Zoo of asymptotic symmetries

For the same bulk theory there could be inequivalent boundary theories, depending on the chosen boundary conditions. Even for a given fall-off behavior of the fields near the boundary the asymptotic symmetry algebra may change if we allow for state-dependent transformation parameters.

In these lectures, we address why it can be useful (or even necessary) to allow for state-dependence in the transformation parameters, how we deal with such a situations, and what it implies for the asymptotic symmetries. We can understand these issues in terms of different state space slicings. Finally, we study these somewhat abstract notions for AdS_3 Einstein gravity, where we shall find numerous alternatives to Brown–Henneaux boundary conditions, including all the explicit examples discovered in the literature of the past decade. These considerations explain the zoo of asymptotic symmetries highlighted at the end of lecture 4.

1 State-dependent parameters

Let us first clarify the notation: by state-dependent we mean that the corresponding quantity is allowed to vary on our state-space, while state-independence negates this possibility. For example, in the usual Feferman–Graham expansion of locally asymptotically AdS_3 metrics,

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + \big(e^{2\rho/\ell}\,\gamma^{(0)}_{\mu\nu} + \gamma^{(2)}_{\mu\nu} + \mathcal{O}(e^{-2\rho/\ell})\big)\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \tag{1}$$

the leading order term is state-independent, $\delta\gamma^{(0)} = 0$, while the subleading order term is state-dependent, $\delta\gamma^{(2)} \neq 0$. The sub-subleading terms can be state-dependent, but this state-dependence is inherited from the one contained in $\gamma^{(2)}$ and does not bring any relevant new information into the system.

So far, we assumed that the transformation parameters (appearing, e.g., in asymptotic Killing vectors) are state-independent, which seems eminently reasonable. Thus, before delving into details of how to deal with state-dependent transformation parameters we explain why it can be necessary to do this in the first place.

1.1 Why?

The simplest reason for considering state-dependent parameters is the consistency between metric and gauge-theoretic formulations of gravity. For concreteness, consider the example we studied in detail, 3d gravity.

In the metric formulation, after imposing some boundary conditions we obtain some asymptotic Killing vector vector fields ξ^μ. Let us assume that all functions appearing therein are state-independent.

In the gauge-theoretic formulation we similarly obtain some asymptotic gauge parameters ϵ^a. Let us assume that all functions appearing therein are state-independent as well.

Translating between these two formulations requires the identity

$$\epsilon^a = A^a_\mu\,\xi^\mu \tag{2}$$

where A^a_μ is the gauge connection. If both ϵ^a and ξ^μ were state-independent, $\delta\epsilon^a = 0 = \delta\xi^\mu$, then also A^a_μ must be state-independent, $\delta A^a_\mu = 0$. However, this is not true for any interesting set of boundary conditions: in any physically interesting scenario, the gauge-connection always allows for some state-dependent function(s), like the functions $\mathcal{L}^\pm$ in the Brown–Henneaux case.

Thus, we have to allow some state-dependence, either in ϵ^a or in ξ^μ or in both. Once we permit this possibility, some of the choices we made in the previous lectures should be reconsidered.

1.2 Why was this not an issue so far?

Actually, when discussing Brown–Henneaux boundary conditions in the Chern–Simons formulation, we had state-dependence in our gauge transformation parameters, since we found (I am using here notations and conventions from lectures 3-4)

$$\hat{\epsilon} = \epsilon\, L_1 - \epsilon'\, L_0 + \big(\tfrac{1}{2}\epsilon' - \mathcal{L}\epsilon\big)\, L_{-1}\,. \tag{3}$$

So even when we assume the parameter function ϵ to be state-independent, we get a non-vanishing variation

$$\delta\hat{\epsilon} = -(\delta\mathcal{L})\,\epsilon\, L_{-1} + \delta\epsilon\text{-terms} = -(\delta\mathcal{L})\,\epsilon\, L_{-1} \text{ if } \delta\epsilon = 0\,. \tag{4}$$

However, this state-dependence did not affect our charges or asymptotic symmetries, because in the result for the variation of the charges

$$\delta Q[\hat{\epsilon}] = \frac{k}{2\pi}\oint_{S^1} \operatorname{tr}\big(\hat{\epsilon}(x^+)\,\delta a(x^+)\big) = -\frac{k}{2\pi}\oint_{S^1} \operatorname{tr}\big(\hat{\epsilon}(x^+)\, L_{-1}\big)\,\delta\mathcal{L}(x^+)\;\mathrm{d}x^+ \tag{5}$$

the trace took care that only the L_1-component of $\hat{\epsilon}$ contributed. As evident from (4), that component is state-independent, and this is why we never had an issue so far with integrating the charges in field space.

In other words, so far we were lucky. But what shall we do if our luck runs out?

1.3 How?

As long as we specify how precisely the gauge parameters depend on the state-dependent functions we might still be able to end up with integrable charges.

In the BH-example above, consider for example a state-dependence in the function ϵ of the form

$$\epsilon(x^+) = \mathcal{L}(x^+)\,\tilde{\epsilon}(x^+) \qquad\qquad \delta\tilde{\epsilon} \stackrel{!}{=} 0\,. \tag{6}$$

In that case, the variation analogous to (3),

$$\delta\hat{\epsilon} = \delta\mathcal{L}\,\tilde{\epsilon}\, L_1 + \dots \tag{7}$$

yields a non-zero L_1 component. Thus, in the variation of the charges (5),

$$\delta Q[\tilde{\epsilon}] = \frac{k}{2\pi}\oint_{S^1} \tilde{\epsilon}\,\mathcal{L}\,\delta\mathcal{L}\;\mathrm{d}x^+ \tag{8}$$

there is now a non-trivial state-dependent prefactor in front of $\delta\mathcal{L}$. Fortunately, we chose this prefactor such that the charges remain integrable in field space:

$$Q[\tilde{\epsilon}] = \frac{k}{4\pi}\oint_{S^1} \tilde{\epsilon}\,\mathcal{L}^2\;\mathrm{d}x^+ \tag{9}$$

So state-dependent gauge parameters can be fine and compatible with integrable charges, as long as we choose some appropriate state-dependence (we shall be more general and explicit in the next section.)

Of course, integrability is not automatic. A simple counter example is the modified state-dependence

$$\epsilon(x^+) = \big(\partial_{x^+}\mathcal{L}(x^+)\big)\,\tilde{\epsilon}(x^+) \qquad\qquad \delta\tilde{\epsilon} \stackrel{!}{=} 0 \tag{10}$$

yielding the variation

$$\delta Q[\tilde{\epsilon}] = \frac{k}{2\pi}\oint_{S^1} \tilde{\epsilon}\,\mathcal{L}'\,\delta\mathcal{L}\;\mathrm{d}x^+\,. \tag{11}$$

Since there is no functional of $\mathcal{L}$ the variation of which yields $\mathcal{L}'\,\delta\mathcal{L}$, the charges (11) are not integrable in field space.

2 Slicings of state space

While we intend to return to Chern–Simons theories in the next sections for detailed and physically motivated examples, in this section we step back and consider generic gauge- or gravity theories with boundaries and address the issue of state-dependent transformation parameters in full generality, if somewhat abstractly.

Denote the gauge parameter(s) by ϵ and the field(s) by ϕ. The variation of the co-dimension 2 boundary charges

$$\delta Q = \int_{\partial\Sigma} \epsilon(\phi)\, k(\phi)\, \delta\phi \tag{12}$$

is in general not integrable in field space if ϵ is some generic functional of ϕ. We parametrize ϵ as

$$\epsilon = \tilde\epsilon\, c(\phi) \qquad\qquad \delta\tilde\epsilon = 0 \tag{13}$$

with some generic functional $c(\phi)$. If the functional k formally can be written as

$$k = \frac{\delta F}{\delta\phi} + F\,\frac{\delta \ln c}{\delta\phi} \tag{14}$$

then the associated charges

$$Q = \int_{\partial\Sigma} \epsilon\, F(\phi) \tag{15}$$

are integrable in field space.

The proof of this statement is straightforward: varying the charges (15) yields

$$\delta Q = \int \Big(\tilde\epsilon\, c(\phi)\,\frac{\delta F(\phi)}{\delta\phi} + \tilde\epsilon\,\frac{\delta c(\phi)}{\delta\phi}\,F(\phi)\Big)\,\delta\phi = \int \tilde\epsilon\, c(\phi)\,\Big(\frac{\delta F(\phi)}{\delta\phi} + F(\phi)\,\frac{\delta \ln c(\phi)}{\delta\phi}\Big)\,\delta\phi$$

which by virtue of (14) and (13) recovers precisely (12).

Note that in the discussion above we have not altered the boundary conditions. Instead, we have injected (or changed) the state-dependence of the transformation parameters appearing the boundary-condition preserving gauge transformations. We refer to such a procedure as a "change of slicing of the state space". The idea behind this nomenclature is as follows. Our state space is specified by the bulk theory and the boundary conditions, so we are not changing the state space when changing the state dependence of the transformation parameters. However, we change the boundary charges, so the same state can have different values for the boundary charges depending on our choice of state space slicing.

Changes of slicings are in general *not* a change of basis in the asymptotic symmetry algebra. To see this, one counter example is sufficient. Consider a situation where the boundary charges

$$Q[\epsilon] = \frac{k}{2\pi}\oint \epsilon(\varphi)\, J(\varphi)\,\mathrm{d}\varphi \tag{16}$$

obey a $\hat u(1)$ current algebra

$$\delta_\epsilon J = \epsilon' \qquad \Rightarrow \qquad \delta_{\epsilon_1} Q[\epsilon_2] = \{Q[\epsilon_1],\, Q[\epsilon_2]\} = \frac{k}{2\pi}\oint \epsilon_2\epsilon_1'\,\mathrm{d}\varphi\,. \tag{17}$$

Changing the slicing as

$$\epsilon = \tilde\epsilon\, J \qquad\qquad \delta\tilde\epsilon \overset{!}{=} 0 \tag{18}$$

yields new charges

$$Q[\tilde\epsilon] = \frac{k}{4\pi}\oint \tilde\epsilon\, J^2\,\mathrm{d}\varphi \tag{19}$$

that produce a new asymptotic symmetry algebra (using $\delta J = \tilde\epsilon J' + \tilde\epsilon' J$)

$$\delta_{\tilde\epsilon_1} Q[\tilde\epsilon_2] = \{Q[\tilde\epsilon_1],\, Q[\tilde\epsilon_2]\} = Q[\tilde\epsilon_2\tilde\epsilon_1' - \tilde\epsilon_1\tilde\epsilon_2'] \tag{20}$$

recognized as Witt algebra. There is no change of basis from a $\hat u(1)$ current algebra (abelian but with center) to a Witt algebra (without center but non-abelian).

3 Most general boundary conditions for AdS_3

In this section, we return to AdS_3 Einstein gravity, with the intention to go beyond BH boundary conditions. In fact, we are aiming for the loosest set of boundary conditions, in the sense that we obtain the maximal number of towers of charges (for BH this number was 2).

For inspiration, we start with the quantum Hall system, which is also described effectively by Chern–Simons theory, since the electrons are essentially confined to a plane in such systems (hence, we are effectively dealing with a $(2+1)$-dimensional gauge theory with boundaries).

While the physical requirements imposed on quantum Hall systems differ from gravity (for instance, in quantum Hall systems we do not mind if the connection trivializes locally, $a = 0$, whereas in gravity such a trivialization would render the metric singular and is thus forbidden), we will still get more concrete ideas how to generalized BH boundary conditions from that analysis.

3.1 Inspiration from quantum Hall system

Some aspects of quantum Hall physics are captured by an effective low-energy description of an abelian gauge field in 2+1 dimensions, for details see these lecture notes by Tong. In a derivative expansion, the dominant part of the low-energy effective action is the Chern–Simons term[1]

$$I_{\text{CS}}[a] = \frac{k}{4\pi} \int a \wedge \mathrm{d}a \,. \tag{21}$$

In what follows, we use Minkowski coordinates t, x, y.

We assume an interface at $y = 0$ between the quantum Hall phase and the vacuum. Thus, we need to impose boundary conditions at $y = 0$ on the connection a to get a well-defined variational principle. Indeed, the first variation of the action

$$\delta I_{\text{CS}}[a] \approx \frac{k}{4\pi} \int \mathrm{d}\big(a \wedge \delta a\big) \sim \big(a_t\, \delta a_x - a_x\, \delta a_t\big)\big|_{y=0} \tag{22}$$

is on-shell a boundary term that we need to enforce to be zero for consistency, unless we add some suitable boundary contributions to the action (21); see below for an exploration of this possibility.

Some simple possibilities for consistent boundary conditions are $a_t|_{y=0} = 0$ or $a_x|_{y=0} = 0$. They can be combined into a 1-parameter family of boundary conditions

$$\big(a_t - v\, a_x\big)\big|_{y=0} = 0 \tag{23}$$

where v is some fixed but arbitrary parameter with the dimension of a velocity. This parameter is not present in the action but enters solely through our choice of boundary conditions. For simplicity and concreteness we set $v = 0$.

We are interested in boundary excitations, i.e., gauge modes that become physical at the boundary. Since the bulk EOM dictate vanishing field strength, $f = 0$, we can write the connection locally as $a = \mathrm{d}\phi$ with some scalar field ϕ (which is a gauge parameter). Inserting $a = \mathrm{d}\phi$ into the Chern–Simons action (21) yields a boundary term

$$I_{\text{FJ}}^{v=0}[\phi] = \frac{k}{4\pi} \int_{y=0} \mathrm{d}t\, \mathrm{d}x\, \partial_t\phi\, \partial_x\phi \,. \tag{24}$$

[1]The Chern–Simons level is quantized in integers to render this action gauge invariant under large gauge transformations. This aspect is important for quantum Hall physics but not for the point we are making here.

This action is the $v = 0$ case of the Floreanini–Jackiw action and it describes a self-dual scalar field in two dimensions. (The crucial detail is the single time-derivative in the action, as opposed to actions of Klein–Gordon type with two time-derivatives that propagate both chiralities.) Thus, the boundary excitations of the effective quantum Hall description provided by the low energy effective action (21) with the boundary conditions (23) are described by a Floreanini–Jackiw scalar field that lives at the boundary. While we have shown this above only for $v = 0$, the same statement is true for finite v, see section 6.1.2 in Tong's lecture notes.

There is a relevant subtlety if we assume periodicity in x: we should not enforce the Floreanini–Jackiw scalar field ϕ to be periodic in x but instead allow for a winding mode, i.e., a term linear in x. The usual nomenclature is to call such a field "quasi-periodic": $\phi(x + 2\pi) = \phi(x) + 2\pi\, w$, where w is an arbitrary constant associated with the linear term in ϕ. Physically, the winding mode is necessary to allow for a zero mode of the charge density, $\rho := \partial_x\phi$, so that the total charge $Q = \oint \mathrm{d}x\, \rho = \oint \mathrm{d}x\, \partial_x\phi$ can be non-zero. In group-theoretic slang, what we allow here is that the group element g in the gauge connection $a = g^{-1}\, \mathrm{d}g$ is not necessarily single-valued. We shall keep this in mind for our gravity considerations below.

Finally, let us consider an enticing alternative to the boundary conditions (23). If we add to the Chern–Simons action (21) a boundary term proportional to

$$\int_{y=0} \mathrm{d}t\, \mathrm{d}x\, a_t a_x \tag{25}$$

then by choosing the coefficient suitably we can either cancel the $a_t\, \delta a_x$ term or the $a_x\, \delta a_t$ term in the variation (22) (the remaining boundary term then acquires a factor 2). Assuming we do the former, we end up with a boundary term proportional to $a_x\, \delta a_t$. To get a well-defined variational principle we can thus impose the boundary conditions

$$\delta a_t = 0 \qquad\qquad \delta a_x \neq 0\,. \tag{26}$$

Thus, a_t is fixed and a_x is allowed to vary. Different names for these quantities exist in various parts of the physics literature: a_t is called "source", "chemical potential", "state-independent", "non-normalizable", "intensive"; a_x is called "vev", "charge", "state-dependent", "normalizable", "extensive". Note that it is natural in gauge theories to identify a_t as chemical potential — indeed, this is how the chemical potential usually is introduced in gauge theories.

This rudimentary study of quantum Hall physics reinforces the necessity of physical input when choosing boundary conditions. However, it also suggests some "natural" choices. In the following, we apply these lessons to 3d gravity in the Chern–Simons formulation.

3.2 Chern–Simons formulation

In gravity, we do not want our connection to become zero at the boundary, like in (23). However, we would not mind fixing the variation of the time component of the connection, like in (26). Thus, let us make the ansatz (pioneered in `1608.01308`)

$$A^+ = b^{-1}\big(\,\mathrm{d}{+}a^+(t,\varphi)\big)b \qquad\qquad A^- = b\big(\,\mathrm{d}{+}a^-(t,\varphi)\big)b^{-1} \tag{27}$$

with the group element

$$b = e^{L_{-1}}e^{\rho L_0} \tag{28}$$

and the boundary connections

$$a_t^\pm = \mu_1^\pm(t,\varphi)L_1 + \mu_0^\pm(t,\varphi)L_0 + \mu^\pm(t,\varphi)L_{-1} \tag{29}$$

$$a_\varphi^\pm = \mathcal{L}_1^\pm(t,\varphi)L_1 + \mathcal{L}_0^\pm(t,\varphi)L_0 + \mathcal{L}_{-1}^\pm(t,\varphi)L_{-1} \tag{30}$$

where $\delta\mu_a^\pm = 0$ and $\mathcal{L}_a^\pm \neq 0$. Thus, we have a total of 6 state-independent functions and 6 state-dependent functions, and therefore should expect 6 towers of charges.

To obtain the charges, we study first the boundary condition preserving gauge transformations (for brevity, consider just the +-chirality and drop the sign decorations)

$$\delta_\epsilon A = \mathrm{d}\epsilon + [A, \epsilon] \stackrel{!}{=} \mathcal{O}(\delta A) \tag{31}$$

As in previous lectures, it is convenient to redefine the gauge parameters with the same group element b,

$$\epsilon = b^{-1}\,\hat{\epsilon}\,b = b^{-1}\big(\epsilon_1(t,\varphi)L_1 + \epsilon_0(t,\varphi)L_0 + \epsilon_{-1}(t,\varphi)L_{-1}\big)b \tag{32}$$

which reduces the condition (31) to

$$\delta_\epsilon a = \mathrm{d}\hat{\epsilon} + [a, \hat{\epsilon}] \stackrel{!}{=} \mathcal{O}(\delta a) = \mathcal{O}(1)\ \mathrm{d}\varphi\,. \tag{33}$$

The φ-component of the equations (33) lead no constraints at all since we switched on all algebraic components in a_φ and allowed them to vary arbitrarily and independently from each other. The t-component establishes three constraints of the form $\partial_t\epsilon_a = \dots$ that fix the behavior of the gauge parameters ϵ_a under time evolution.

Using the background independent result for the charge variation in Chern–Simons theories,

$$\delta Q[\epsilon] = \frac{k}{2\pi}\oint_{S^1}\mathrm{tr}\big(\epsilon\,\delta A\big) \tag{34}$$

yields charge variations

$$\delta Q[\epsilon] = \frac{k}{2\pi}\oint_{S^1}\epsilon_a\,\delta\mathcal{L}_b\,\kappa^{ab} \tag{35}$$

where κ^{ab} is the Cartan–Killing metric for $\mathrm{sl}(2,\mathbb{R})$. Assuming the parameter functions ϵ_a to be state-independent (our "natural" choice of slicing) renders the charges (35) integrable in field space.

Introducing Fourier modes J^a_n (and replacing Poisson brackets by commutators, with suitable factors i attached) yields two copies of $\mathrm{sl}(2,\mathbb{R})$ current algebras

$$[J^a_n,\, J^b_m] = (a-b)\,J^{a+b}_{n+m} - k\,n\,\kappa^{ab}\,\delta_{n+m,\,0} \tag{36}$$

as asymptotic symmetry algebra for AdS_3 Einstein gravity with the loosest set of boundary conditions. The algebra (36) is non-abelian and has a central extension proportional to k.

For our choice of slicing, $\delta\epsilon_a = 0$, the charges (35) are integrable in field space. Moreover, they are manifestly finite (since they do not depend on the radial coordinate ρ), and non-zero on all states where at least one of the functions $\mathcal{L}_a$ is non-zero. So apart from conservation, we have checked all the usual nice properties of boundary charges. Conservation of the charges,

$$\partial_t\delta Q[\epsilon] = 0 \tag{37}$$

can be proven using the EOM. For details see section 4.1 in 1608.01308.

Moreover, adding a boundary term analogous to (25),

$$\Gamma = I_{\textsc{cs}} - \frac{k}{4\pi}\int\mathrm{d}t\,\mathrm{d}\varphi\,\mathrm{tr}\big(A_tA_\varphi\big) \tag{38}$$

leads to a well-defined variational principle for the action Γ,

$$\delta\Gamma \approx -\frac{k}{2\pi}\int\mathrm{d}t\,\mathrm{d}\varphi\,\mathrm{tr}\big(a_\varphi\,\delta a_t\big) \tag{39}$$

since our boundary conditions enforce fixed chemical potentials, $\delta a_t = 0$. This is completely analogous to the example discussed in the last few paragraphs of the quantum Hall section.

Before recovering known examples of boundary conditions with fewer towers of charges (including the original BH), we translate the results above into the metric formulation.

3.3 Metric formulation

There are some interesting subtleties in the metric formulation. First of all, had we chosen the group element $b = e^{\rho L_0}$ instead of (28) the metric would not feature all six state-dependent functions. This may seem confusing since the Chern–Simons formulation does not care about the particular choice of the group element b (indeed, the boundary charges are independent of b), but recall that perfectly regular Chern–Simons field configurations can correspond to singular metric configurations. The simplest pertinent example is to choose the group element $b = \mathbb{1}$, in which case the metric degenerates to a 2d metric and is thus singular from a 3d perspective. In short, we chose the group element b as in (28) to get non-degenerate metrics that feature all six state-dependent functions and all six chemical potentials. (Having said this, other such choices of b are possible.)

With our choices, the boundary conditions on the metric are given by a generalized Fefferman–Graham expansion,

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + 2\Big(e^\rho N_i^{(0)} + N_i^{(1)} + e^{-\rho} N_i^{(2)} + \mathcal{O}(e^{-2\rho})\Big)\,\mathrm{d}\rho\,\mathrm{d}x^i + \Big(e^{2\rho} g_{ij}^{(0)} + e^\rho g_{ij}^{(1)} + g_{ij}^{(2)} + \mathcal{O}(e^{-\rho})\Big)\,\mathrm{d}x^i\,\mathrm{d}x^j \tag{40}$$

where the quantities $N_t^{(n)}$ and $g_{tt}^{(n)}$ (with $n = 0, 1, 2$) are fixed and determined by the six chemical potentials, while $N_\varphi^{(n)}$ and $g_{\varphi\varphi}^{(n)}$ are allowed to vary independently and are determined by the six state-dependent functions. The mixed components $g_{t\varphi}^{(n)}$ are also given in terms of the chemical potentials and state-dependent functions, but do not contain additional indepedent functions. For further details see section 3.2 of 1608.01308. The lesson from the result (40) is that gauge-fixing to Fefferman–Graham gauge (where all the N_i vanish) comes with loss of generality!

The AKVs follow from solving the defining property

$$\mathcal{L}_\xi g_{\mu\nu} \overset{!}{=} \mathcal{O}(\delta g_{\mu\nu}) \tag{41}$$

leading to an expansion

$$\xi^\mu(t,\varphi,\rho) = \xi^\mu_{(0)}(t,\varphi) + e^{-\rho}\,\xi^\mu_{(1)}(t,\varphi) + e^{-2\rho}\,\xi^\mu_{(2)}(t,\varphi) + \mathcal{O}(e^{-3\rho}) \tag{42}$$

where explicit results for the $\xi^\mu_{(n)}$ can be found in section 3.3 of 1608.01308.

If we assume that the leading order functions appearing in $\xi^\mu_{(0)}$ are all state-independent we obtain the usual Lie bracket algebra of the AKVs as asymptotic symmetry algebra.

However, if we assume that instead the gauge parameters Λ_I in the Chern–Simons formulation are state-independent we have to use instead the adjusted (sometimes also called "modified") bracket

$$[\xi_1,\,\xi_2]^\mu_{\text{adjusted}} = \mathcal{L}_{\xi_1}\xi_2^\mu - \delta^g_{\xi_1}\xi_2^\mu + \delta^g_{\xi_2}\xi_1^\mu \tag{43}$$

where $\delta^g_{\xi_1}\xi_2^\mu$ denotes the change induced in $\xi_2^\mu(g)$ due to the variation $\delta^g_{\xi_1} g_{\mu\nu} := \mathcal{L}_{\xi_1} g_{\mu\nu}$. The last two terms on the right hand side of (43) are absent in the usual Lie bracket.

It can be shown that the ASA generated by the Lie bracket is different from the ASA generated by the modified Lie bracket. Given what we know already about changes of slicings, this is not a surprise. It was checked in 1608.01308 that the modified Lie brackt algebra of the AKVs recovers the centerless version of the sl$(2,\mathbb{R})$ current algebras (36), as anticipated on general grounds.

3.4 Recovering the AdS$_3$ zoo

Since we have now the most general set of AdS$_3$ boundary conditions available to us (up to changes of slicings, of course), we should be able to recover all known special cases, like BH boundary conditions. This is indeed the case. For a detailed survey of all relevant cases see section 4 of 1608.01308.

Here, we confine ourselves to recovering BH boundary conditions. Starting from the most general boundary conditions of section 3.2, we impose additional constraints on the state-depedent functions.

$$\mathcal{L}_1^+ = \mathcal{L}_{-1}^- = 1 \qquad \mathcal{L}_0^+ = \mathcal{L}_0^- = 0 \qquad \mathcal{L}_{-1}^+, \mathcal{L}_1^- \;:\; \text{arbitrary} \tag{44}$$

This procedure is also known as "Drinfeld–Sokolov reduction" in the literature. The chemical potentials are constrained by the EOM, see section 4.2.1 of 1608.01308 for details.

An important consequence of the Drinfeld–Sokolov reduction is that the canonical realization of the asymptotic symmetries reduces the two sl$(2,\mathbb{R})$ current algebras (36) to two Virasoro algebras with the BH values of the central charge, $c = 6k$.

4 Boundary actions

Inspired by the Floreanini–Jackiw boundary action (24) encountered in the quantum Hall system it is natural to ponder what are the boundary actions of various gravity theories, e.g., for AdS$_3$ Einstein gravity with BH boundary conditions. You can find the answer to this, for instance, in section 3 of 1906.10694. Here are some key steps.

The spatial part of the connection can be locally represented in terms of a group element G,

$$A_i = G^{-1}\partial_i G \qquad G(t, \varphi + 2\pi, \rho) = h(t)\, G(t, \varphi, \rho) \tag{45}$$

which, however, is globally allowed to be non-trivial, i.e., the holonomy function $h(t)$ need not be the identity. It turns out that in the boundary action one can get rid of the radial dependence, so that effectively for our purposes the group element G can be assumed to depend only on the boundary coordinates t, φ.

Technically, it is convenient to Gauss-decompose the SL$(2,\mathbb{R})$ group element,

$$G = e^{XL_1} e^{\Phi L_0} e^{YL_{-1}}\,. \tag{46}$$

The (quasi-)periodicity properties of the three functions are $Y(t, \varphi + 2\pi) = Y(t, \varphi)$,

$$X(t, \varphi + 2\pi) = e^{-2\pi J_0(t)} X(t, \varphi) \qquad \Phi(t, \varphi + 2\pi) = \Phi(t, \varphi) + 2\pi\, J_0(t) \tag{47}$$

where $h(t) = \exp\left(2\pi\, J_0(t)\, L_0\right)$.

For BH boundary conditions it turns out that the boundary action reduces to the Floreanini–Jackiw action of a self-dual scalar field with finite velocity

$$I_{\text{BH}}[\Phi] \sim I_{\text{FJ}}^{v=0}[\Phi] + \mu \int_{\partial\mathcal{M}} \mathrm{d}t\, \mathrm{d}\varphi \left((\Phi')^2 + 2\Phi''\right) \sim I_{\text{FJ}}^{v=\mu}[\Phi]\,. \tag{48}$$

See section 4.2 in 1906.10694 for details. The combination appearing in the Hamiltonian,

$$\mathcal{L} = (\Phi')^2 + 2\Phi'' \tag{49}$$

up to normalization can be interpreted as the Sugawara-constructed stress tensor appearing in BH boundary conditions. The fields X and Y are determined by constraints as

$$X' = e^{-\Phi} \qquad \Phi' = -2Y\,. \tag{50}$$

Expressing the boundary action instead as functional depending on the field X yields the Alekseev–Shatashvili boundary action, featuring the famous Schwarzian derivative $\{X, \varphi\} = X'''/X'' - \frac{3}{2}(X''/X')^2$ as Hamiltonian term. This is the geometric action of the Virasoro group on its coadjoint orbit.