Flat space holography

The holographic principle, if correct, should apply beyond its most famous incarnation, the AdS/CFT correspondence. Whether or not this is the case is one of the big open research questions: How general is holography? If it always works, how does it work precisely? If it does not always work, when does it work?

When facing such big questions, it can be a useful strategy to break them down into smaller but sharper questions, such as: (How) does holography work in asymptotically flat spacetimes?

In this lecture, we address this question from the perspective of asymptotic symmetries. Hence, our first task is to figure out the asymptotic symmetries of asymptotically flat spacetimes.

1 BMS asymptotic symmetries

The first thing we could try is to simply take some AdS boundary conditions and send the AdS-radius to infinity, i.e., to take the flat space limit of asymptotically AdS. So let us try this.

$$\lim_{\ell\to\infty}\Big(\mathrm{d}\rho^2+\big(e^{2\rho/\ell}\,\gamma^{(0)}_{\mu\nu}+\gamma^{(2)}_{\mu\nu}+\mathcal{O}(e^{-2\rho/\ell})\big)\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu\Big)=\mathrm{d}\rho^2+\mathcal{O}(1)_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \tag{1}$$

The result looks strange: all terms are of the same order in ρ, so we cannot accommodate even simple solutions like Schwarzschild(–Tangherlini) in such an expansion. Clearly, the naive limit of asymptotically AdS fails to produce anything of relevance.

The second thing to try is lifting our 2d example of asymptotically flat spaces from lecture 1 to higher dimensions. If you were not present in lecture 1, we found asymptotically flat space boundary conditions in 2d using Eddington–Finkelstein (EF) coordinates,

$$\mathrm{d}s^2=-2\,\mathrm{d}u\,\mathrm{d}r+\big(g_{-1}(u)\,r+g(u)\big)\,\mathrm{d}u^2+\mathcal{O}(1/r) \tag{2}$$

that are preserved by the AKVs

$$\xi=\epsilon(u)\,\partial_u+\big(-\epsilon'(u)\,r+\eta(u)\big)\,\partial_r+\mathcal{O}(1/r)\,. \tag{3}$$

The function $\epsilon(u)$ generates diffeomorphisms of the line, and the function $\eta(u)$ generates time-dependent radial translations. Since we did get something nontrivial, both in terms of metrics and asymptotic symmetries, we are encouraged to generalize the expansion (2) to higher dimensions.

Thus, the third thing we do is to formulate asymptotically flat space boundary conditions in 3d, by analogy to (2). With hindsight, we guess ($\varphi\sim\varphi+2\pi$)

$$\mathrm{d}s^2=\Big(-2\,\mathrm{d}u\,\mathrm{d}r+h_{uu}(u,\,\varphi)\,\mathrm{d}u^2+2h_{u\varphi}(u,\,\varphi)\,\mathrm{d}u\,\mathrm{d}\varphi+r^2\,\mathrm{d}\varphi^2\Big)\big(1+\mathcal{O}(1/r)\big) \tag{4}$$

The rationale for the choice (4) is that we use again EF gauge (since this worked like a charm in 2d) and that we switch on all metric fluctuations compatible with asymptotically vanishing curvature invariants. Since we want to maintain r as radial coordinate (and u is retarded time), we are led to spherical coordinates for the remainder, which in 3d is just one azimuthal angle φ. The boundary (plus gauge) conditions (4) are preserved by the AKVs ξ^L and ξ^M,

$$\xi^L=uL'(\varphi)\,\partial_u+\big(L(\varphi)-\frac{u}{r}\,L''(\varphi)+\mathcal{O}(1/r^2)\big)\,\partial_\varphi-\big(rL'(\varphi)+\mathcal{O}(1)\big)\,\partial_r \tag{5}$$

$$\xi^M=M(\varphi)\,\partial_u-\Big(\frac{1}{r}\,M'(\varphi)+\mathcal{O}(1/r^2)\Big)\,\partial_\varphi+\mathcal{O}(1)\,\partial_r\,. \tag{6}$$

The subleading terms, represented by $\mathcal{O}(r^n)$-expressions, generate proper gauge transformations and are modded out in the asymptotic symmetry algebra, which is generated by two arbitrary functions on the celestial 1-sphere, $L(\varphi)$ and $M(\varphi)$.

Their Lie bracket algebra (the ellipses refer to subleading terms in the AKVs)

$$[\xi^L(L_1),\,\xi^L(L_2)]_{\text{Lie}} = \xi^L\big(L_1L_2' - L_2L_1'\big) + \dots \tag{7}$$

$$[\xi^L(L_1),\,\xi^M(M_2)]_{\text{Lie}} = \xi^M\big(L_1M_2' - M_2L_1'\big) + \dots \tag{8}$$

$$[\xi^M(M_1),\,\xi^M(M_2)]_{\text{Lie}} = 0 + \dots \tag{9}$$

has again infinitely many generators. Comparing the first line (7) with AdS_3 results, we see the Witt algebra as a subalgebra. Geometrically, this makes sense since to leading order, the asymptotic Killing vector ξ^L generates diffeomorphisms of the celestial S^1. The last line (9) shows that the asymptotic Killing vectors ξ^M commute with each other (up to subleading terms). Since their zero mode, $\xi^M_0 = \partial_u$, generates time-translations (which are part of Poincaré), the asymptotic Killing vectors ξ^M are known as "supertranslations" (note: no relation to supersymmetry). In order to have suggestive names, sometimes the Witt algebra generators ξ^L are called "superrotations", as the zero mode $\xi^L_0 = \partial_\varphi$ generates rotations. Superrotations and supertranslations do not commute (8), but instead, yield something reminiscent of a Witt algebra. Neglecting the $\mathcal{O}(1/r)$-terms, the algebra (7)-(9) is known as BMS_3, where the acronym stands for Bondi, van der Burgh, Metzner, and Sachs who discovered the 4d analog of this algebra in the 1960ies.

All central extensions of the algebra (7)-(9) are known. To present them, it is convenient to introduce again Fourier modes, in terms of which the centrally extended version of BMS_3 reads

$$[L_n,\,L_m] = (n-m)\,L_{n+m} + \frac{c_L}{12}\,\big(n^3-n\big)\,\delta_{n+m,\,0} \tag{10a}$$

$$[L_n,\,M_m] = (n-m)\,M_{n+m} + \frac{c_M}{12}\,\big(n^3-n\big)\,\delta_{n+m,\,0} \tag{10b}$$

$$[M_n,\,M_m] = 0\,. \tag{10c}$$

The central charge c_L is a Virasoro central charge, while c_M is referred to as BMS central charge. Their specific values depend, of course, on the theory that we are considering. We shall discuss these values later, in section 3.1.

In four or more spacetime dimensions, similar stories work: we use again EF gauge, using retarded or advanced time and radius as two of our coordinates, we place again the asymptotic boundary at the locus corresponding to $r\to\infty$, we use again spherical coordinates for the remaining $D-2$ directions (in practice, it is often more convenient to use stereographic coordinates, but we will not delve that deeply into details), and we make again an asymptotic expansion compatible with the vanishing of all curvature invariants. In 4d, this asymptotic expansion yields

$$\mathrm{d}s^2 = -2\,\mathrm{d}u\,\mathrm{d}r + \frac{2M}{r}\,\mathrm{d}u^2 + \Big(\tfrac12\,\partial_b C^b_a + \frac{2}{3r}\,\big(N_a + \tfrac14\,C^b_a\partial_c C^c_b\big)\Big)\,\mathrm{d}u\,\mathrm{d}x^a$$
$$+ \Big(r^2\,\Omega_{ab} + r\,C_{ab} + \mathcal{O}(1/r)\Big)\,\mathrm{d}x^a\,\mathrm{d}x^b + \mathcal{O}(1/r^2) \tag{11}$$

where all functions depend on u and x^a but not on r. The functions are called Bondi mass aspect (M), angular momentum aspect (N_a), asymptotic shear (C_{ab}), and Bondi news tensor ($N_{ab} = \partial_u C_{ab}$). The leading order metric Ω_{ab} is fixed and describes the celestial 2-sphere. All other functions are allowed to fluctuate. Bondi mass and angular momentum aspects are additionally constrained to obey evolution equations, e.g., $\partial_u M = -\frac18\,N_{ab}N^{ab} + \frac14\,\partial_a\partial_b N^{ab}$. For more details, extensions, and literature see, e.g., 2009.01926.

The AKVs preserving (11) contain again an infinite set of supertranslations, $\xi^M = M(x^a)\,\partial_u$, i.e., angle-dependent translations into the direction of (retarded or advanced) time. The supertranslations mutually commute, and this feature persists in any higher dimension as well. BMS were shocked by their discovery since they expected to only get Poincaré as asymptotic symmetries, given that their spacetimes asymptote to Minkowski, not an infinite enhancement thereof.

2 Conformal Carrollian symmetries

Let us switch gears and consider conformal Carrollian symmetries. It will become clear at the end of this section why we are doing this.

The Carroll algebra is the $c \to 0$ İnönü–Wigner contraction of the Poincaré algebra. Generators unaffected by this limit are time translations $H = \partial_t$, spatial translations $P_i = \partial_i$, and rotations $J_{ij} = x_i\,\partial_j - x_j\,\partial_i$. Thus, the only generators affected are boosts, $B_i = c\,t\,\partial_i + \frac{1}{c}\,x_i\,\partial_t$. To be able to take the desired limit, we rescale them by c before taking the limit,

$$C_i := \lim_{c\to 0} c\,B_i = x_i\,\partial_t\,. \tag{12}$$

Carroll boosts C_i commute among themselves and also with the Hamiltonian H. This means there is no Thomas precession, and boosting does not generate energy, unlike the Poincaré or Galilei cases. We have absolute space but relative time.

The finite conformal Carroll algebra has additional generators: dilatations $D = t\partial_t + x^i\partial_t$, spatial special conformal transformations $K_i = -2x_j(t\,\partial_t + x^i\,\partial_i) + x^j x_j\,\partial_i$, and temporal special conformal transformations $K = x^i x_i\,\partial_t$. For their commutation relations, see, e.g., section 2.1 in 2202.01172.

In any dimension, it is possible to give the finite conformal Carroll algebra an infinite lift by introducing (Carroll-)time translation generators that depend on all the spatial coordinates, $M_f = f(x^i)\,\partial_t$. These (supertranslation) generators mutually commute. We refer to the infinite version as conformal Carroll algebra.

It is possible to give the conformal Caroll algebra a geometric meaning. To do so, let us step back and consider first a geometric interpretation of the Carroll algebra without conformal symmetries.

Since we obtained Carroll symmetries as $c \to 0$ limit of Poincaré symmetries, we naturally expect Carroll spacetime to emerge as $c \to 0$ limit of Minkowski spacetime. Taking the $c \to 0$ limit of $\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = -c^2\,\mathrm{d}t^2 + \delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$ yields a degenerate metric of signature $(0,+,\dots,+)$, i.e., the Carroll metric

$$h_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = 0\cdot\mathrm{d}t^2 + \delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j \tag{13}$$

has no inverse. Thus, to get a complete geometric description, we also take the $c \to 0$ limit of the (suitably rescaled) inverse of the Minkowski metric, $c^2\,\eta^{\mu\nu}\partial_\mu\partial_\nu = -\partial_t^2 + c^2\,\delta^{ij}\,\partial_i\partial_j$, which yields a bi-vector $-v^\mu v^\nu\partial_\mu\partial_\nu$. So as the second ingredient for defining a Carrollian structure, we use a vector field

$$v^\mu\partial_\mu = \partial_t \qquad\qquad h_{\mu\nu}v^\nu = 0 \tag{14}$$

that lies in the kernel of the Carroll metric. These definitions naturally generalize to curved Carroll manifolds. See appendix A of . for more details.

The vector fields ξ generating the Carroll algebra are all Killing vectors of the Carroll structure, i.e., they preserve the Carollian structure defined above.

$$\mathcal{L}_\xi h_{\mu\nu} = 0 = \mathcal{L}_\xi v^\mu \tag{15}$$

However, there is an interesting difference to Minkowski and Poincaré: the Poincaré Killing vectors not only preserve the Minkowski metric, but they are also the only Killing vectors doing so. However, the Carroll symmetries above are not the only vectors obeying (15). In particular, all vector fields $\xi = f(x^i)\,\partial_t$ also obey (15). So the infinite lift we gave to the finite Caroll algebra naturally arises geometrically. The conformal Carroll algebra generalizes (15) by only conformally preserving the Carrollian structure, $\mathcal{L}_\xi h_{\mu\nu} \propto h_{\mu\nu}$ and $\mathcal{L}_\xi v^\mu \propto v^\mu$.

The punch line of this discussion is that the conformal Carroll algebra in D spacetime dimensions is isomorphic to the BMS algebra in $D+1$ dimensions, see 1402.5894. Thus, in the same way that the asymptotic symmetries of AdS provide us with a natural candidate for the dual field theory, namely a conformal field theory in one lower dimension, **the asymptotic symmetries of BMS provide us with a natural candidate for the dual field theory, namely a conformal Carrollian field theory (CCFT) in one lower dimension.**

3 3d flat space holography: $\mathrm{BMS}_3/\mathrm{CCFT}_2$

Let us verify if the idea expressed at the end of the previous section withstands scrutiny. To perform calculations, it is again convenient to consider 3d gravity. In this section, we aim to test the conjectured $\mathrm{BMS}_3/\mathrm{CCFT}_2$ correspondence.

3.1 İnönü–Wigner contraction of $\mathrm{AdS}_3/\mathrm{CFT}_2$ and basic checks

In the last section, we reviewed the standard İnönü–Wigner contraction sending the speed of light to zero, pioneered by Lévy-Leblond. However, for the purpose of flat space holography, we are more interested in the limit where the AdS radius tends to infinity.

Let us check if we can recover the $\mathrm{BMS}_3/\mathrm{CCFT}_2$ symmetries as a limit from two Virasoro algebras.

$$[\mathcal{L}_n^\pm,\, \mathcal{L}_m^\pm] = (n-m)\,\mathcal{L}_{n+m}^\pm + \frac{c^\pm}{12}\,\big(n^3-n\big)\,\delta_{n+m,\,0} \tag{16}$$

Defining the generators

$$L_n := \mathcal{L}_n^+ - \mathcal{L}_{-n}^- \qquad\qquad M_n := \frac{1}{\ell}\,\big(\mathcal{L}_n^+ + \mathcal{L}_{-n}^-\big) \tag{17}$$

and taking the limit $\ell \to 0$ after evaluating all commutators in terms of the new generators L_n, M_n yields

$$[L_n,\, L_m] = (n-m)\,L_{n+m} + \frac{c_L}{12}\,\big(n^3-n\big)\,\delta_{n+m,\,0} \tag{18a}$$

$$[L_n,\, M_m] = (n-m)\,M_{n+m} + \frac{c_M}{12}\,\big(n^3-n\big)\,\delta_{n+m,\,0} \tag{18b}$$

$$[M_n,\, M_m] = 0\,. \tag{18c}$$

with

$$c_L = \lim_{\ell\to\infty}\big(c^+ - c^-\big) \qquad\qquad c_M = \lim_{\ell\to\infty}\frac{c^+ + c^-}{\ell}\,. \tag{19}$$

The contracted algebra (18) is identical to BMS_3 (10).

The procedure above gives us a prediction for the central charges in flat space Einstein gravity. Inserting the BH values of the central charges, $c^\pm = 3\ell/(2G)$, into the result for the BMS central charges (19) yields

$$\text{3d Einstein gravity:} \qquad c_L = 0 \qquad\qquad c_M = \frac{3}{G}\,. \tag{20}$$

So for flat space Einstein gravity, there is no Virasoro central charge but the BMS central charge is non-zero. An asymptotic symmetry analysis confirms this `gr-qc/0610130`. (In TMG both central charges are non-zero 1208.1658.)

If you are concerned that the BMS central charge is dimensionful and thus its value has no meaning: there is a change of basis, $M_n \to \alpha\, M_n$, that changes its value multiplicatively by α. However, this does not mean there is no content in the value of c_M. There are invariant quantities independent from this change of basis, for instance, the ratio of the M_0 eigenvalue of some state and c_M. We just have to make sure in the future when we discuss observables that they depend only on such basis-independent ratios (and if they do not, we have to be aware that the corresponding result is basis-dependent).

In CCFT_2, one can label states by their L_0 and M_0 eigenvalues, analogous to the labeling of CFT_2 states by their $\mathcal{L}_0^\pm$ eigenvalues, the conformal weights. Similarly, there is again the concept of highest weight states in CCFT_2. Since CCFT_2 are isomorphic to Galilean CFT_2 (just exchange space with time), one can exploit results like the representation theory of the latter, see 0912.1090.

3.2 Thermodynamical checks

In AdS_3, thermal states are given by BTZ black holes ($\varphi \sim \varphi + 2\pi$)

$$\mathrm{d}s^2_{\text{BTZ}} = -\frac{(r^2 - r_+^2)(r^2 - r_-^2)}{\ell^2 r^2}\,\mathrm{d}t^2 + \frac{\ell^2 r^2\,\mathrm{d}r^2}{(r^2 - r_+^2)(r^2 - r_-^2)} + r^2\left(\mathrm{d}\varphi - \frac{r_+ r_-}{\ell r^2}\,\mathrm{d}t\right)^2. \quad (21)$$

While taking the naive $\ell \to \infty$ limit of (21) produces nonsense, this can be averted by redefining first $r_+ \to \ell\,\hat{r}_+$, yielding after the limit (we keep fixed $\hat{r}_+$ and r_-)

$$\mathrm{d}s^2_{\text{FSC}} = \frac{\hat{r}_+^2 (r^2 - r_-^2)}{r^2}\,\mathrm{d}t^2 - \frac{r^2\,\mathrm{d}r^2}{\hat{r}_+^2 (r^2 - r_-^2)} + r^2\left(\mathrm{d}\varphi - \frac{\hat{r}_+ r_-}{r^2}\,\mathrm{d}t\right)^2. \quad (22)$$

By construction, the solutions (22) solve the vacuum Einstein equations $R_{\mu\nu} = 0$ and are known as "flat space cosmologies". In the same way that BTZ is an orbifold of AdS, FSC is an orbifold of Minkowski space, the so-called shifted-boost orbifold hep-th/0203031. Note that unlike BTZ, the FSC solutions only have one Killing horizon at $r = r_-$. This horizon is not a black hole horizon but rather a cosmological horizon since r has now the meaning of time for $r > r_-$ (have a look at the signs in (22) to convince yourself). The solutions are regular and without closed timelike curves on and outside the Killing horizon. Moreover, by transforming to EF gauge one can show that they obey the asymptotically flat boundary conditions (4). So FSCs are admissible states in our theory.

FSCs are indeed thermal states, and their macroscopic thermodynamical quantities can be determined in a variety of standard ways, see 1305.2919 for results on free energy, entropy, temperature, angular momentum, angular potential, and Hawking–Page like phase transitions to hot flat space. The Bekenstein–Hawking entropy

$$S = \frac{4\pi\,r_-}{4G} \quad (23)$$

is reproduced microscopically by a Cardy-like formula for CCFT_2 (with $c_L = 0$)

$$S = 2\pi\,L_0\sqrt{\frac{c_M}{2M_0}} \quad (24)$$

where L_0 and M_0 are the corresponding zero mode eigenvalues of the FSC whose entropy is being counted (they are respectively, angular momentum and mass of the FSC solution). See 1208.4372 and 1208.4371 for derivations. Note that the entropy (24) depends on c_M and M_0 only through their ratio and thus is basis-independent, as it should be.

In summary, thermodynamical considerations confirm the $\mathrm{BMS}_3/\mathrm{CCFT}_2$ correspondence.

3.3 Stress tensor correlation functions

In lecture 4 I explained in detail how to holographically calculate the stress tensor correlation functions by first checking the 2-point function and then establishing a BPZ recursion relation (see lecture 4 for details). This was a check of the $\mathrm{AdS}_3/\mathrm{CFT}_2$ correspondence.

The short version of a longer story is that essentially the same calculation works again in the context of $\mathrm{BMS}_3/\mathrm{CCFT}_2$. For details, see 1507.05620.

3.4 BMS descendants of the vacuum

If the previous subsection was short, this one will be even shorter: see 1502.06185.

3.5 Entanglement entropy

While in the corresponding AdS_3/CFT_2 check, Ryu and Takayanagi had CFT_2-results available to guide them, here we need to first establish the $CCFT_2$ results for entanglement entropy before attempting a holographic derivation thereof.

This was done in 1410.4089. For an entangling interval with separations Δu and $\Delta\varphi$ in a $CCFT_2$ on the plane with central charges c_L and c_M entanglement entropy turns out to be given by

$$S_{\text{EE}} = \frac{c_L}{6}\,\ln\frac{\Delta\varphi}{\epsilon_\varphi} + \frac{c_M}{6}\left(\frac{\Delta u}{\Delta\varphi} - \frac{\epsilon_u}{\epsilon_\varphi}\right) \tag{25}$$

where ϵ_ℓ and ϵ_φ are UV cutoffs. For $c_M = 0$ the result above recovers a chiral half of the CFT_2 result, as it must be. However, if $c_M \neq 0$ the result (25) looks qualitatively different from the CFT_2 result: both Δu and $\Delta\varphi$ are present, there are two cutoffs, and the dependence on the intervals is monomial rather than logarithmic.

Despite the differences to the CFT_2 results, it is again possible to give a Ryu–Takayanagi-like prescription to determine entanglement entropy (25) holographically. This was done first in the Chern–Simons formulation (see appendix) in terms of Wilson lines (with specifically chosen boundary conditions) in 1410.4089 (see 1511.08662 for more details) and later in the metric formulation in terms of geodesics by Jiang, Song, and collaborators in 1706.07552 and 2006.10740.

Similarly to CFT_2, there is again a uniformization map that allows determining entanglement entropy for all states dual to solutions of flat space Einstein gravity in 3d, see 1907.06650. On the gravity side, this uniformization map captures the flat space analog of all Bañados geometries,

$$\mathrm{d}s^2 = -2\,\mathrm{d}u\,\mathrm{d}r + M(\varphi)\,\mathrm{d}u^2 + \big(L(\varphi) + u\,\partial_\varphi M(\varphi)\big)\,\mathrm{d}u\,\mathrm{d}\varphi + r^2\,\mathrm{d}\varphi^2\,. \tag{26}$$

For constant M, L and positive M, the geometries (26) describe FSCs in EF coordinates. The Fourier modes of M, L are essentially the BMS_3 charges.

3.6 Further remarks on $\mathbf{BMS_3/CCFT_2}$

Essentially, any calculation done in the context of AdS_3/CFT_2 could be transposed to a corresponding $BMS_3/CCFT_2$ calculation. Sometimes, taking limits works nicely (like in the derivation of the $CCFT_2$ symmetry algebra from the CFT_2 algebra), whereas at other times, it can be more fruitful to perform calculations directly in flat space (on the gravity side) or the CCFT (on the field theory side).

Some further checks and developments along these lines include the flat space chiral gravity proposal 1208.1658, a flat version of the Liouville boundary theory 1210.0731, induced representations 1403.5803, the addition of chemical potentials 1411.3728, BMS modules 1603.03812, BMS bootstrap 1612.01730, Poincaré blocks 1712.07131, semiclassical BMS blocks 1805.00949, quantum energy conditions 1907.06650, saturation of the chaos bound 2106.07649, etc.

The morale appears to be that anything that can be done in AdS_3/CFT_2 can also be done in $BMS_3/CCFT_2$, though it is often not evident how. The non-triviality of the limit of vanishing cosmological constant makes further checks and developments desirable to get a firm understanding of the inner workings of flat space holography.

3.7 Link to flat space holography in 2d

Given the recent success of the SYK/JT correspondence that can be viewed as a holographic description of AdS_2 dilaton gravity (see, e.g., 1801.09605 and refs. therein) it is natural to wonder whether there is a flat space version of the story. The answer is affirmative, and there is a flat space analog of the Schwarzian boundary action featured in this correspondence. See 1911.05739 for details.

4 4d flat space holography: Carrollian vs. celestial

As suggested on the previous six pages, a natural candidate for the holographic dual to flat space (Einstein) gravity is a CCFT in one lower dimension, since the symmetries match. This viewpoint is known as the Carrollian approach to flat space holography. It is developed to a reasonable degree of maturity in lower dimensions but far less developed in 4d or higher.

Alternatively, there is the celestial approach to flat space holography, specifically for 4d, developed mostly by Strominger and collaborators, see e.g. 2107.02075 and refs. therein. In the celestial approach, the focus is put on scattering amplitudes and their translations into correlation functions on the celestial 2-sphere.

One of the open questions in the celestial program is whether or not there is an independent definition of the celestial CFT_2 that does not rely on a mere translation of scattering data into CFT language. Evidence that the celestial CFT_2, if it exists, is a logarithmic CFT was presented in 2305.08913.

If both approaches capture the physics of asymptotically flat spacetimes adequately, they must be related. In other words, there should be a map between Carrollian and celestial observables. This turns out to be true and was shown independently in 2202.04702 and 2202.08438.

5 Final words

Apart from my bonus lectures, which I will not convert into lecture notes, this is the end of my OIST lectures on asymptotic symmetries, given in July/August 2023. If you listened to my lectures I hope you could take something useful away for your own research program. If you are just reading these lecture notes online, I hope they are reasonably understandable without my additional explanations on the blackboard.

If you have some questions, corrections, or comments on my lecture notes please let me know by e-mail: `grumil@hep.itp.tuwien.ac.at`.

A Chern–Simons formulation for flat space

In lecture 3, we discussed in detail the Chern–Simons formulation for AdS_3 Einstein gravity. For 3d flat space Einstein gravity, there is a Chern–Simons formulation as well, reviewed in this appendix.

Taking the naive $\ell \to \infty$ limit does not work since all this achieves is to send the Chern–Simons level to infinity. It is more fruitful to work directly in flat space without invoking any limits. For AdS_3 we took the isometry algebra of the maximally symmetric solution (global AdS_3) as gauge algebra, $so(2,2) \simeq so(2,1) \oplus so(2,1) \simeq sl(2,\mathbb{R}) \oplus sl(2,\mathbb{R})$. This suggests taking the isometry algebra of Minkowski space as gauge algebra for flat space Einstein gravity in the Chern–Simons formulation, $iso(2,1) \simeq isl(2,\mathbb{R})$. And indeed, this works.

So the Chern–Simons formulation of 3d Einstein gravity without cosmological constant is given by the bulk action

$$I_{\textrm{CS}}[A] = \frac{k}{4\pi} \int \langle A \wedge \mathrm{d}A + \frac{2}{3}\, A \wedge A \wedge A \rangle \tag{27}$$

where the connection 1-form expands as

$$A = \omega^a\, L_a + e^a\, M_a \tag{28}$$

with the $isl(2,\mathbb{R})$ generators $(a,b \in \{1,\, 0,\, -1\})$

$$[L_a,\, L_b] = (a-b)\, L_{a+b} \qquad [L_a,\, M_b] = (a-b)\, M_{a+b} \qquad [M_a,\, M_b] = 0 \tag{29}$$

and the bilinear form ($\eta_{ab} = \mathrm{antidiag}(1, -\frac{1}{2}, 1)$ is the 3d Minkowski metric)

$$\langle L_a,\, M_b \rangle = -2\eta_{ab}\,. \tag{30}$$

We use suggestive notation to make clear that the L-part of the connection is interpreted as (dualized) spin connection $\omega^a = \frac{1}{2}\, \epsilon^{abc}\, \omega_{bc}$ and the M-part as dreibein e^a. (Latin indices are lowered with η_{ab}.)

As expected, the Chern–Simons equations of motion, i.e., the gauge flatness conditions

$$\mathrm{d}A + A \wedge A = 0 \tag{31}$$

reproduce the 3d Einstein–Hilbert–Palatini equations of motion

$$R^a = 0 = T^a \tag{32}$$

where $T^a = \mathrm{d}e^a + \epsilon^a{}_{bc}\, \omega^b \wedge e^c$ is the torsion 2-form and $R^a = \mathrm{d}\omega^a + \frac{1}{2}\, \epsilon^a{}_{bc}\, \omega^b \wedge \omega^c$ is the (dualized) curvature 2-form.

To obtain the flat space analog of the Bañados geometries (26) (in EF gauge) as part of the solution space allowed by the boundary conditions, we take inspiration from BH and impose the boundary conditions

$$A = b^{-1}\left(\mathrm{d} + a\right) b \qquad\qquad b = \exp\left(\tfrac{r}{2}\, M_{-1}\right) \tag{33}$$

with the boundary connection

$$a = \Big(M_1 - \frac{M(\varphi)}{4}\, M_{-1}\Big)\, \mathrm{d}u + \Big(L_1 - \frac{M(\varphi)}{4}\, L_{-1} - \frac{u\, M'(\varphi) + 2L(\varphi)}{4}\, M_{-1}\Big)\, \mathrm{d}\varphi\,. \tag{34}$$

A difference to the AdS case is that the metric is no longer given by a trace,[1] but it still reads

$$g_{\mu\nu} = e_\mu^a e_\nu^b\, \eta_{ab}\,. \tag{35}$$

Inserting the field configuration (33) with (34) and (28) into the metric (35) recovers the solution space (26).

[1] It is possible to define some twisted version of a trace so that the metric is the twisted trace of the bilinear in the Chern–Simons connection; see 1411.3728, which discusses also the generalization to include arbitrary chemical potentials and/or spin-3 fields.