I. NOISE IN ELECTRICAL CIRCUITS

A. Origin of noise

Every real electrical circuit generates electrical noise in addition to a signal that we want to measure. For example, consider a simple circuit shown in Fig. 1a. Here, we want to measure the voltage $u(t)$ across the resistor R_2. Let's assume that the signal generator produces an ideal harmonic output $u_0 \cos \omega t$, then we expect to observe the voltage signal across the resistor $u(t) = (u_0 R_2/(R_1 + R_2)) \cos \omega t$. However, in reality there is always an additional voltage across the resistor, $u_n(t)$, that we call *noise*, which adds to the signal that we want to measure. So, the total signal that we measure is actually $u(t) + u_n(t)$, see Fig. 1b. When the noise in your circuit is too large, it can significantly affect your measurements of $u(t)$. Therefore, understanding the origin of noise and different ways to reduce it in your measurements is a very important skill for any experimentalists.

The noise can be either the *random noise* or *coherent interference*. The random noise is typically intrinsic, that is it is associated with different physical phenomena occurring within your instruments, e.g. thermal motion of charge carriers in your resistors, diffusion of atoms in semiconductors that affects the device resistivity etc. Because it is intrinsic in nature, we can not get read of the random noise. However, due to random nature of this noise, we can significantly reduce its effect on the result of our measurement. This can be done by the *signal averaging*, which is also related to the *bandwidth narrowing* of your measuring instrument, as will be discussed below.

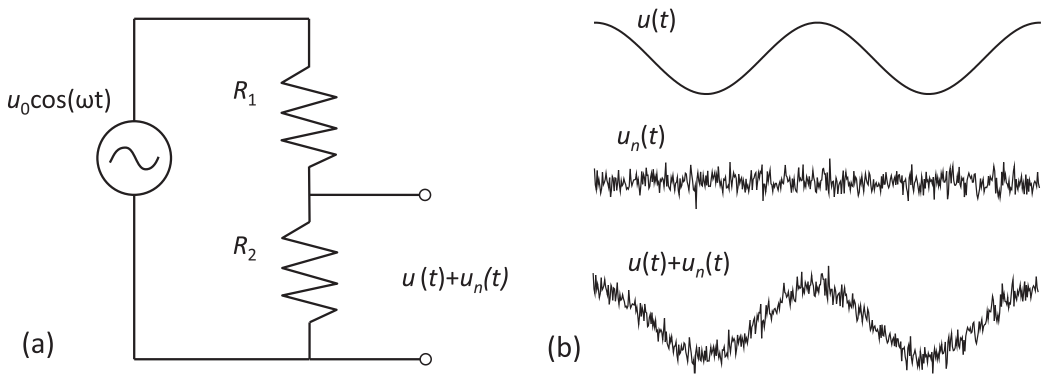

FIG. 1: Noisy electrical signal.

Unlike the random noise, the coherent interference is typically extrinsic to your measuring devices and generally arises from periodic, ”man-made” disturbances, such as electromagnetic field produced by power line and radio/TV broadcast antennas, current fluctuations in the power line due to big running motors etc. The interference noise can be significantly reduced by proper shielding and grounding of your electrical circuits, the use of isolating transformers etc. This is largely determined by your specific experimental setup, and finding the proper ways to reduce the interference noise in your experiment is almost an art. Another way to reduce effect of the interference noise is to chose the working frequency of your experiment far from the noise frequency, and use filtering to suppress unwanted noise signal.

B. Random noise

The random noise occurring in your circuit must be described by statistical methods. For example, we can describe the random noise in quantity x by the probability density function $\rho(x)$, which gives us the probability for the instantaneous value of x. For example, in the circuit in Fig. 1a, the probability that at time t the value of noise voltage across the resistor R_2 lies within the interval from u_n to $u_n + du_n$ is given by $\rho(u_n)du_n$. Most often the probability density function for random noise is given by the normal distribution, see Fig. 2

$$\rho(u_n) = \frac{1}{\sqrt{2\pi}u_\sigma} e^{-u_n^2/2u_\sigma^2}, \tag{1}$$

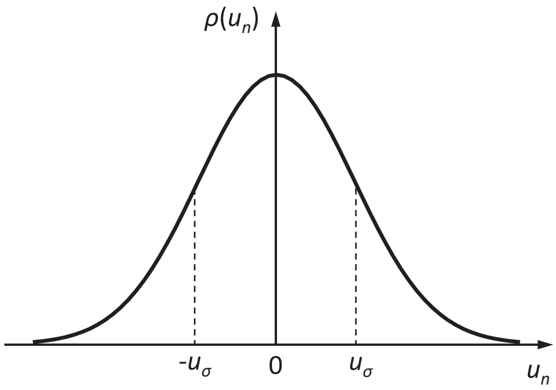

FIG. 2: Normal distribution.

so the noise is characterized by zero mean values, $\langle u_n \rangle = 0$, and the mean square value determined by the dispersion, u_σ^2

$$\langle u_n^2 \rangle = u_\sigma^2 + \langle u_n \rangle^2 = u_\sigma^2. \tag{2}$$

Obviously, when the standard deviation (SD) u_σ of the noise signal is comparable to or larger than the amplitude of the voltage signal that you want to measure (that is most of the time the noise voltage u_n is comparable to the measured signal u), the noise presents a big problem for your measurements, see Fig. 3a. Fortunately, you can reduce SD of noise signal by repeating measurements several times and calculating the average values of your measurements. Let's show that in this case SD of the average value of the noise voltage will decrease as $1/\sqrt{N}$, where N is the number of repeated measurements. Strictly speaking, we need to consider an ensemble of N *identical* and *independent* setups and measure our total signal at time t in each setup. Because the setups are identical, the deterministic signal $u(t)$ is the same in each setup, so its average value is equal to $u(t)$. However, because the setups are independent and the noise is random, the noise signal will be different in each setup, see Fig. 3b. Let's consider the average value u_n^{aver} of noise signal measurements u_n^i $(i = 1, 2, .. , N)$ all done at a certain moment of time t, see Fig. 3b. It is given by

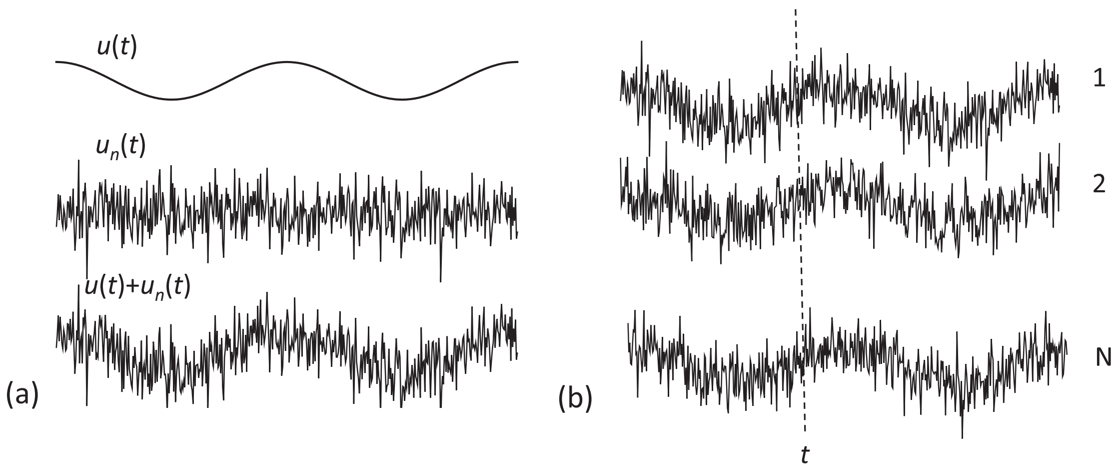

FIG. 3: In (a) the standard deviation of noise signal $u_n(t)$ is equal to the amplitude of signal $u(t)$. In (b) N measurements are done on N identical and independent setups. Note that in each measurement the deterministic signal $u(t)$ will be the same, but the random noise signal $u_n(t)$ will be different and distributed according to the probability density function $\rho(u_n)$.

$$u_n^{aver} = \frac{1}{N}\sum_{i=1}^{N} u_n^i. \tag{3}$$

Averaging over the probability distribution, we calculate its dispersion as

$$\begin{aligned}(u_n^{aver})^2_\sigma &= \langle (u_n^{aver})^2\rangle - \langle (u_n^{aver})\rangle^2 = \frac{1}{N^2}\langle \left(\sum_i\sum_j u_n^i u_n^j\right)\rangle - \frac{1}{N^2}\left(\sum_i \langle u_n^i\rangle\right)^2 = \\ &= \frac{1}{N^2}\langle\left(\sum_i \left(u_n^i\right)^2 + \sum_{i\neq j} u_n^i u_n^j\right)\rangle = \frac{1}{N^2}\left(\sum_i \langle \left(u_n^i\right)^2\rangle + \sum_{i\neq j}\langle u_n^i\rangle\langle u_n^j\rangle\right) = \\ &= \frac{1}{N^2}\sum_i \langle\left(u_n^i\right)^2\rangle = \frac{u_\sigma^2}{N}. \end{aligned} \tag{4}$$

Note that above we used the fact that all setups in the ensemble are independent. Thus, SD of the averaged value of the noise signal is $\left(u_n^{aver}\right)_\sigma = u_\sigma/\sqrt{N}$ and goes to zero as $N \to \infty$.

This procedure of reducing SD of a random noise signal is called the *averaging*. Of cause, in reality you work with a single setup, not an ensemble of identical and independent setups (which most often is impossible to do anyway!). However, most often the averaging of instantaneous measurements done in an ensemble of identical and independent systems is equivalent to averaging of measurements done in the same system over time. This important property of statistical ensembles is called *ergodicity*, and the systems that obey this property are called *ergodic*. Thus in the case of *stationary* and ergodic random noise, which is true for most of the real-world situations, we can average the signals obtained in the same setup by repeating the measurements over periods of time during which the signal of interest repeats itself. It is clear that the noise signals at different periods of time must be *uncorrelated*, which is equivalent to having *independent* setups in the ensemble averaging.

C. Auto-correlation function and power spectral density of random noise

So far we mostly considered characteristics of a random noise signal at a certain time t. In particular, the mean value of u_n^2 at any moment of time is given by the dispersion of the probability density function, that is $\langle u_n^2\rangle = u_\sigma$. The last sentence of the previous paragraph brings us to the question: What is relation between noise signals at different moments of time. Let us define the *auto-correlation* function of an ergodic noise signal $u_n(t)$ as

$$\psi(\tau) = \langle u_n(t)u_n(t+\tau)\rangle = \lim_{T\to\infty}\frac{1}{T}\int\limits_{-T/2}^{T/2} u(t)u(t+\tau)dt. \tag{5}$$

Note that at $\tau = 0$ the auto-correlation function coincides with the dispersion. Let us relate the auto-correlation function to the power characteristics of the noise signal. Note that $\langle u_n^2\rangle$ can be considered as a mean value of power dissipated by the noise signal in a 1 Ω resistor. Let us use the Fourier transform of $u(t)$ and write Eq. (5) as

$$\psi(\tau) = \lim_{T\to\infty}\frac{1}{T}\int\limits_{-T/2}^{T/2} u_n(t)\left(\int\limits_{-\infty}^{+\infty} u_T(\omega)e^{j\omega(t+\tau)}d\omega\right)dt =$$
$$= \lim_{T\to\infty}\frac{1}{T}\int\limits_{-\infty}^{\infty} u_T(\omega)\left(\int\limits_{-T/2}^{+T/2} u(t)e^{j\omega t}dt\right)e^{j\omega\tau}d\omega = \lim_{T\to\infty}\frac{1}{T}\int\limits_{-\infty}^{\infty} u_T(\omega)u_T(-\omega)e^{j\omega\tau}d\omega =$$
$$= \int\limits_{-\infty}^{\infty}\left(\lim_{T\to\infty}\frac{1}{T}|u_T(\omega)|^2\right)e^{j\omega\tau}d\omega, \tag{6}$$

where $u_T(\omega)$ is the Fourier transform of the signal lasting from $-T/2$ to $T/2$. The quantity

$$S(\omega) = \lim_{T\to\infty}\frac{1}{T}|u_T|^2, \tag{7}$$

can be considered as the mean power dissipated in a 1 Ω resistor by the spectral components of the noise signal from the spectrum interval of width $2\pi\times 1$ Hz centered at ω. This quantity is called the *power spectral density*. Thus we see that

$$S(\omega) = \int\limits_{-\infty}^{\infty} \psi(\tau)e^{-j\omega\tau}d\tau,$$
$$\psi(t) = \int\limits_{-\infty}^{\infty} S(\omega)e^{j\omega\tau}d\omega. \tag{8}$$

Thus the auto-correlation function and power spectral density are related to each other by the Fourier transform. This important relation is called the *Wiener-Khinchin theorem*.

D. White noise and measurement bandwidth

The important approximation often used to describe real noise signals is the so called *white noise*, that is the noise that has a constant power spectral density over all frequencies,

$$S(\omega) = S_0/2, \quad -\infty < \omega < \infty \tag{9}$$

The factor of 1/2 comes from the convention to use both positive and negative values of frequencies, which is traced back to our convention to use complex numbers to describe real harmonic signals. From Eq. (8) we can immediately find the auto-correlation function of the white noise

$$\psi(\tau) = \frac{S_0}{2} \int_{-\infty}^{\infty} e^{j\omega t} d\omega = \frac{S_0}{2} \delta(\tau). \tag{10}$$

Thus, the white noise is completely uncorrelated for any, no matter how close, moments of time. Also, at $\tau = 0$ the auto-correlation function is infinite. That is the mean power of white noise diverges, which can also be seen from the integral of $S(\omega)$ over the whole frequency range from $-\infty$ to ∞. Obviously, no real source of noise can have this property. Fortunately, no real signals have infinitely large frequencies. In reality, any electrical circuit or instrument will have a finite band of frequencies that it can pass through, that is a finite *bandwidth*. Let us see what happens to the dispersion of the white noise if we pass it through an ideal low-pass filter that transmits completely unattenuated signals at frequencies $|\omega| < \omega_c$, and completely attenuates signals at $|\omega| > \omega_c$, where ω_c is the filter cut-off frequency, see Fig. 4. In order to measure the noise signal dispersion, let's assume that we have a broadband voltmeter that measures the root-mean-square (RMS) of the input signal. Let's find the auto-correlation function of the noise signal at the output of the low-pass filter (that is at the input of the voltmeter)

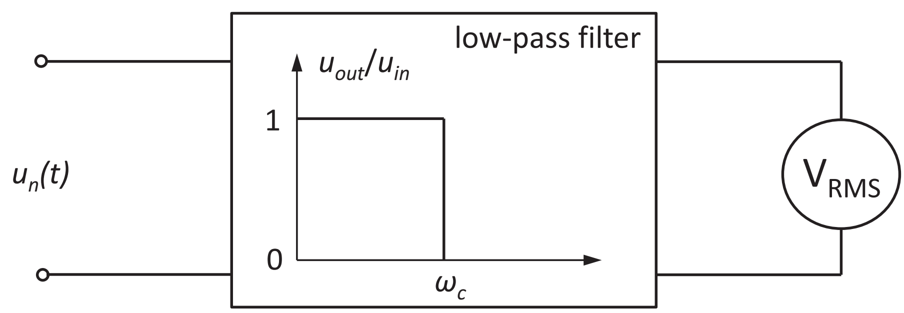

FIG. 4: Measurement of the real spectral power density of a white noise signal $u_n(t)$.

$$\psi(\tau) = \frac{S_0}{2}\int_{-\omega_c}^{\omega_c} e^{j\omega\tau} d\omega_c = S_0\omega_c \frac{\sin(\omega_c\tau)}{\omega_c\tau}. \tag{11}$$

Thus the dispersion of the white noise signal at the low-pass filter output is given by $u_\sigma^2 = \psi(0) = S_0\omega_c$, where we used

$$\lim_{x\to 0} \frac{\sin x}{x} = 1. \tag{12}$$

Thus, the RMS signal measured by the voltmeter is $\sqrt{\langle u_n^2\rangle} = u_\sigma = \sqrt{S_0\omega_c}$.

More generally, it is easy to show that if the white noise is passed through a real filter of effective bandwidth B, the dispersion of the noise signal at the filter output is given by

$$u_\sigma^2 = \langle u_n^2\rangle = S_0 B. \tag{13}$$

This is a simple but very important equation that described the white noise in the real experimental setups. Before we consider two most important sources of white noise in electrical circuits, the so called *Johnson noise* and *shot noise*, let us look at the auto-correlation function of the white noise after it is passed through the low-pass filter, see Eq. (11). It is easy to see that the auto-correlation function differs significantly from zero only in the narrow interval of τ from $-\tau_c/2$ to $\tau_c/2$, where $\tau_c \approx 2\pi/\omega_c$. Thus, the noise can be considered uncorrelated at any two moments of time separated by more than τ_c. This time is sometimes called the *correlation time*, and as we see it is simply related to the cut-off frequency of the filter. More generally, the correlation time is given approximately by the inverse of the bandwidth (in Hz) of your experimental setup, $\tau_c \approx B^{-1}$.

E. Johnson noise

In any resistor, the thermal motion of charge carriers produces fluctuations in the voltage across the resistor, which is known as the Johnson noise. Using the *equipartition theorem* from statistical mechanics, it can be shown that the dispersion of the Johnson noise is given by (Nyquist, 1928)

$$u_\sigma^2 = \langle u^2\rangle = 4Rk_BTB, \tag{14}$$

where R is the resistance in Ω, T is the temperature of the resistor in K, $k_B = 1.39 \times 10^{-23}$ J/K, and B is the bandwidth in Hz. For derivation of the above equation, please see a very beautiful in its simplicity paper by Nyquist [1].

Alternatively, we can write it as the current noise with the dispersion

$$i_\sigma^2 = \langle i^2 \rangle = \langle \left(\frac{u}{R} \right)^2 \rangle = \frac{\langle u^2 \rangle}{R^2} = \frac{4k_B T}{R} B. \tag{15}$$

Let us see how to combine the Johnson noise from several resistors connected in series or parallel. For series connection we add the squares of the voltages (that is the powers generated by the Johnson noise in each resistor) because the fluctuations of voltages across each resistor are completely uncorrelated. Thus

$$u_\sigma^2 = 4(R_1 + R_2 + ..)k_B T B. \tag{16}$$

Similarly, for parallel connection we add together the squares of the currents, thus

$$i_\sigma^2 = 4\left(\frac{1}{R_1} + \frac{1}{R_2} + .. \right) k_B T B. \tag{17}$$

Obviously, you can rewrite the last equation in terms of voltage noise as

$$u_\sigma^2 = \left(\frac{R_1 R_2 ...}{R_1 + R_2 + ...} \right)^2 4\left(\frac{1}{R_1} + \frac{1}{R_2} + .. \right) k_B T B = 4\left(\frac{R_1 R_2 ...}{R_1 + R_2 + ...} \right) k_B T B. \tag{18}$$

F. Shot noise

Due to discreteness of charge carries the amount of charge flowing through the cross section per unit time, that is the electrical current, fluctuates with time. This fluctuations are called the shot noise, and its dispersion is given by (Schottky, 1918)

$$i_\sigma^2 = \langle i^2 \rangle = 2qIB, \tag{19}$$

where q is the charge of carries in C (e.g. $e = 1.6 \times 10^{-19}$ C for electrons), and I is the current in A. The shot noise can appear as the noise in voltage across a resistor R when the current I is passed through it. In this case, we can rewrite the above equation as

$$u_\sigma^2 = \langle (iR)^2 \rangle = 2qIR^2 B. \tag{20}$$

For those who are interested in details, the derivation of the Eq. (19) is given in Supplemental Materials at the end of these notes.

G. Flicker or $1/f$ noise

In addition to the white noise, such as the Johnston and shot noise, electrical circuits are often affected by random noise that have a nonuniform power spectral density $S(\omega)$. One of the most well-known example is the so called $1/f$ (or flicker) noise that have spectral dependence $S(\omega) \propto 1/\omega$. It often occurs due to internal processes in the electronic devices, such as deterioration of the materials, diffusion of atoms and reorganization of atomic structure at the surfaces, etc. For example, fluctuations in resistance often cause the voltage noise across a resistor when the current I is passed though it, and the dispersion of this noise is given by

$$u_\sigma^2 = \frac{AI^2}{f} B, \tag{21}$$

where A is a constant that depends on the resistor material. This noise adds to the Johnston and shot noise.

II. SUPPLEMENTAL MATERIALS

A. Derivation of the expression for short noise dispersion

Let's observe current through the cross section of a conductor (e.g. a piece of wire) during time interval from $t = 0$ to T. Because of the discreteness of charge, the actual current will look like a series of spikes happening during times t_k, $k = 1, .. , N$, which are randomly distributed across the interval, see Fig. 5. Note that the current $i_k(t)$ due to individual charge q can be well approximated by a δ-function, $i_k(t) = q\delta(t - t_k)$. Indeed, the time integral will give q, that is total charge passed through the cross section. Thus, the total current i observed during the time interval $[0, T]$ is given by the sum of all i_k, $i = \sum_k I_k(t)$. This quantity has to be averaged over an ensemble of identical systems, or, equivalently (see discussion at the end of Section I(B)), over many time intervals $[t, t + T]$.

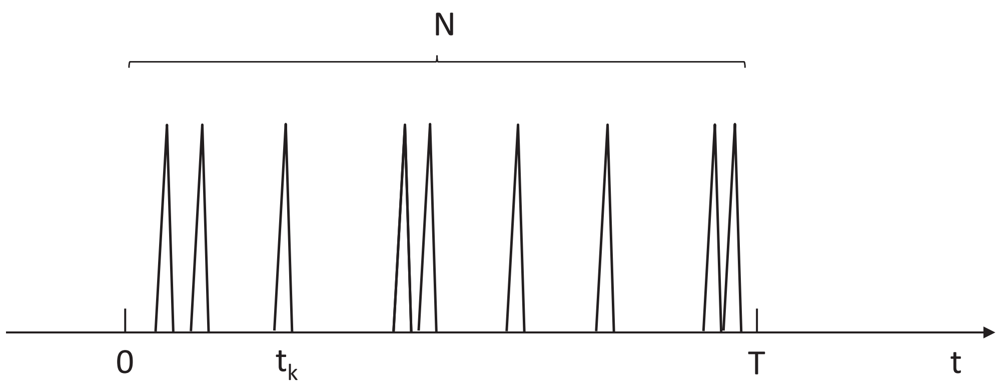

FIG. 5: Current through a conductor during time interval $[0, T]$ due to individual charges passing the conductor's cross section at tomes t_k, $k = 1, .. ,$N.

In order to work out the ensemble average, it is convenient to use the Fourier transform of $i_k(t)$. Using the property of the δ-function, we can immediately write

$$i_k(t) = \frac{q}{T} + \frac{2q}{T}\sum_{n=1}^{\infty} \cos\left(\frac{2\pi n(t - t_k)}{T}\right), \tag{22}$$

and, therefore, for the total current

$$i = \sum_{k=1}^{N} i_k = \frac{qN}{T} + \frac{2q}{T}\sum_{n=1}^{\infty}\sum_{k=1}^{N} \cos\left(\frac{2\pi nt}{T} - \phi_{nk}\right), \tag{23}$$

where we introduced the phase $\phi_{nk} = 2\pi n t_k/T$. Note that the first term in the above equation gives just the average current $I = qN/T$. The second term gives the fluctuating term that we are interested with.

Next, let us take the square of Eq. (23) and average over ensemble. Here, we can use a well-known theorem that the mean square of the sum of all Fourier components is equal to the sum of the mean squares of all the individual components, that is

$$\langle i^2 \rangle = I^2 + \left(\frac{2q}{T}\right)^2 \sum_{n=1}^{\infty} \langle \left(\sum_{k=1}^{N} \cos\left(\frac{2\pi nt}{T} - \phi_{nk}\right)\right)^2 \rangle. \tag{24}$$

Note that in the above equation the phase ϕ_k is distributed randomly in the interval $[0, 2\pi]$. It is well known that the cross terms in square of $\sum_k$ will average to zero, while the average of each term of the form $\cos^2(2\pi f_n t - \phi_{nk})$ will give $1/2$. Thus, we obtain

$$\langle i^2 \rangle = I^2 + \left(\frac{2q}{T}\right)^2 \sum_{n=1}^{\infty} \frac{N}{2}. \tag{25}$$

Note that the sum in the above equation diverges. This is because we included all possible frequencies $f_n = n/T$, $n = 1, .. , \infty$. The real systems have a finite bandwidth B, see discussion in the Section 1(D). Therefore, we have to terminate the sum at $n_{\text{max}} = T \times B$. Thus, we obtain

$$\langle i_\sigma^2 \rangle = \langle i^2 \rangle - I^2 = \left(\frac{2q}{T}\right)^2 \frac{N}{2} TB = 2qIB. \tag{26}$$

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[1] H. Nyquist, Phys. Rev. **32**, 110 (1928).