where $\omega = 2\pi f$, and f is the frequency in Hz. Plugging this into the above equation, we can figure out what is the relation between time-dependent voltage and current at the particular element. For example, for a resistor the current is $i(t) = (u_0/R)\cos(\omega t)$, that is the current has exactly the same time dependence as the voltage (we say that they are in phase). For a capacitor we have $i(t) = (-u_0\omega C)\sin(\omega t) = (u_0\omega C)\cos(\omega t+\pi/2)$, that is the time-dependent current is shifted in phase by $\pi/2$ with respect to the voltage. Finally, for an inductor we have $i(t) = (u_0/\omega L)\cos(\omega t - \pi/2)$, that is the time-dependent current is shifted by $-\pi/2$ with respect to the voltage.

It is very convenient to use complex notations where the voltage and current are represented by complex numbers. For example, instead of the above time dependent voltage $u(t) = u_0\cos(\omega t)$ we write

$$u(t) = u_0 e^{j\omega t}. \tag{2}$$

Obviously, the "real" voltage signal is just the real part of Eq. (2), that is $u(t) = \mathrm{Re}\,(u_0 e^{j\omega t})$. The complex notations greatly simplify calculation analysis of the AC circuits. For example, from Eq. (1) we can write

$$u = Ri, \quad u = \frac{i}{j\omega C}, \quad u = j\omega Li. \tag{3}$$

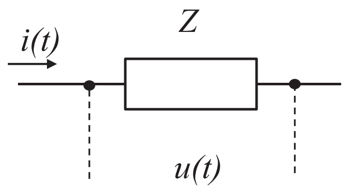

FIG. 1: Complex impedance and Ohm's law.

This allows to generalize the Ohm's law for AC circuits by introducing the *complex impedance* Z (see Fig. 1)

$$u(t) = Zi(t). \tag{4}$$

From Eq.(3) we see that the impedance of a resistor, capacitor and inductor is given by R, $1/(j\omega C)$ and $j\omega L$, respectively. In AC circuits, you combine complex impedances in series and in parallel, and follow the same rules in the analysis as for the DC circuits. For example, the total impedance Z of two complex impedances Z_1 and Z_2 connected in parallel is given by

$$\frac{1}{Z} = \frac{1}{Z_1} + \frac{1}{Z_2}. \tag{5}$$

For example, for two capacitances connected in parallel we obtain $1/Z = j\omega C_1 + j\omega C_2 = j\omega(C_1 + C_2)$, that is it is equivalent to a capacitor $C = C_1 + C_2$. And so on.

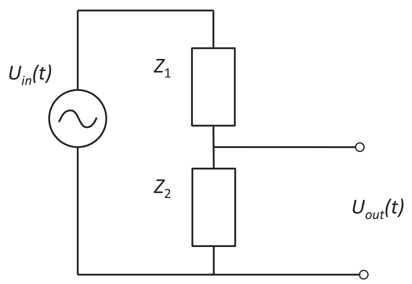

FIG. 2: Generalized voltage divider.

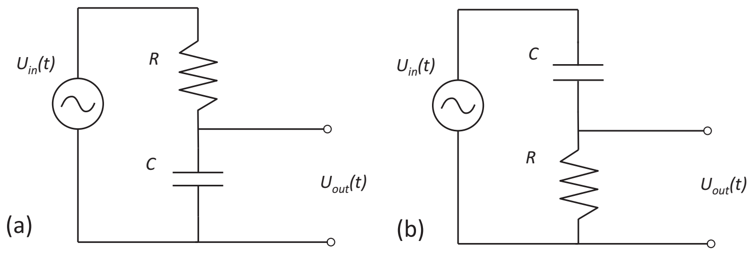

FIG. 3: Low-pass (a) and high-pass (b) filters.

B. Generalized voltage divider. Low-pass and high-pass filters.

Let us consider a simple but very important circuit, which is the voltage divider consisting of two impedances Z_1 and Z_2 connected in series, see Fig. 2. Again, you can analyze it completely the same way as in the case of a DC voltage divider consisting of resistances. Thus, the output voltage signal u_{out} is related to the input voltage signal $u_{in} = u_0 e^{j\omega t}$ (e.g. from a signal generator) as

$$u_{out} = \frac{Z_2}{Z_1 + Z_2} u_{in}. \tag{6}$$

For example, if the two impedances in Fig. 2 are just two resistances R_1 and R_2, the output voltage signal has an amplitude $R_2 u_0/(R_1 + R_2)$, which is independent of the frequency ω of the input signal. So, the circuit acts as a simple voltage divider. Similarly, you can use two capacitors, or two inductors, to build a simple voltage divider.

The situation is more interesting when the two elements in Fig. 2 are not the same type. For example, consider a circuit in Fig. 3a. It is straightforward to find that

$$u_{out} = \frac{u_{in}}{1 + j\omega CR} = \frac{u_0 e^{j(\omega t + \phi)}}{\sqrt{1 + \omega^2 C^2 R^2}}. \tag{7}$$

The amplitude and phase (with respect to the input signal) of the output signal are shown schematically in Fig. 4a. Note that the amplitude of the output signal is significantly decreased below the "cut-off" frequency given by $\omega_c = 1/RC$. Thus this circuit acts as a *low-pass filter*, that is it transmits an unattenuated signal only at sufficiently low frequency, and attenuates signal at frequency higher than the cut-off frequency.

Similarly, consider a circuit in Fig. 3b. It is straightforward to find that

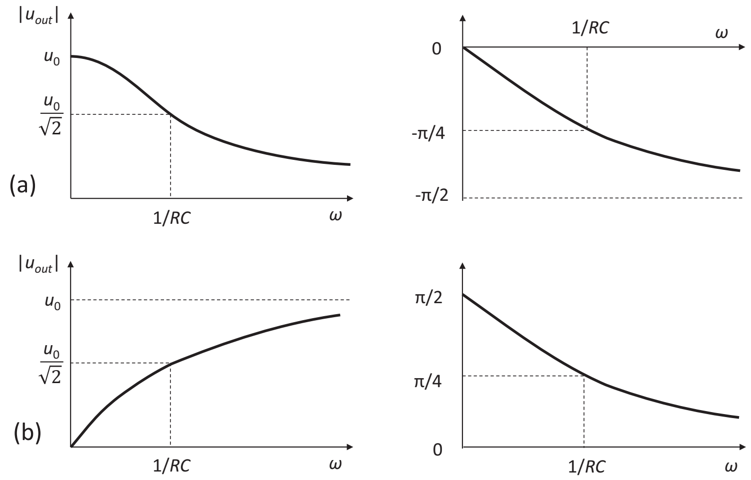

FIG. 4: Amplitude and phase characteristics of the low-pass (a) and high-pass (b) filters.

$$u_{out} = \frac{j\omega CRu_{in}}{1 + j\omega CR} = \frac{\omega CRu_0 e^{j(\omega t+\phi)}}{\sqrt{1 + \omega^2 C^2 R^2}}. \tag{8}$$

The amplitude and phase (with respect to the input signal) of the output voltage signal are shown schematically in Fig. 4b. Obviously, the circuit acts as a *high-pass filter*, that is it transmits unattenuated signal only at frequencies significantly higher than the cut-off frequency ω_c.

C. Exercise

Consider two circuits comprised of resistance R and inductance L, see Fig. 5. Which one forms a low-pass filter, and which one forms a high-pass filter? Draw schematically the amplitude and phase characteristics of these filters.

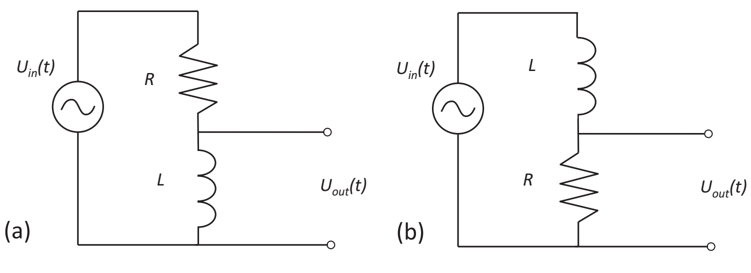

FIG. 5: Circuits for exercise.

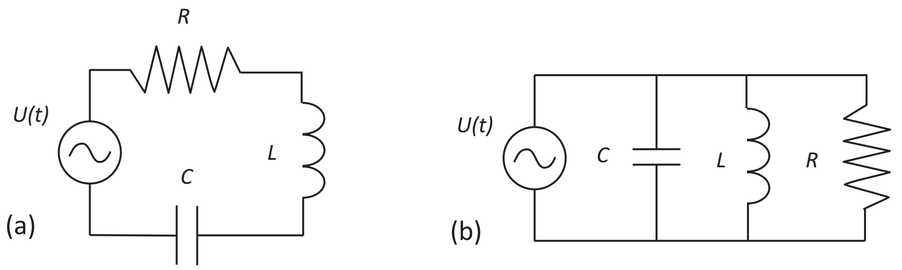

FIG. 6: Series and parallel resonant circuits.

D. Resonant Circuits

Another simple but important circuit is a resonant circuit comprised of an inductor, capacitor and resistor. The *series resonant circuit* contains these elements connected in series, while the *parallel resonant circuit* connects them in parallel, see Fig. 6. By definition, the *resonance frequency* ω_0 of the resonance circuit is the frequency at which the total impedance of the circuit is real. For example, for the series resonant circuit, see Fig. 6a, the total impedance is

$$Z = R + j\left(\omega L - \frac{1}{\omega C}\right). \tag{9}$$

Thus, the resonant frequency of this circuit is $\omega_0 = 1/\sqrt{LC}$.

Exercise

Show that the resonant frequency of the parallel resonant circuit, see Fig. 6b, is also given by $\omega_0 = 1/\sqrt{LC}$.

By definition, the *quality factor* Q of the series resonant circuit is the ratio of the voltage amplitude across the inductor (or the capacitor) to the voltage amplitude across the resistor when the frequency of the driving signal ω is equal to the resonant frequency. Because at $\omega = \omega_0$ the total impedance of the circuit is equal to R, the current i in the circuit is $i = u/R$, and the voltage across the resistor is $u_R = u$. Thus, the voltage across the inductor is $u_L = j\omega_0 Li$, and the quality factor Q is

$$Q = \frac{|u_L|}{|u_R|} = \frac{\omega_0 L}{R} = \frac{1}{R}\sqrt{\frac{L}{C}}. \tag{10}$$

From the above equation, we can give an alternative definition of the quality factor. Let's multiply both the numerator an denominator by the square of the current amplitude, i_0^2, then

$$Q = \frac{\omega_0 L i_0^2}{R i_0^2} = 2\pi \frac{L i_0^2/2}{T(R i_0^2/2)}. \tag{11}$$

Thus, the quality factor is equal to 2π times the ratio of average energy stored in the inductor to the Joule heat dissipated in the resistor during one period of oscillations $T = 1/f$.

Similarly, we can define the quality factor Q of the parallel resonant circuit, see Fig. 6b, as the ratio of the current amplitude through the inductor (or capacitor) to the current amplitude through the resistor when the frequency of the driving signal ω is equal to the resonant frequency.

Exercise

Show that the quality factor of the parallel resonant circuit, see Fig. 6b, is given by

$$Q = \frac{R}{\omega_0 L} = R\sqrt{\frac{C}{L}} \tag{12}$$

Also, show that it is given by the 2π times the ratio of average energy stored in the inductor to the Joule heat dissipated in the resistor during one period of oscillations T.

Note that the definition of the quality factor in terms of the ratio between the stored and dissipated energies coincides for both series and parallel resonant circuits, while it is different when expressed through the values of R, L and C.

Like other elements of AC circuits, both series and parallel resonance circuits can be used in a voltage divider to make very useful circuits.

E. Exercise

Show that the two circuits given in Fig. 7 form an ideal band-pass (a) and notch (b) filters centered at the resonance frequency ω_0. Draw schematically the amplitude characteristics

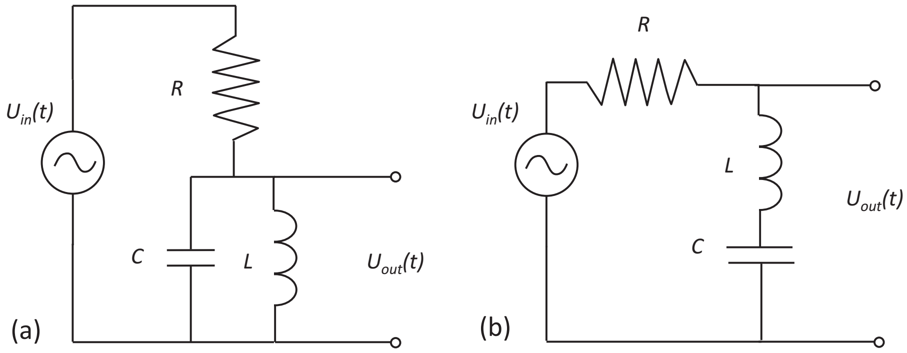

FIG. 7: Band-pass and notch filters.

for such filters.