Near horizon symmetries and soft Heisenberg hair

In this lecture, we impose near horizon boundary conditions and choose a specific state space slicing to obtain the simplest non-trivial symmetry algebra in AdS_3 gravity, namely two copies of $\hat{u}(1)$ current algebras.

This will lead us to the concept of soft Heisenberg hair, where "soft" refers to the fact that all descendants have the same energy as the parent state, in stark contrast to typical descendants (e.g., Virasoro descendants). The expression "soft hair" was coined by Hawking, Perry and Strominger and refers to zero energy excitations on black holes that nevertheless carry physical information (hence "hair"). The attribute "Heisenberg" comes from the specific form of the asymptotic symmetry algebra that we shall encounter, see 1603.04824.

1 Near horizon symmetries

Near horizon symmetries are essentially the same as asymptotic symmetries, i.e., the AKVs are again given by solutions to

$$\left(\mathcal{L}_\xi g\right)_{\mu\nu} = \mathcal{O}(\delta g_{\mu\nu}) \qquad \forall\; g_{\mu\nu} = \bar{g}_{\mu\nu} + \delta g_{\mu\nu} \tag{1}$$

except that $\bar{g}$ is not some asymptotic background but rather a Rindler-type ("near horizon") metric and δg does not come from some asymptotic expansion but from some near horizon expansion of the metric.

1.1 Why near horizon symmetries?

Whenever one is interested in asking conditional questions, like "given a black hole, what are the scattering amplitudes in a given channel?" or "given a cosmological horizon, what are the allowed states that remain in the physical Hilbert space and how could they be related through symmetries?" or "given a black hole or cosmological horizon, can we microscopically account for the Bekenstein–Hawking entropy in the classical limit?" it is crucial to impose boundary conditions that make sure that the condition in the question is met. Near horizon symmetries achieve this goal.

1.2 How to choose the boundary conditions?

Like in asymptotic studies, there is again a lot of choice involved in the precise choice of the boundary conditions. We are inspired by the universality of the Rindler approximation of non-extremal Killing horizons and thus make an ansatz of the form

$$\mathrm{d}s^2 = -\kappa^2\,\rho^2\,\mathrm{d}t^2 + \mathrm{d}\rho^2 + \Omega_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b + \dots \tag{2}$$

where in the time-independent case κ is surface-gravity, ρ is the radial Rindler coordinate, Ω_{ab} the co-dimension-2 metric transversal to the 2d Rindler space, and the ellipsis denotes higher order terms and rotation terms proportional to $\mathrm{d}t\,\mathrm{d}x^a$. In principle, κ and Ω_{ab} could depend on Rindler time t and the transversal coordinates x^a, but not on the radial coordinate ρ.

Even if we grant that (2) might be what we want, we have to make up our minds about the allowed fluctuations. The key question is whether or not κ is allowed to fluctuate in our state space. A version of near horizon boundary conditions where κ is allowed to fluctuate was proposed (in 2+1 dimensions) in 1512.08233. However, the physical interpretation of this setup remained rather unclear. A physically more transparent choice, first made in 1511.08687, is to demand $\delta\kappa = 0$ and to allow $\delta\Omega_{ab} \neq 0$. In these lectures, we are choosing the latter.

For the sake of specificity, we analyze first the AdS_3 case (with unit AdS radius) and generalize in the final section to arbitrary dimensions.

1.3 What are the near horizon symmetries?

Using all the tools from previous lectures, we can quickly answer the question in the heading of this subsection. We use again the Chern–Simons formulation of 3d gravity and the convenient split of the connection

$$A^{\pm} = b_{\pm}^{-1} \left(\mathrm{d} + a^{\pm} \right) b_{\pm} \tag{3}$$

into some group elements that are state-independent and only depend on the radial coordinate, $b_{\pm} = \exp\left(\pm\rho\,(L_1 - L_{-1})/2\right)$, and into boundary connections

$$a^{\pm} = \left(\mu^{\pm}(t,\,\varphi)\;\mathrm{d}t \pm \mathcal{J}^{\pm}(t,\,\varphi)\;\mathrm{d}\varphi \right) L_0 \tag{4}$$

that only depend on the boundary coordinates $(t, \varphi) \sim (t, \varphi + 2\pi)$ and also only have legs in these directions.

Inspired by our analysis of the most general boundary conditions, we anticipate that $\mu^{\pm}$ are chemical potentials, i.e., $\delta\mu^{\pm} = 0$, while $\mathcal{J}^{\pm}$ are charges, i.e., allowed to vary, $\delta\mathcal{J}^{\pm} \neq 0$.

It is not at all obvious why we used only the Cartan subalgebra generator L_0 in (4). This was obtained by trial-and-error in 1603.04824. Indeed, using the field configuration above and extracting from it the metric recovers the expansion (2), where $\kappa = -(\mu^+ + \mu^-)/2$ is indeed fixed, while $\Omega = (\mathcal{J}^+ + \mathcal{J}^-)^2/4$ is free to vary. Assuming the chemical potentials are constant, the equations of motion (a.k.a. "holographic Ward identities") establish charge conservation,

$$\partial_t \mathcal{J}^{\pm} = 0\,. \tag{5}$$

The boundary condition preserving gauge transformations

$$\delta_{\epsilon^{\pm}} a^{\pm} = \mathrm{d}\epsilon^{\pm} + [a^{\pm},\, \epsilon^{\pm}] \tag{6}$$

modulo trivial gauge transformations are given by

$$\epsilon^{\pm} = \eta^{\pm}(\varphi)\, L_0\,. \tag{7}$$

The state-dependent functions transform as

$$\delta_{\epsilon^{\pm}} \mathcal{J}^{\pm} = \pm(\eta^{\pm})'\,. \tag{8}$$

Using the background independent result for the co-dimension-2 charges of Chern–Simons theory yields

$$Q^{\pm}[\eta^{\pm}] = \mp\frac{k}{4\pi} \oint \eta^{\pm} \mathcal{J}^{\pm}\;\mathrm{d}\varphi \tag{9}$$

where we used $\mathrm{tr}(L_0^2) = \frac{1}{2}$.

Expanding in Fourier modes

$$J_n^{\pm} = \frac{k}{4\pi} \oint \mathcal{J}^{\pm} e^{\pm in\varphi}\;\mathrm{d}\varphi \tag{10}$$

yields the mode version of the canonical realization of the asymptotic symmetry algebra (I replaced already $-i\{,\} \to [,]$)

$$\boxed{[J_n^{\pm},\, J_m^{\pm}] = \frac{k}{2}\, n\, \delta_{n+m,\,0} \qquad\qquad [J_n^{+},\, J_m^{-}] = 0} \tag{11}$$

Thus, the asymptotic symmetry algebra associated with our near horizon boundary conditions (for the chosen slicing of the state space) is given by two $\hat{u}(1)$ current algebras. This is arguably the simplest set of non-trivial infinite-dimensional asymptotic symmetries and can serve as a building block for more complicated algebras.

Historical note: 1511.08687 found a different asymptotic symmetry algebra for the same boundary conditions, namely a non-abelian algebra without central extension, whereas our algebra is abelian apart from the central extension. This difference comes from a change of slicing, made explicit in 1611.09783.

2 Soft Heisenberg hair

The main purpose of this section is to explain its title, which we do in reverse order.

2.1 Hair

I assume you know what "hair" means in a black hole context and that black holes are supposed to have none. The black holes that are part of our state space can have hair, namely boundary excitations generated by the charges $\mathcal{J}^\pm$. We have shown in the previous section that $\mathcal{J}^\pm \neq 0$ corresponds to non-trivial states (since their associated charges are non-zero), which means that black holes carrying $\mathcal{J}^\pm$ charges have hair.

Geometrically, we can generate such hair by diffeomorphisms of the S^1,

$$\mathrm{d}\varphi \to \frac{\mathcal{J}^+(\varphi) + \mathcal{J}^-(\varphi)}{2}\,\mathrm{d}\varphi \tag{12}$$

so as long as the quantities $\mathcal{J}^\pm$ are compatible with periodicity and their sum is positive these diffeomorphisms are globally well-defined. Thus, there is no reason to exclude black holes with $\mathcal{J}^\pm$-hair from our state space if the parent black hoke was regular, as no singularities are induced by equipping these black holes with (near horizon) hair.

This explains why the expression "hair" is appropriate in our context.

2.2 Heisenberg

Linearly combining the generators as

$$X_n = J_n^+ - J_{-n}^- \qquad\qquad P_n = \frac{i}{kn}\left(J_{-n}^+ + J_n^-\right)\; n \neq 0 \qquad\qquad P_0 = J_0^+ + J_0^- \tag{13}$$

converts the two $\hat{u}(1)$ current algebras (11) into an infinite tower of Heisenberg algebras

$$[X_n,\, P_m] = i\,\delta_{n,\,m}\; n \neq 0 \tag{14}$$

while all undisplayed commutators vanish, in particular the ones with P_0.

This explains why the hair was labeled as "Heisenberg" (which sounds more digestible than "$\hat{u}(1)$-hair".)

2.3 Soft

Start with some reference state $|\psi\rangle$ and consider some arbitrary descendant

$$|\psi(\{n_i^\pm\})\rangle = \prod_{n_i^\pm > 0} J_{-n_i^+}^+ J_{-n_i^-}^- |\psi\rangle \tag{15}$$

labeled by a set of integers $\{n_i^\pm\}$. The energy as measured by the near horizon Hamiltonian

$$E = Q[\partial_t] = \kappa\left(J_0^+ + J_0^-\right) \tag{16}$$

commutes with all elements of the near horizon symmetry algebra (11). Therefore, the action of raising operators $J^\pm_{-n_i^\pm}$ does not change the energy of the state.

This means that all the Heisenberg hair excitations are without energy, which explains the attribute "soft", hence "soft Heisenberg hair".

Historical note: the term "soft hair" was coined by Hawking, Perry and Strominger and also refers to excitations on black holes that do not carry energy.

3 Applications

3.1 Entropy and near horizon first law

You may remember that the Cardy-formula

$$S = 2\pi\left(\sqrt{\frac{c^+L_0^+}{6}} + \sqrt{\frac{c^-L_0^-}{6}}\right) \tag{17}$$

contains the square root of the asymptotic zero mode charges. It is natural to ask if there is a Cardy-like formula that features the near horizon zero mode charges $J_0^\pm$. The answer is yes.

The entropy in near horizon variables

$$S = 2\pi\left(J_0^+ + J_0^-\right) \tag{18}$$

turns out to be linear in the near horizon zero mode charges, which is a slight simplification as compared to Cardy's formula (17). It turns out that the near horizon entropy law (18) is far more universal than the Bekenstein–Hawking or the Wald entropy, for more on this see below.

The near horizon first law

$$\mathrm{d}E = T\ \mathrm{d}S \tag{19}$$

relates the entropy (18) to the temperature $T = \frac{\kappa}{2\pi}$ and the near horizon energy (16). As a consequence of our choice to have κ state-independent, the near horizon first law (19) trivially integrates to $E = TS$. Notably, there are no work terms in the near horizon first law (19).

You may wonder whether or not there is a Cardy-like derivation of the Cardy-like formula (18). Again, the answer is yes. The main technical difference is that we no longer have an isotropic scale invariance with respect to the boundary coordinates, as we did for the CFT_2 case. Instead, there is an anisotropic scale invariance of Lifshitz type,

$$t \to \lambda^z t \qquad\qquad \varphi \to \lambda\varphi \tag{20}$$

with Lifshitz exponent $z = 0$. The fact that t is not allowed to scale stems again from our assumption in the boundary conditions that κ is fixed. For positive z one can show that the Cardy-like formula for the entropy is given by (see, e.g., 1611.09783)

$$S = 2\pi(1+z)\sum_\pm \Delta_\pm^{1/(1+z)}\ \exp\left(\frac{z}{1+z}\ln\left(\Delta_0^\pm/z\right)\right) \tag{21}$$

where $\Delta_\pm$ are the zero mode charges of the state whose entropy is calculated and $\Delta_0^\pm$ are the zero mode charges of some ground state. In the limit $z \to 0^+$ the latter drop out (assuming they remain either finite in the limit or diverge not worse than polynomially in $1/z$) and the Cardy-like formula (21) reduces to the near horizon entropy (18) with $\Delta_\pm = J_0^\pm$.

3.2 Relation to asymptotic symmetries of BH

We can relate the near horizon results to asymptotic results, where for the latter the connection is given by (for brevity we discuss only one chiral sector)

$$\hat{A} = \hat{b}^{-1}\left(\mathrm{d}+\hat{a}\right)\hat{b} \qquad\qquad \hat{b} = e^{\rho L_0} \tag{22}$$

$$\hat{a}_t = \zeta\, L_1 - \zeta'\, L_0 + \left(\frac{1}{2}\zeta'' - \frac{1}{2}\mathcal{L}\zeta\right) L_{-1} \qquad\qquad \hat{a}_\varphi = L_1 - \frac{1}{2}\mathcal{L}\, L_{-1} \tag{23}$$

Here, ζ is the asymptotic chemical potential and $\mathcal{L}$ the asymptotic (Brown–Henneaux) charge. One can show that the configuration above is gauge-equivalent to (3), (4) upon identifying

$$\zeta' - \mathcal{J}\,\zeta = -\mu \qquad\qquad \mathcal{L} = \frac{1}{2}\,\mathcal{J}^2 + \mathcal{J}' \tag{24}$$

Similarly, the gauge parameter η in (7) relates to the BH-parameter ϵ as

$$\epsilon' - \mathcal{J}\epsilon = -\eta\,. \tag{25}$$

Inserting this relation together with (24) into the near horizon transformation law (8) recovers the infinitesimal Schwarzian

$$\delta_\epsilon \mathcal{L} = \epsilon\mathcal{L}' + 2\epsilon'\mathcal{L} - \epsilon'''\,. \tag{26}$$

In Fourier modes, the second equality (24),

$$L_n = \frac{1}{k}\sum_{m\in\mathbb{Z}} J_{n-m}J_m + in\,J_n \tag{27}$$

is recognized as a twisted Sugawara construction. Using the near horizon symmetry algebra (11), it is straightforward to show that the twisted Sugawara stress tensor (27) obeys the Virasoro algebra with central charge $c = 6k$.

Yet another way to understand the relation between near horizon and asymptotic variables is as a change of slicing. To see this, consider the variation of the codimension-2 charges

$$\delta Q = -\frac{k}{4\pi}\oint \mathrm{d}\varphi\,\epsilon\,\delta\mathcal{L} = -\frac{k}{4\pi}\oint \mathrm{d}\varphi\,\eta\,\delta\mathcal{J}\,. \tag{28}$$

For the BH-slicing, ϵ is state-independent (by assumption), while for our Heisenberg-slicing, η is state-independent.

3.3 Generalizations in 3d

While the story above was first developed for AdS_3 Einstein gravity, it generalizes within 3d in several ways to

- flat space cosmologies
- higher derivative theories [no Bekenstein–Hawking]
- higher spin black holes [no Bekenstein–Hawking, no Wald formula]
- warped black holes [no Bekenstein Hawking]

Algebraically, a key aspect is that we recover asymptotic symmetry algebras like BMS_3 or warped conformal symmetries or even higher spin algebras from specific Sugawara-like constructions, that are induced by the same logical flow as in the previous subsection.

As an example and application, consider higher spin black holes within spin-3 gravity, see 1607.05360 for details of this analysis. The entropy of such black holes obeys some complicated "Cardy-formula" in terms of spin-2 ($\mathcal{L}_\pm$) and spin-3 charges ($\mathcal{W}_\pm$)

$$S = 2\pi\Bigg[\sqrt{\mathcal{L}_+}\,\cos\bigg(\frac{1}{3}\,\arcsin\frac{\mathcal{W}_+}{\mathcal{L}_+^{3/2}}\bigg) + \sqrt{\mathcal{L}_-}\,\cos\bigg(\frac{1}{3}\,\arcsin\frac{\mathcal{W}_-}{\mathcal{L}_-^{3/2}}\bigg)\Bigg] \tag{29}$$

In the near horizon version, the entropy of these higher spin black holes reduces to the universal form (18).

4 Generalizations to higher dimensions

Given the universality and simplicity of the near horizon symmetries and the associated entropy law, one may wonder if this story extends to higher dimensions. The answer is essentially yes, though there will be additional technical and physical aspects to consider. The main difference is that in higher dimensions we have local gravitational degrees of freedom that can fall into the black hole (gravitational wave absorption by a black hole). So when setting up near horizon boundary conditions, one has to decide whether or not the boundary conditions should allow such processes.

The simplest approach is to forbid gravitational wave absorption (or emission) by black holes. In this case, the boundary conditions imposed in higher dimensions are again given by (2) with state-independent κ and state-dependent Ω_{ab}.

$$\begin{aligned} g_{tt} &= -\kappa^2\rho^2 + \mathcal{O}(\rho^3) & g_{\rho\rho} &= 1 + \mathcal{O}(\rho) \\ g_{t\rho} &= \mathcal{O}(\rho^2) & g_{\rho a} &= f_{\rho a}\,\rho + \mathcal{O}(\rho^2) \\ g_{ta} &= f_{ta}\,\rho^2 + \mathcal{O}(\rho^3) & g_{ab} &= \hat{\Omega}_{ab} + \mathcal{O}(\rho^2)\,. \end{aligned} \tag{30}$$

The near horizon expansion (30) is preserved by diffeomorphisms generated by vector fields $\xi = \xi^\mu\,\partial_\mu$ with

$$\xi^t = \frac{\eta}{\kappa} + \mathcal{O}(\rho), \qquad \xi^\rho = \mathcal{O}(\rho^2), \qquad \xi^a = \eta^a + \mathcal{O}(\rho^2) \tag{31}$$

where η^a depends arbitrarily on x^a, while η depends additionally on t subject to the condition $\partial_t\eta + \eta^a\partial_a\kappa = \delta\kappa$. The dynamical fields, $\mathcal{P}$ and $\mathcal{J}_a$, defined by

$$\mathcal{P} := \frac{\sqrt{\Omega}}{8\pi G} \qquad\qquad \mathcal{J}_a := \frac{\sqrt{\Omega}}{16\pi G\kappa}\Big(\partial_t f_{\rho a} - 2 f_{ta}\Big) \tag{32}$$

transform as

$$\delta\mathcal{P} = \eta^a\partial_a\mathcal{P} + \mathcal{P}\partial_a\eta^a \tag{33a}$$

$$\delta\mathcal{J}_a = \mathcal{P}\partial_a\eta + \eta^c\partial_c\mathcal{J}_a + \mathcal{J}_c\partial_a\eta^c + \mathcal{J}_a\partial_c\eta^c\,. \tag{33b}$$

The associated codimension-2 charges

$$\delta Q[\eta,\,\eta^a] = \int \mathrm{d}^{D-2}x\,\big[\eta\,\delta\mathcal{P} + \eta^a\,\delta\mathcal{J}_a\big] \tag{34}$$

turn out to be non-trivial and finite. For more details and numerous different state-space slicings (including BMS-slicing, higher-spin versions of BMS, and the Heisenberg-slicing) as well as examples like Kerr–NUT black holes, see 1908.09833. In BMS-slang, the charges associated with $\mathcal{P}$ generate supertranslations, and the charges associated with $\mathcal{J}$ (some version of) superrotations.

The entropy is again linear in the near horizon charges,

$$S = 2\pi\,\mathcal{P}_0 \tag{35}$$

with $\mathcal{P}_0 = \int \mathrm{d}^{D-2}x\,\mathcal{P}$, which may have been anticipated from the 3d universality. For the Kerr BH in Boyer–Lindquist coordinates it reads $\mathcal{P} = \frac{r_+(r_++r_-)}{8\pi G}\,\sin\theta$.

Generalizing this story to allow for gravitational wave absorption (or, semiclassically, also emission) is not completely trivial, but possible. See 2110.04218 and Refs. therein. A technical challenge is that the charges are neither integrable nor conserved, which necessarily happens when dealing with an open system. Notably, there is a flux-balance law that relates the non-conservation of the charges to the flux through the horizon, which physically is sensible.