Bonus lectures

By majority vote, apart from some sections that we did not discuss in detail in lecture 4 (section 7 and appendix C), the topic in the bonus lectures is generalized near-horizon symmetries in 4d. Physically, the motivation is to allow gravitational waves to interact with a black hole and determine how this affects the near-horizon symmetry analysis we did in lecture 6.

1 Kerr geometry and near-bifurcation expansion

For once, we are interested in black holes that actually exist, i.e., Kerr black holes. The Kerr metric in Boyer–Lindquist coordinates

$$\mathrm{d}s^2 = -\frac{\Delta}{\rho^2}\big(\,\mathrm{d}t - a\sin^2\theta\,\mathrm{d}\varphi\big)^2 + \frac{\rho^2}{\Delta}\,\mathrm{d}r^2 + \rho^2\,\mathrm{d}\theta^2 + \frac{(r^2+a^2)^2\sin^2\theta}{\rho^2}\big(\,\mathrm{d}\varphi - \frac{a}{r^2+a^2}\,\mathrm{d}t\big)^2$$

with

$$\Delta = r^2 - 2Mr + a^2 \qquad\qquad \rho^2 = r^2 + a^2\cos^2\theta \tag{1}$$

depends parametrically on mass M and angular momentum $J = Ma$. The angular coordinates (also called "angular part") are the usual polar angle $\theta \in [0, \pi]$ and the azimuthal angle $\varphi \sim \varphi + 2\pi$. The time and radial coordinates (also referred to as "spacetime part") are non-compact, $t \in (-\infty,\,\infty)$ and $r \in [0,\,\infty)$.

The inner and outer horizon radii $r_\pm$ are respectively the smaller and bigger roots of $\Delta = 0$,

$$r_\pm = M \pm \sqrt{M^2 - a^2}\,. \tag{2}$$

The loci $r = r_\pm$ are bifurcate Killing horizons with a bifurcation 2-sphere $\mathcal{B}$. The outer one is simultaneously a black hole event horizon and will be our main region of interest. Surface gravity at the black hole horizon is given by

$$\kappa = \frac{r_+ - r_-}{2(r_+ + r_-)r_+}\,. \tag{3}$$

Since Boyer–Lindquist coordinates are not well-adapted to near-horizon expansions we introduce new coordinates below, following closely 2002.08346.

1.1 Near horizon coordinates

For near-horizon analyses various coordinate choices have been used in the literature, e.g., Eddington–Finkelstein (EF) types of coordinates, conformal types of coordinates, and Rindler coordinates. Of course, there is also the venerable set of Kruskal coordinates or simplifications thereof. Each of these has its own merits and drawbacks. For our purpose, the most striking difference between various choices is the way the horizon is approached and whether or not the bifurcation 2-sphere can be covered. We found none of these coordinate systems suitable for our purposes.

Thus, we introduce a Kruskal–Israel-inspired coordinate system well-adapted to study not only the black hole horizon but specifically the region near the bifurcation 2-sphere. The main features of our coordinates are

- co-rotation with the horizon (as opposed to Kruskal- or Israel-coordinates)
- no mixing of angular and spacetime coordinates (as opposed to HHPS)
- cover open region around the bifurcation 2-sphere (as opposed to EF or Rindler)

We describe now explicitly the new coordinates, starting with the angular part.

Since we are interested in an expansion around the outer horizon, $r \sim r_+$, it is useful to transform the azimuthal angle so that our coordinate frame is co-rotating with the outer horizon

$$\phi = \varphi - \Omega_{\text{H}}\, t \qquad\qquad \Omega_{\text{H}} = \frac{a}{r_+^2 + a^2}\,. \tag{4}$$

As we are content with the polar angle θ we do not transform it.

We address now the spacetime part. We introduce Kruskal–Israel-like coordinates that we denote by $x^\pm$, defined by

$$x^+ = \pm\sqrt{\left|\frac{r-r_+}{r_+-r_-}\right|}\, e^{\kappa t} \qquad\qquad x^- = \pm\sqrt{\left|\frac{r-r_+}{r_+-r_-}\right|}\, e^{-\kappa t} \tag{5}$$

the inverse of which is

$$t = \frac{1}{2\kappa}\,\ln\left|\frac{x^+}{x^-}\right| \qquad\qquad r = r_+ - (r_+ - r_-)\, x^+ x^- \tag{6}$$

and hence

$$\Delta = (r_+ - r_-)^2 x^+ x^- (x^+ x^- - 1) \qquad\quad \rho^2 = (r_+ - (r_+ - r_-)x^+x^-)^2 + r_+ r_- \cos^2\theta\,. \tag{7}$$

In this coordinate system $x^\pm \in \mathbb{R}$. The signs in (5) are fixed as follows. As in the usual Kruskal coordinates, the outside region corresponds to $x^+ > 0, x^- < 0$, the black hole region to $x^\pm > 0$, the white hole region to $x^\pm < 0$, and the second outside region to $x^+ < 0, x^- > 0$. The locus $x^+x^- = 0$ describes the bifurcate black hole horizon and $x^+ = x^- = 0$ the bifurcation 2-sphere $\mathcal{B}$. The inner (Cauchy) horizon corresponds to $x^+x^- = 1$ and our coordinates break down on it. Thus, the only restriction on the range of the coordinates $x^\pm$ is that their product is smaller than unity, $x^+x^- < 1$.

In these new coordinates, the Kerr black hole is described by metrics of the form

$$\begin{aligned} \mathrm{d}s^2 = \rho^2\,\mathrm{d}\theta^2 - \rho^2\,\frac{(x^-\,\mathrm{d}x^+ + x^+\,\mathrm{d}x^-)^2}{x^+x^-(1-x^+x^-)} + \frac{\sin^2\theta}{\rho^2}\Big(a\big(x^+\,\mathrm{d}x^- - x^-\,\mathrm{d}x^+\big)\big(2r_+ - (r_+ - r_-)x^+x^-\big) \\ + \big(r_+^2(1-x^+x^-)^2 + r_-^2(x^+x^-)^2 + r_+r_-(1 + 2x^+x^-(1-x^+x^-))\big)\,\mathrm{d}\phi\Big)^2 \\ + \frac{1-x^+x^-}{\rho^2\,x^+x^-}\Big(\rho_+^2\,\big(x^-\,\mathrm{d}x^+ - x^+\,\mathrm{d}x^-\big) - a\sin^2\theta\,(r_+ - r_-)\,x^+x^-\,\mathrm{d}\phi\Big)^2 \end{aligned} \tag{8}$$

with $a = \sqrt{r_+r_-}$ and $\rho_+^2 = r_+(r_+ + r_-\cos^2\theta)$. In the extremal case, $r_+ = r_- > 0$, the metric (8) simplifies to the near-horizon extremal Kerr (NHEK) geometry

$$\begin{aligned} \mathrm{d}s^2_{\text{NHEK}} = r_+^2\,(1+\cos^2\theta)\Big(-\frac{(x^-\,\mathrm{d}x^+ + x^+\,\mathrm{d}x^-)^2}{x^+x^-(1-x^+x^-)} + \frac{(x^-\,\mathrm{d}x^+ - x^+\,\mathrm{d}x^-)^2(1-x^+x^-)}{x^+x^-} \\ + \mathrm{d}\theta^2 + \frac{4\sin^2\theta}{(1+\cos^2\theta)^2}\,\big(\mathrm{d}\phi - x^-\,\mathrm{d}x^+ + x^+\,\mathrm{d}x^-\big)^2\Big) \end{aligned} \tag{9}$$

The reason why we obtain NHEK rather than extremal Kerr is that the coordinate transformation (6) is singular in the extremal case and zooms into the region $r = r_+$ for any finite values of $x^\pm$. So, our coordinate system captures generic and near-extremal cases. For convenience, we collect the ranges of the coordinates: $x^\pm \in \mathbb{R}$, $x^+x^- < 1$, $\theta \in [0, \pi]$ and $\phi \sim \phi + 2\pi$.

In our new coordinates (8), the Kerr Killing vectors read

$$\zeta_{\text{H}} = \kappa\,\big(x^+\partial_+ - x^-\partial_-\big) \qquad\qquad \zeta_\phi = \partial_\phi\,. \tag{10}$$

Consistently, ζ_{H} is null at the bifurcate horizon $x^+x^- = 0$ and vanishes at the bifurcation 2-sphere $\mathcal{B}$.

1.2 Fall-off conditions

To motivate our first attempt at 4d fall-off conditions, we expand the Kerr metric (8) near the bifurcation surface $\mathcal{B}$ at $x^\pm = 0$,

$$\mathrm{d}s^2 = -4\rho_+^2\, \mathrm{d}x^+\, \mathrm{d}x^- - 8Ma \left(\frac{r_+^2}{\rho_+^2} + \kappa r_+\right) \sin^2\theta \left(x^-\, \mathrm{d}x^+ - x^+\, \mathrm{d}x^-\right) \mathrm{d}\phi + \cdots \tag{11}$$

where the ellipsis denotes the induced metric on $\mathcal{B}$ and higher order terms in $x^\pm$. In this expansion, we assume $x^\pm$ to be small and of the same order.

Suggested by the expansion (11), we postulate near-bifurcation fall-off conditions

$$g_{\pm\pm} = \mathcal{O}(x^2) \qquad\qquad g_{\pm A} = x^\mp C_A^\pm(x^B) + \mathcal{O}(x^3) \tag{12a}$$

$$g_{+-} = \eta(x^B) + \mathcal{O}(x^2) \qquad\qquad g_{AB} = \Omega_{AB}(x^C) + \mathcal{O}(x^2) \tag{12b}$$

where $x^A = (\theta, \phi)$ denote the coordinates on $\mathcal{B}$. To avoid clutter we use $\mathcal{O}(x^n)$ as a shorthand for $\mathcal{O}((x^\pm)^n)$. The near-bifurcation expansion functions, $\Omega_{AB}, C_A,\ \eta$, are not constrained by the Einstein field equations to leading order. The fall-off conditions (12) are the most general ones (subject to Taylor-expandability) preserving the bifurcation 2-sphere at $x^\pm = 0$.

1.3 Near bifurcation Killing vectors

The diffeomorphisms that keep the near-bifurcation expansion (12) intact are generated by "near-bifurcation Killing vectors" given by

$$\xi^\pm = \pm x^\pm T^\pm(x^A) + \mathcal{O}(x^3) \qquad\qquad \xi^A = Y^A(x^B) + \mathcal{O}(x^2)\,. \tag{13}$$

Under a transformation generated by near-bifurcation Killing vector fields (13), the leading order metric functions transform as

$$\delta_\xi \eta = Y^A \partial_A \eta + \left(T^+ - T^-\right) \eta \tag{14a}$$

$$\delta_\xi C_A^\pm = Y^B \partial_B C_A^\pm + C_B^\pm \partial_A Y^B + \left(T^+ - T^-\right) C_A^\pm \mp \eta \partial_A T^\mp \tag{14b}$$

$$\delta_\xi \Omega_{AB} = Y^C \partial_C \Omega_{AB} + \Omega_{AC} \partial_B Y^C + \Omega_{BC} \partial_A Y^C \tag{14c}$$

1.4 Near-bifurcation Killing vector algebra

The near-bifurcation Killing vector fields (13) satisfy the Lie bracket algebra

$$\left[\xi(T_1^+, T_1^-, Y_1^A),\, \xi(T_2^+, T_2^-, Y_2^A)\right] = \xi(T_{12}^+, T_{12}^-, Y_{12}^A) \tag{15}$$

where

$$T_{12}^\pm = Y_1^A \partial_A T_2^\pm - Y_2^A \partial_A T_1^\pm \qquad\qquad Y_{12}^A = Y_1^B \partial_B Y_2^A - Y_2^B \partial_B Y_1^A\,. \tag{16}$$

The generators $T^\pm, Y^A$ are functions on $\mathcal{B}$. As (16) shows, Y^A generate general 2d diffeomrphisms on $\mathcal{B}$. The $T^\pm$ transform as scalars under 2d diffeomorphisms while commuting among themselves and with each other. Thus, the $T^\pm$ generate so-called supertranslations.

For simplicity, we impose a further restriction on the metric, namely conformality to the round S^2 of the codimension-two metric Ω_{AB},

$$\Omega_{AB}\, \mathrm{d}x^A\, \mathrm{d}x^B = \Omega\, \gamma_{AB}\, \mathrm{d}x^A\, \mathrm{d}x^B = \Omega\, \frac{4\, \mathrm{d}z\, \mathrm{d}\bar{z}}{(1 + z\,\bar{z})^2}\,. \tag{17}$$

The assumption (17) allows us to work with a single function Ω instead of Ω_{AB}. At the level of diffeos, this amounts to working with a subclass of the Y^A generating Weyl rescalings of $\mathcal{B}$, i.e., we are restricting to superrotations rather than diff(S^2).

In the coordinates $z, \bar{z}$ defined in (17) the generators expand as

$$\mathcal{T}^{\pm}_{n,m} := \pm z^n \bar{z}^m x^{\pm}\partial_{\pm} \qquad \xi^{\pm}\partial_{\pm} = \sum_{n,m\in\mathbb{Z}} \tau^{\pm}_{nm}\mathcal{T}^{\pm}_{n,m} \tag{18a}$$

$$\mathcal{L}_n := -z^{n+1}\partial_z \ , \quad \bar{\mathcal{L}}_n = -\bar{z}^{n+1}\partial_{\bar{z}} \qquad Y^A(x^B)\partial_A = \sum_{n\in\mathbb{Z}} \left(\mathcal{Y}_n\mathcal{L}_n + \bar{\mathcal{Y}}_n\bar{\mathcal{L}}_n\right) \tag{18b}$$

where $\tau^{\pm}_{nm}$, $\mathcal{Y}_n$ and $\bar{\mathcal{Y}}_n$ are arbitrary numbers.

In the basis spanned by $\mathcal{L}_n, \bar{\mathcal{L}}_n, \mathcal{T}^{\pm}_{n,m}$ the algebra (15) takes the form

$$[\mathcal{L}_n, \mathcal{L}_m] = (n-m)\mathcal{L}_{n+m} \qquad [\bar{\mathcal{L}}_n, \bar{\mathcal{L}}_m] = (n-m)\bar{\mathcal{L}}_{n+m} \tag{19a}$$

$$[\mathcal{L}_k, \mathcal{T}^{\pm}_{n,m}] = -n\ \mathcal{T}^{\pm}_{n+k,m} \qquad [\bar{\mathcal{L}}_k, \mathcal{T}^{\pm}_{n,m}] = -m\ \mathcal{T}^{\pm}_{n,m+k} \tag{19b}$$

$$[\mathcal{L}_n, \bar{\mathcal{L}}_m] = 0 \qquad [\mathcal{T}^{\pm}_{n,m}, \mathcal{T}^{\pm}_{k,l}] = 0 = [\mathcal{T}^{+}_{n,m}, \mathcal{T}^{-}_{k,l}]\,. \tag{19c}$$

The algebra (19a) consists of a Witt $\oplus$ Witt algebra (the "superrotation part"), generated by $\mathcal{L}_n, \bar{\mathcal{L}}_m$, and two towers of supertranslations generated by $\mathcal{T}^{\pm}_{n,m}$. This algebra closely resembles the DGGP algebra, but we have two copies of supertranslations instead of one copy there.

1.5 Conserved charges and their algebra

If the charge is integrable, then the fundamental theorem of the covariant phase space method (see lecture 2) states

$$\delta_{\xi_2}Q_{\xi_1} = \{Q_{\xi_1}, Q_{\xi_2}\} = Q_{[\xi_1,\xi_2]} + K(\xi_1,\xi_2) \tag{20}$$

where the bracket is defined by the first equality and $K(\xi_1,\xi_2)$ is a central extension.

In the present case, the metric $g+\delta g$ is given by the near-bifurcation fall-off (12), the symmetry generators by the near-bifurcation Killing vectors (13), and the theory is general relativity. The action-based covariant phase space method yields

$$\not{\delta} Q_\xi = \frac{1}{8\pi G}\oint_{\mathcal{B}} \mathrm{d}^2x_{\mu\nu}\sqrt{-g}\left(h^{\lambda[\mu}\nabla_\lambda\xi^{\nu]} - \xi^\lambda\nabla^{[\mu}h^{\nu]}_\lambda - \frac{1}{2}h\nabla^{[\mu}\xi^{\nu]} + \xi^{[\mu}\nabla_\lambda h^{\nu]\lambda} - \xi^{[\mu}\nabla^{\nu]}h\right)$$

where $h_{\mu\nu} = \delta g_{\mu\nu}$ is a metric variation allowed by the near-bifurcation fall-off (12).

Assuming that the generators $T^{\pm}$, Y^A are field-independent, the charges are integrable and independent from the difference $T^+ - T^-$,

$$Q(T,\,Y^A) \equiv \int_{\mathcal{B}} \mathrm{d}^2x\left(T\,\mathcal{P} + Y^A\,\mathcal{J}_A\right) \tag{21}$$

with $T = (T^+ + T^-)/2$ and the charge densities

$$\mathcal{P} = \frac{\Omega}{8\pi G} \qquad \mathcal{J}_A = -\frac{\Omega}{16\pi G}\frac{C^+_A - C^-_A}{\eta}\,. \tag{22}$$

Since there are fewer charges in (22) than functions parametrizing our phase space, η, Ω, $C^{\pm}_A$, there is a redundancy in our phase space.

Let us now return to the general result for the charges (21) and derive the algebra generated by them, using (20). The transformation laws (14) yield

$$\delta_\xi\mathcal{P} = Y^A\partial_A\mathcal{P} + \mathcal{P}\partial_A Y^A \tag{23a}$$

$$\delta_\xi\mathcal{J}_A = Y^B\partial_B\mathcal{J}_A + \mathcal{J}_B\partial_A Y^B + \mathcal{J}_A\partial_B Y^B + \mathcal{P}\partial_A T \tag{23b}$$

which is the usual transformation behavior of scalar- and vector-densities of weight one under 2d diffeos generated by Y^A. The algebra above coincides with the DGGP algebra, which notably features no central extension.

Thus, the near-bifurcation boundary conditions recover known results (for more details, see 2002.08346). In particular, it is impossible for gravitational waves to be absorbed or emitted by black holes that obey near-bifurcation boundary conditions, so they are not good enough for our purposes. We wasted 4 pages on this statement!

2 Near null boundary expansion

We want to describe physical processes of gravitational wave absorption through some null hypersurface (which initially might coincide with an isolated horizon of a black hole). We follow closely 2110.04218, see also refs. therein.

Let $\mathcal{N}$ be a given smooth codimension-one null hypersurface in a D dimensional spacetime of signature $(-,+,\dots,+)$. In a neighborhood of any such hypersurface, one can adopt Gaussian null-type coordinates that we set up as follows. Let v be the advanced time coordinate along the null hypersurface such that the null surface is defined by

$$g^{\mu\nu}\,\partial_\mu v\,\partial_\nu v = 0\,. \tag{24}$$

We take the null surface $\mathcal{N}$ to be localized at vanishing affine parameter, $r=0$, as depicted in Fig. 1. The null surface $\mathcal{N}$ is assumed to have the topology $\mathbb{R}_v \ltimes \mathcal{N}_v$, where $\mathcal{N}_v$ is the $D-2$ dimensional constant-v subspace on $\mathcal{N}$ spanned by x^A. We refer to $\mathcal{N}_v$ as transverse surface.

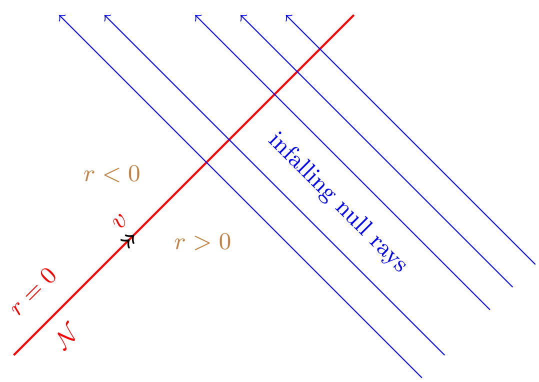

Figure 1: Section of null hypersurface $\mathcal{N}$ at $r=0$ in rv-plane. Infalling null rays traverse $\mathcal{N}$ at different values of advanced time v. Each point on the red line corresponds to a transverse surface $\mathcal{N}_v$.

In these adapted coordinates, inverse metric and metric have the following vanishing components

$$g^{vv} = g^{vA} = g_{rr} = g_{rA} = 0\,. \tag{25}$$

The line-element

$$\mathrm{d}s^2 = -V\,\mathrm{d}v^2 + 2\eta\,\mathrm{d}v\,\mathrm{d}r + g_{AB}\left(\mathrm{d}x^A + U^A\,\mathrm{d}v\right)\left(\mathrm{d}x^B + U^B\,\mathrm{d}v\right) \tag{26}$$

depends on generic functions of all coordinates, V, U^A, g_{AB}, as well as on the function $\eta = \eta(v, x^A) > 0$. (Geodecity, $k\cdot\nabla k = 0$, implies $\partial_r\eta = 0$.)

We assume that the locus of the null surface, $r=0$, is not singular and that the metric coefficients admit a Taylor series expansion in powers of r around $r=0$.

$$V = 2\big(\eta\kappa - \mathcal{D}_v\eta\big)r + \dots \qquad U^A = \mathcal{U}^A - \frac{\eta}{\Omega}\Upsilon^A r + \dots \qquad g_{AB} = \Omega_{AB} - 2\eta\lambda_{AB}\,r + \dots \tag{27}$$

where all functions depend on v, x^A, the ellipses denote $\mathcal{O}(r^2)$ terms, and

$$\Omega := \sqrt{\det\Omega_{AB}} \qquad\qquad \Omega_{AB} = \Omega^{2/(D-2)}\gamma_{AB} \qquad\qquad \det\gamma_{AB} = 1 \tag{28}$$

where γ_{AB} is an arbitrary unimodular matrix. We defined $\mathcal{D}_v := \partial_v - \mathcal{L}_{\mathcal{U}}$. To have a non-degenerate volume form, $\sqrt{-\det g_{\mu\nu}}|_{r=0} = \eta\Omega$, we assume $\Omega, \eta > 0$. The function η yields the volume of the spacetime part of the metric.

3 Null boundary symmetries

We analyze the diffeomorphisms that preserve our null boundary structure and then determine their algebra.

3.1 Null boundary preserving diffeomorphisms

Diffeomorphisms generated by the vector field

$$\xi = T\,\partial_v + \Big(r(\mathcal{D}_v T - W) - r^2\frac{\eta}{2}\Big(\frac{\Upsilon_A}{\Omega} - \frac{\partial_A \eta}{\eta}\Big)\partial^A T + \mathcal{O}(r^3)\Big)\,\partial_r \\ + \Big(Y^A - r\eta\partial^A T - r^2\eta^2\lambda^{AB}\partial_B T + \mathcal{O}(r^3)\Big)\,\partial_A \quad (29)$$

keep $r = 0$ as a null surface, where $T = T(v, x^A)$, $W = W(v, x^A)$ and $Y^A = Y^A(v, x^A)$ are the symmetry generators.

3.2 Algebra of null boundary symmetries

Using the adjusted Lie bracket (see lecture 5) we have

$$[\xi(T_1, W_1, Y_1^A), \xi(T_2, W_2, Y_2^A)]_{\text{adj. bracket}} = \xi(T_{12}, W_{12}, Y_{12}^A) \quad (30)$$

where

$$T_{12} = \big(T_1\partial_v + Y_1^A\partial_A\big)\,T_2 - (1 \leftrightarrow 2), \quad (31a)$$

$$W_{12} = \big(T_1\partial_v + Y_1^A\partial_A\big)\,W_2 - (1 \leftrightarrow 2), \quad (31b)$$

$$Y_{12}^B = \big(T_1\partial_v + Y_1^A\partial_A\big)\,Y_2^B - (1 \leftrightarrow 2). \quad (31c)$$

The above algebra is Diff($\mathcal{N}$) $\Subset$ Weyl($\mathcal{N}$), where Diff($\mathcal{N}$) is generated by T, Y^A and Weyl($\mathcal{N}$) which denotes the Weyl scaling on $\mathcal{N}$, is generated by W. We refer to it as null boundary symmetry algebra.

Note that the null boundary symmetry algebra is considerably larger than the near-bifurcation symmetry algebra (15), since the functions in the latter depended only on coordinates on the bifurcation 2-sphere, whereas here all functions depend additionally on the null time v. This additional v-dependence is necessary if we want to capture time-dependent processes such as gravitational wave absorption.

The null boundary symmetry algebra Diff($\mathcal{N}$) $\Subset$ Weyl($\mathcal{N}$) has several interesting subalgebras. If we turn off Y^A and W sectors, the generator T forms a Witt algebra (diffeomorphisms along v direction) but with an arbitrary dependence in x^A. Turning off T, W sectors, Y^A generate diffeomorphisms of the transverse surface $\mathcal{N}_v$. Nonetheless, one should note that these diffeomorphisms have arbitrary v dependence. A class of subalgebras arises from the fact that our generators are generic functions of v. If the v direction has no special points, one may Taylor-expand the generators around any given point v_0 and keep terms up to the order that still close the algebra. As an example, consider the subalgebra obtained through the following truncation

$$T = t_0 + t_1 v + t_2 v^2 \qquad W = w_0 \qquad Y^A = y_0^A \quad (32)$$

where $t_0, t_1, t_2; w_0, y_0^A$ are only function of x^A. The t_i form an $sl(2,\mathbb{R})$ algebra and w_0 an abelian $u(1)$ algebra, Weyl($\mathcal{N}_v$). This subalgebra is hence (Diff($\mathcal{N}_v$) $\Subset$ $sl(2,\mathbb{R})_{\mathcal{N}_v}$)$\Subset$ Weyl($\mathcal{N}_v$).

4 Surface charge analysis

The surface charge variation associated with a symmetry generator ξ

$$\not{\delta} Q_\xi := \oint_{\partial\Sigma} \mathcal{Q}^{\mu\nu}_\xi \, \mathrm{d}x_{\mu\nu} \tag{33}$$

expands in Einstein gravity as

$$\mathcal{Q}^{\mu\nu}_\xi = \frac{\sqrt{-g}}{8\pi G}\Big(h^{\lambda[\mu}\nabla_\lambda \xi^{\nu]} - \xi^\lambda \nabla^{[\mu} h^{\nu]}_\lambda - \frac{1}{2} h \nabla^{[\mu}\xi^{\nu]} + \xi^{[\mu}\nabla_\lambda h^{\nu]\lambda} - \xi^{[\mu}\nabla^{\nu]} h\Big) \tag{34}$$

where $h_{\mu\nu} = \delta g_{\mu\nu}$, $h = g^{\mu\nu}\delta g_{\mu\nu}$, and $\partial\Sigma$ corresponds to the transverse surface $\mathcal{N}_v$.

Plugging (26) and (29) into (33), yields the surface charge variation

$$\not{\delta} Q_\xi = \frac{1}{16\pi G}\int_{\mathcal{N}_v} \left(W\delta\Omega + Y^A \delta\Upsilon_A + T\not{\delta}\mathcal{A}\right) \tag{35}$$

with (see appendix for various definitions)

$$\not{\delta}\mathcal{A} = -2\Omega\,\delta\Theta_l + \Omega\Theta_l \frac{\delta\eta}{\eta} - \Gamma\,\delta\Omega + \mathcal{U}^A\,\delta\Upsilon_A - \Omega N^{AB}\,\delta\Omega_{AB}\,. \tag{36}$$

The notation $\not{\delta}$ is used to stress that the charge variation is not necessarily integrable in field space. Tackling the question of whether or not the charges are integrable requires specifying which combinations of the symmetry generators are taken to be field-independent, which amounts to a choice of slicing of the phase space.

Only after a slicing is specified one can state whether or not the charges are integrable for this particular slicing. This implies that integrability of the charges is not solely a property of the bulk theory or the boundary conditions, but additionally may depend on the choice of how to slice the phase space.

Physically, non-integrable charges are typically related to a non-vanishing flux through the boundary. Generally, $\not{\delta} Q$ is non-integrable over our null boundary solution space since we allow fluxes through the boundary $\mathcal{N}$. This feature prevents us from working with the Poisson bracket of the charges. We use instead the modified bracket (MB) proposed by Barnich and Troessaert,

$$\delta_{\xi_2} Q^{\mathrm{I}}_{\xi_1} := \left\{Q^{\mathrm{I}}_{\xi_1}, Q^{\mathrm{I}}_{\xi_2}\right\}_{\mathrm{MB}} - F_{\xi_2}(\delta_{\xi_1} g) \tag{37a}$$

$$\left\{Q^{\mathrm{I}}_{\xi_1}, Q^{\mathrm{I}}_{\xi_2}\right\}_{\mathrm{MB}} = Q^{\mathrm{I}}_{[\xi_1,\xi_2]_{\text{adj. bracket}}} + K_{\xi_1,\xi_2} \tag{37b}$$

where K_{ξ_1,ξ_2} is the central term, Q^{I}_ξ the integrable part of the charges and $F_\xi(\delta g)$ the non-integrable part, $\not{\delta} Q_\xi = \delta Q^{\mathrm{I}}_\xi + F_\xi(\delta g)$. The flux term is not necessarily antisymmetric $F_{\xi_2}(\delta_{\xi_1} g) \neq -F_{\xi_1}(\delta_{\xi_2} g)$, which we shall see in examples below.

The split into integrable and non-integrable parts is ambiguous

$$Q^{\mathrm{I}}_\xi \to \tilde{Q}^{\mathrm{I}}_\xi = Q^{\mathrm{I}}_\xi + A_\xi(g) \qquad\qquad F_\xi(\delta g) \to \tilde{F}_\xi(\delta g) = F_\xi(\delta g) - \delta A_\xi(g) \tag{38}$$

and leads to a shift-ambiguity in the central term K_{ξ_1,ξ_2},

$$K_{\xi_1,\xi_2} \to \tilde{K}_{\xi_1,\xi_2} = K_{\xi_1,\xi_2} + \delta_{\xi_2} A_{\xi_1}(g) - \delta_{\xi_1} A_{\xi_2}(g) - A_{[\xi_1,\,\xi_2]}(g) \tag{39}$$

To partially fix this ambiguity, we require the central term to be state-independent.

An important aspect is that the integrability of the charges and the presence or absence of fluxes depends on the slicing. In the following, to shed light on this issue, we discuss two classes of slicings.

The first one, studied in section 4.1, is dubbed "thermodynamic slicing". In this slicing, W, T, Y^A are state-independent ($\delta W = \delta T = \delta Y^A = 0$). The second one is a specific "genuine slicing". By this, we mean any slicing in which the charges are integrable in the absence of bulk fluxes through the boundary, i.e., when there is no physical radiation through the boundary.

4.1 Thermodynamical slicing

The thermodynamic slicing is defined by state-independence of W, T, Y^A in the vector field (29), $\delta W = \delta T = \delta Y^A = 0$.

Applying the MB method discussed above and separating the integrable and flux parts, $\not\delta Q_\xi = \delta Q^{\rm I}_\xi + F_\xi(\delta g)$, yields the integrable part

$$Q^{\rm I}_\xi = \frac{1}{16\pi G}\int_{\mathcal{N}_v}\left(W\ \Omega + Y^A\ \Upsilon_A + T\ (-\Gamma\Omega + \mathcal{U}^A\Upsilon_A)\right) \tag{40}$$

and the flux

$$F_\xi(\delta g; g) = \frac{1}{16\pi G}\int_{\mathcal{N}_v} T\Big(\underbrace{\Omega\Theta_l\frac{\delta\eta}{\eta} - 2\Omega\delta\Theta_l + \Omega\delta\Gamma - \Upsilon_A\delta\mathcal{U}^A}_{\text{fake news}} - \underbrace{\Omega N^{AB}\delta\Omega_{AB}}_{\text{news}}\Big). \tag{41}$$

Straightforward but long computations show that the integrable part of the charges (40) satisfy the same algebra as the symmetry generators (30), (31), i.e. Diff($\mathcal{N}$) $\Subset$ Weyl($\mathcal{N}$). In particular, there is no central extension. Explicitly, if we denote the charges associated with the symmetry generators $\xi(T,0,0)$, $\xi(0,W,0)$ and $\xi(0,0,Y^A)$ by $\boldsymbol{\mathcal{T}}(T), \boldsymbol{\mathcal{W}}(W)$ and $\boldsymbol{\mathcal{J}}(Y^A)$, respectively, then the MB bracket algebra reads

$$\{\boldsymbol{\mathcal{T}}(T_1), \boldsymbol{\mathcal{T}}(T_2)\}_{\rm MB} = \boldsymbol{\mathcal{T}}(T_1\partial_v T_2 - T_2\partial_v T_1), \tag{42a}$$
$$\{\boldsymbol{\mathcal{J}}(Y_1^A), \boldsymbol{\mathcal{J}}(Y_2^B)\}_{\rm MB} = \boldsymbol{\mathcal{J}}(Y_1^A\partial_A Y_2^B - Y_2^A\partial_A Y_1^B), \tag{42b}$$
$$\{\boldsymbol{\mathcal{T}}(T), \boldsymbol{\mathcal{J}}(Y^A)\}_{\rm MB} = -\boldsymbol{\mathcal{T}}(Y^A\partial_A T) + \boldsymbol{\mathcal{J}}(T\partial_v Y^A), \tag{42c}$$
$$\{\boldsymbol{\mathcal{W}}(W_1), \boldsymbol{\mathcal{W}}(W_2)\}_{\rm MB} = 0, \tag{42d}$$
$$\{\boldsymbol{\mathcal{T}}(T), \boldsymbol{\mathcal{W}}(W)\}_{\rm MB} = \boldsymbol{\mathcal{W}}(T\partial_v W), \tag{42e}$$
$$\{\boldsymbol{\mathcal{W}}(W), \boldsymbol{\mathcal{J}}(Y^A)\}_{\rm MB} = -\boldsymbol{\mathcal{W}}(Y^A\partial_A W). \tag{42f}$$

Consistently, in the absence of flux of bulk gravitons, $N_{AB} = 0$, and in a co-rotating frame, $\mathcal{U}^A = 0$, we recover the results of the first section.

4.2 Genuine and Heisenberg slicing

The expression of the flux in the thermodynamic slicing (41) is non-zero even in the absence of a graviton flux encoded in the news tensor N_{AB} (see appendix for its definition). This flux depends on the slicing, and one would expect that there should exist genuine slicings such that the flux is manifestly zero for vanishing genuine news, by which we mean $N_{AB} = 0$.

Direct-sum genuine slicings. Starting from the thermodynamic slicing, consider a one-parameter family change of slicings

$$\tilde{W} = W - \Gamma T - \left(Y^A + T\mathcal{U}^A\right)\bar{\nabla}_A\mathcal{P}, \tag{43a}$$
$$\tilde{T}^{(s)} = e^{-s\mathcal{P}}\Omega\Theta_l T + e^{-s\mathcal{P}}\bar{\nabla}_A(\Omega(Y^A + T\mathcal{U}^A)) \tag{43b}$$
$$\tilde{Y}^A = Y^A + T\mathcal{U}^A \tag{43c}$$

where s is a real number and $\mathcal{P} := \ln(\eta/\Theta_l^2)$. The change of slicing (43) takes the original symmetry generators to a linear combination thereof, with coefficients that depend on the fields on the solution space and their derivatives. The change of slicing then amounts to taking $\delta\tilde{W} = \delta\tilde{T}^{(s)} = 0 = \delta\tilde{Y}^A$. Therefore, the original symmetry generators, W, T, Y^A, have non-zero variations in the new slicing, dictated by the requirement of new tilde-generators to have vanishing variations over the solution space. As a result, the charges transform to a certain (in general non-linear) combination of the original charges.

The charge variation can be written as $\not\delta Q_\xi = \delta \tilde{Q}^{\mathrm{I}}_\xi + \tilde{F}_\xi(\delta g)$, with the integrable part

$$\tilde{Q}^{\mathrm{I}}_\xi = \frac{1}{16\pi G}\int_{\mathcal{N}_v}\left(\tilde{W}\Omega + \tilde{Y}^A \mathcal{J}_A + \tilde{T}^{(s)}\mathcal{P}_{(s)}\right) \tag{44}$$

and the flux

$$\tilde{F}_\xi(\delta g) = -\frac{1}{16\pi G}\int_{\mathcal{N}_v}\left[e^{s\mathcal{P}}\tilde{T}^{(s)} - \bar{\nabla}_C(\Omega\tilde{Y}^C)\right]\Theta_l^{-1}N^{AB}\,\delta\Omega_{AB} \tag{45}$$

where

$$\mathcal{J}_A = \Upsilon_A + \bar{\nabla}_A(\Omega\mathcal{P})\,, \qquad \mathcal{P}_{(s)} = \begin{cases}\frac{1}{s}\,e^{s\mathcal{P}} = \frac{1}{s}\left(\frac{\eta}{\Theta_l^2}\right)^s & \text{if } s\neq 0\\ \mathcal{P} & \text{if } s = 0\,.\end{cases} \tag{46}$$

We call $\Omega, \mathcal{P}_{(s)}, \mathcal{J}_A$, respectively, entropy aspect, expansion aspect, and angular momentum aspect. The expressions above make manifest that the flux proportional to the traceless news tensor N_{AB} is not integrable and vanishes if there is no genuine news, $N_{AB} = 0$. Therefore, this slicing is in the family of genuine slicings.

Using the MB, the charge algebra is

$$\{\Omega(v,x), \Omega(v,x')\} = 0 \tag{47a}$$
$$\{\mathcal{P}_{(s)}(v,x), \mathcal{P}_{(s')}(v,x')\} = 0 \tag{47b}$$
$$\{\Omega(v,x), \mathcal{P}_{(s)}(v,x')\} = 16\pi G\left(s\mathcal{P}_{(s)}(v,x) + \delta_{s,0}\right)\delta^{D-2}\left(x-x'\right) \tag{47c}$$
$$\{\mathcal{J}_A(v,x), \mathcal{J}_B(v,x')\} = 16\pi G\left(\mathcal{J}_A(v,x')\partial_B - \mathcal{J}_B(v,x)\partial'_A\right)\delta^{D-2}\left(x-x'\right) \tag{47d}$$
$$\{\mathcal{J}_A(v,x), \Omega(v,x')\} = \{\mathcal{J}_A(v,x), \mathcal{P}_{(s)}(v,x')\} = 0\,. \tag{47e}$$

This algebra is the direct sum $\mathcal{C}_2^{(s)} \oplus$ Diff$(\mathcal{N}_v)$, where $\mathcal{C}_2^{(s)}$ is generated by the $\Omega(v,x), \mathcal{P}_{(s)}(v,x)$-towers of charges and Diff$(\mathcal{N}_v)$ by $\mathcal{J}_A(v,x)$. We call this slicing a direct-sum genuine slicing.

Heisenberg slicing. For $s = 0$ case the charge algebra (47) takes a simple form of Heisenberg $\oplus$ Diff$(\mathcal{N}_v)$. The Heisenberg slicing is a fundamental slicing since the other genuine slicings in the s-family (and many others) may be constructed from it. Due to its importance as an algebraic building block, we display the charges

$$\tilde{Q}^{\mathrm{I}}_\xi = \frac{1}{16\pi G}\int_{\mathcal{N}_v}\left(\tilde{W}\Omega + \tilde{Y}^A \mathcal{J}_A + \tilde{T}\mathcal{P}\right)\,, \tag{48}$$

and flux

$$\tilde{F}_\xi(\delta g) = -\frac{1}{16\pi G}\int_{\mathcal{N}_v}\left[\tilde{T} - \bar{\nabla}_C(\Omega\tilde{Y}^C)\right]\Theta_l^{-1}N^{AB}\,\delta\Omega_{AB} \tag{49}$$

where $\tilde{T} = \tilde{T}^{(0)}$. The associated transformation laws

$$\delta_\xi\Omega = \tilde{T} \tag{50a}$$
$$\delta_\xi\mathcal{P} \approx -\tilde{W} + \frac{2T}{\Theta_l}N_{AB}N^{AB} \tag{50b}$$
$$\delta_\xi\mathcal{J}_A \approx \mathcal{L}_{\tilde{Y}}\mathcal{J}_A - 2\bar{\nabla}^B(\Omega T N_{AB}) + 2\bar{\nabla}_A(\Omega T\Theta_l^{-1}N_{BC}N^{BC}) \tag{50c}$$

yield the charge algebra

$$\boxed{\{\Omega(v,x), \Omega(v,x')\} = \{\mathcal{P}(v,x), \mathcal{P}(v,x')\} = 0} \tag{51a}$$
$$\boxed{\{\Omega(v,x), \mathcal{P}(v,x')\} = 16\pi G\,\delta^{D-2}\left(x-x'\right)} \tag{51b}$$
$$\boxed{\{\mathcal{J}_A(v,x), \Omega(v,x')\} = \{\mathcal{J}_A(v,x), \mathcal{P}(v,x')\} = 0} \tag{51c}$$
$$\boxed{\{\mathcal{J}_A(v,x), \mathcal{J}_B(v,x')\} = 16\pi G\left(\mathcal{J}_A(v,x')\partial_B - \mathcal{J}_B(v,x)\partial'_A\right)\delta^{D-2}\left(x-x'\right)\,.} \tag{51d}$$

The brackets in the first two lines above justify the name Heisenberg slicing.

5 Null surface balance equation

In the presence of flux, surface charges are not integrable. Moreover, non-integrability and the presence of flux are closely related to the charge non-conservation. While integrability is slicing-dependent, as discussed, there are genuine slicings for which the flux is proportional to the genuine news N_{AB} associated with infalling gravitons. Conservation, too, depends on the choice of phase space slicing. The relation between charge integrability and conservation is captured by the generalized conservation equation, which in the more standard null infinity analyses is called the "flux balance equation". In this section, we briefly discuss the null surface balance equation for the thermodynamic and the Heisenberg slicings.

5.1 Balance equation in thermodynamic slicing

For the thermodynamic slicing, the generator of translations along the advanced time ∂_v is among the symmetry generators $\partial_v = \xi(T = 1, W = 0, Y^A = 0)$. The associated integrable part of the charge (40) and the flux (41)

$$\mathbf{H}_v := Q^{\mathrm{I}}_{\partial_v} = \frac{1}{16\pi G} \int_{\mathcal{N}_v} \left(-\Gamma\Omega + \mathcal{U}^A \Upsilon_A\right) \tag{52a}$$

$$F_{\partial_v}(\delta g; g) = \frac{1}{16\pi G} \int_{\mathcal{N}_v} \Big(-2\Omega\delta\Theta_l + \Omega\Theta_l \frac{\delta\eta}{\eta} + \Omega\delta\Gamma - \Upsilon_A \delta\mathcal{U}^A - \Omega N^{AB}\delta\Omega_{AB}\Big) \tag{52b}$$

obey the null surface energy balance equation

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}v}\mathbf{H}_v \approx -F_{\partial_v}(\delta_{\partial_v} g)} \tag{53}$$

where $\approx$ denotes on-shell equality and $F_{\partial_v}(\delta_{\partial_v} g) := F_{\partial_v}(\delta_\xi g; g)|_{\xi=\partial_v}$. This flux receives two contributions, one from the bulk modes, the $N_{AB}N^{AB}$ term in F, and the other from boundary modes. The latter is essentially a reflection of the fact that in the thermodynamic slicing, the coordinate system adopted (26) corresponds to a non-inertial frame for the boundary dynamics. As viewed by the observer adopting the coordinate system v, r, x^A, the quantity $\mathbf{H}_v = \mathbf{H}_v(v)$ is the boundary Hamiltonian. Thus, a suggestive interpretation of (53) is that it describes an open system, the Hamiltonian of which is time-dependent as a consequence of leakage.

Similarly, one may study the time variation of all other charges, in particular of the zero mode charges, angular momentum, associated with the symmetry generator $\partial_A = \xi(T = 0, W = 0, Y^A = 1)$,

$$\mathbf{J}_A := Q^{\mathrm{I}}_{\partial_A} = \frac{1}{16\pi G} \int_{\mathcal{N}_v} \Upsilon_A \qquad\qquad F_{\partial_A}(\delta g) = 0 \tag{54a}$$

and entropy, associated with the symmetry generator $-r\partial_r = \xi(T = 0, W = 1, Y^A = 0)$,

$$\mathbf{S} := 4\pi Q^{\mathrm{I}}_{-r\partial_r} = \frac{1}{4G} \int_{\mathcal{N}_v} \Omega \qquad\qquad F_{-r\partial_r}(\delta g) = 0\,. \tag{54b}$$

Both obey null surface balance equations similar to (53).

The algebraic relations (42) imply

$$\{\mathbf{H}_v, Q^{\mathrm{I}}_\xi\}_{\mathrm{MB}} = Q^{\mathrm{I}}_{\partial_v \xi} \qquad\qquad \{\mathbf{S}, Q^{\mathrm{I}}_\xi\}_{\mathrm{MB}} = 0 \tag{55}$$

and in particular

$$\{\mathbf{H}_v, \mathbf{S}\}_{\mathrm{MB}} = \{\mathbf{H}_v, \mathbf{J}_A\}_{\mathrm{MB}} = \{\mathbf{S}, \mathbf{J}_A\}_{\mathrm{MB}} = 0\,. \tag{56}$$

As expected, $\mathbf{H}_v$ generates time translations. Moreover, the entropy $\mathbf{S}$ commutes with all the charges. The zero mode charges $\mathbf{H}_v, \mathbf{S}, \mathbf{J}_A$ mutually commute.

On can show that balance equations for zero-mode charges can be generalized to all null boundary charges for generic symmetry generator ξ as

$$\frac{\mathrm{d}}{\mathrm{d}v} Q^{\mathrm{I}}_{\xi} = \delta_{\partial_v} Q^{\mathrm{I}}_{\xi} + Q^{\mathrm{I}}_{\partial_v \xi} \approx -F_{\partial_v}(\delta_\xi g) \tag{57}$$

by virtue of (55), where we used the definition of the MB (37) and that $F_{\partial_v}(\delta_\xi g)$ is given by $F_{\partial_v}(\delta g, g)$ in (52) evaluated at $\delta_\xi g$.

To derive (57), we have used that ∂_v is among our field-independent symmetry generators. The null surface balance equation (52) shows that the flux $F_{\partial_v}(\delta_\xi g)$ receives contributions from the genuine flux, the term proportional to N^{AB}, as well as from terms only involving boundary fields, referred to as fake flux. As for the angular momentum, the latter is generically there because the v, x^A coordinates do not correspond to an inertial observer at the boundary.

A pictorial way to represent a physical process that can be described using the flux balance equations above is shown in fig. 3 at the end of these lecture notes.

5.2 Balance equation in Heisenberg slicing

Unlike the thermodynamic slicing (56), the zero mode charges in the Heisenberg slicing

$$\tilde{\mathbf{H}} := \tilde{Q}^{\mathrm{I}}_{\tilde{T}=1} = \frac{1}{16\pi G}\int_{\mathcal{N}_v} \mathcal{P} \tag{58a}$$

$$\tilde{\mathbf{S}} := 4\pi \tilde{Q}^{\mathrm{I}}_{\tilde{W}=1} = \frac{1}{4G}\int_{\mathcal{N}_v} \Omega \tag{58b}$$

$$\tilde{\mathbf{J}}_A := \tilde{Q}^{\mathrm{I}}_{\tilde{Y}^A=1} = \frac{1}{16\pi G}\int_{\mathcal{N}_v} \mathcal{J}_A \tag{58c}$$

do not commute with each other. Nor does the entropy generically commute with the remaining charges,

$$\{\tilde{\mathbf{S}}, \tilde{\mathbf{H}}\} = \frac{1}{4G}\int_{\mathcal{N}_v} 1 \qquad\qquad \{\tilde{\mathbf{S}}, \tilde{\mathbf{J}}_A\} = \{\tilde{\mathbf{H}}, \tilde{\mathbf{J}}_A\} = 0\,. \tag{59}$$

Notably, $\tilde{\mathbf{S}}$ and $\tilde{\mathbf{H}}$ are Heisenberg pairs with an effective $\hbar$ proportional to $1/G$. One can therefore change the entropy of the system by injecting a $\tilde{\mathbf{H}}$-charge. Recall that $\tilde{\mathbf{H}}$ is the charge associated with the symmetry generator $\tilde{W} = 0 = \tilde{Y}^A$ and $\tilde{T} = \Omega\Theta_l T = 1$, but not with unit v-translations, so we do not refer to it as energy. Moreover, there are no other local combinations of charges playing this role. Thus, in the Heisenberg slicing the zero-mode charge $\tilde{\mathbf{H}}$ should not be viewed as a Hamiltonian, but rather as the Heisenberg conjugate of the entropy.

Since ∂_v is not among the symmetry generators in the Heisenberg slicing, we do not have a null surface balance equation like in thermodynamic slicing (57). The zero-mode charge dynamics is given by

$$\mathcal{D}_v \Omega = \Omega\Theta_l \tag{60a}$$

$$\mathcal{D}_v \mathcal{P} = \Gamma + \frac{2N_{AB}N^{AB}}{\Theta_l} \tag{60b}$$

$$\mathcal{D}_v \mathcal{J}_A = 2\Omega\bar{\nabla}_A(\Theta_l^{-1} N_{BC}N^{BC}) - 2\Omega\bar{\nabla}^B N_{AB}\,. \tag{60c}$$

A Details and news of null hypersurfaces

To decompose the bulk metric adapted to null hypersurfaces, it is standard to define two null vector fields l^μ, n^μ ($l^2 = n^2 = 0$) such that $l \cdot n = -1$, l^μ is outward pointing and n^μ inward pointing. In adapted coordinates the associated 1-forms read

$$l := l_\mu \,\mathrm{d}x^\mu = -\frac{1}{2}V \,\mathrm{d}v + \eta \,\mathrm{d}r \qquad\qquad n := n_\mu \,\mathrm{d}x^\mu = -\,\mathrm{d}v \tag{61}$$

and the corresponding vector fields are given by

$$l^\mu \partial_\mu = \partial_v - U^A \partial_A + \frac{V}{2\eta}\partial_r \qquad\qquad n^\mu \partial_\mu = -\frac{1}{\eta}\partial_r\,. \tag{62}$$

From (62) we see that $\mathcal{D}_v := \partial_v - \mathcal{L}_\mathcal{U}$ is the Lie derivative along the vector l evaluated on $\mathcal{N}$. In terms of l, n, the induced codimension-two metric

$$q_{\mu\nu} = g_{\mu\nu} + l_\mu n_\nu + l_\nu n_\mu \qquad\qquad q_{\mu\nu} l^\mu = q_{\mu\nu} n^\mu = 0 \tag{63}$$

yields the line-element on $\mathcal{N}$

$$\mathrm{d}s^2_\mathcal{N} = \Omega_{AB}\left(\mathrm{d}x^A + \mathcal{U}^A \,\mathrm{d}v\right)\left(\mathrm{d}x^B + \mathcal{U}^B \,\mathrm{d}v\right)\,. \tag{64}$$

As depicted in Fig. 2, $\Omega_{AB} = \Omega_{AB}(v, x^A)$ is the metric over $\mathcal{N}_v$. The inverse of the $D-2$ dimensional metric Ω_{AB} is denoted by Ω^{AB}, $\Omega^{AB}\Omega_{BC} = \delta^A_C$, and A, B indices are raised or lowered by them.

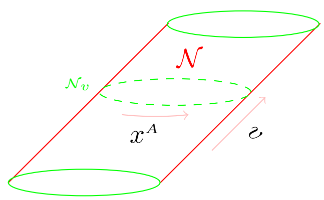

Figure 2: Codimension-one null boundary $\mathcal{N}$ has topology $\mathbb{R}_v \ltimes \mathcal{N}_v$. Transverse surface $\mathcal{N}_v$ is $(D-2)$-dimensional spacelike compact surface.

The deviation tensors,

$$B^l_{\mu\nu} := \left(q^\alpha_\mu q^\beta_\nu \nabla_\beta l_\alpha\right)\big|_{r=0} \qquad\qquad B^n_{\mu\nu} := \left(q^\alpha_\mu q^\beta_\nu \nabla_\beta n_\alpha\right)\big|_{r=0} \tag{65}$$

provide a convenient parametrization. One can decompose them into trace (=expansion), symmetric trace-less (=shear) and anti-symmetric (=twist) parts

$$B^l_{\mu\nu} = \frac{1}{D-2}\Theta_l\, q_{\mu\nu} + N_{\mu\nu} + \omega^l_{\mu\nu} \qquad\qquad B^n_{\mu\nu} = \frac{1}{D-2}\Theta_n\, q_{\mu\nu} + L_{\mu\nu} + \omega^n_{\mu\nu}\,. \tag{66}$$

One can show that the twists $\omega^l_{\mu\nu}, \omega^n_{\mu\nu}$ are zero, and the expansions on $\mathcal{N}$ are

$$\Theta_l = \left(q^{\mu\nu}\nabla_\mu l_\nu\right)\big|_{r=0} = \frac{\mathcal{D}_v\Omega}{\Omega} = \frac{\partial_v\Omega}{\Omega} - \bar{\nabla}_A\mathcal{U}^A \qquad \Theta_n = \left(q^{\mu\nu}\nabla_\mu n_\nu\right)\big|_{r=0} = \Omega^{AB}\lambda_{AB} \tag{67}$$

In the expression for charges, we often use a scalar defined as

$$\Gamma := -2\kappa + \frac{2}{D-2}\Theta_l + \frac{\mathcal{D}_v\eta}{\eta} \tag{68}$$

The shears are given by

$$N_{AB} = \frac{1}{2}\mathcal{D}_v\Omega_{AB} - \frac{\Theta_l}{D-2}\Omega_{AB} = \frac{1}{2}\Omega^{\frac{2}{D-2}}\mathcal{D}_v\gamma_{AB} \qquad\qquad L_{AB} = \lambda_{AB} - \frac{\Theta_n}{D-2}\Omega_{AB} \tag{69}$$

The quantity N_{AB} is called news tensor, and its inverse reads $N^{AB} = -\frac{1}{2}\mathcal{D}_v\Omega^{AB} - \frac{1}{D-2}\Theta_l\Omega^{AB}$. Note that N_{AB} need not be small, i.e., we do not use a linearized approximation here. For further details, see 2110.04218.

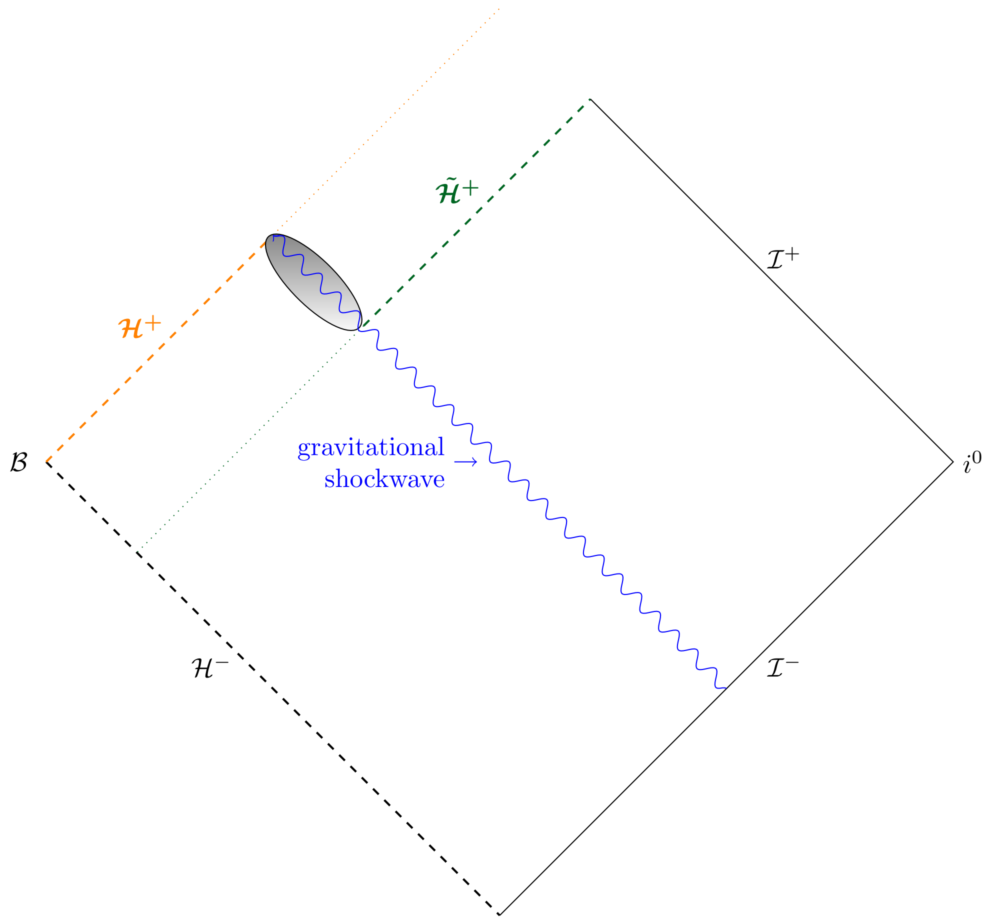

Figure 3: Penrose diagram for shockwave entering black hole. Shaded oval denotes absorption. Dashed orange (green) line is initial (final) horizon $\mathcal{H}^+$ ($\tilde{\mathcal{H}}^+$). Null surfaces used are either the orange or the green line, including the dotted parts.

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