# $\mathbf{AdS_3/CFT_2}$

If you have followed all my OIST lecture, you will have noticed some redundancy in my lecture notes. This is by design, so that individual lecture notes can be read independently, especially for people who only come to some of the lectures. So do not be alarmed when you see some stuff on the next pages that you have already seen in the first three lectures.

The main point of this lecture is to show in which sense Einstein gravity in three spacetime dimensions with negative cosmological constant ($AdS_3$) is equivalent to a two-dimensional conformal field theory ($CFT_2$), a special case of Maldacena's AdS/CFT correspondence. After having shown this, we will mention some alternative boundary conditions where the conclusion about the holographic dual changes.

## 1 Précis of $\mathbf{AdS_3}$ Einstein gravity á la Brown–Henneaux

The bulk action of $AdS_3$ Einstein gravity $I_{\text{AdS}_3} = I_{\text{CS}}[A^+] - I_{\text{CS}}[A^-]$ is the difference of two $\mathrm{sl}(2,\mathbb{R})$ Chern–Simons actions

$$I_{\text{CS}}[A^\pm] = \frac{k}{4\pi}\int_{\mathcal{M}} \mathrm{Tr}\left(A^\pm \wedge \mathrm{d}A^\pm + \tfrac{2}{3}\,A^\pm \wedge A^\pm \wedge A^\pm\right) \tag{1}$$

with level $k = \frac{\ell}{4G}$, where $\ell$ is the AdS radius (related to the cosmological constant by $\Lambda = -\frac{1}{\ell^2}$) and $G$ is Newton's constant in 3d. In a convenient basis for the Lie algebra generators $L_a$ appearing, e.g., in $A^\pm = A^{a\,\pm} L_a$ their commutators read $[L_a, L_b] = (a-b)L_{a+b}$ with $a, b \in \{-1, 0, 1\}$. In this basis, the non-vanishing traces of bilinears are $\mathrm{Tr}(L_0^2) = \frac{1}{2}$ and $\mathrm{Tr}(L_1 L_{-1}) = \mathrm{Tr}(L_{-1}L_1) = -1$. Dualized spin-connection and dreibein follow from the connections as linear combinations, $A^{a\,\pm} = \omega^a \pm \frac{1}{\ell}\,e^a$. The metric is then determined by a bilinear in the dreibein

$$g_{\mu\nu} = \frac{\ell^2}{2}\,\mathrm{Tr}\Big(\left(A^+_\mu - A^-_\mu\right)\left(A^+_\nu - A^-_\nu\right)\Big)\,. \tag{2}$$

Brown–Henneaux (BH) boundary conditions expressed in highest-weight gauge for the connection are given by (btw, we assume $\mathcal{M}$ to be a filled cylinder or torus)

$$A^\pm = e^{\mp\rho/\ell\,L_0}\left(\mathrm{d} + a^\pm(x^+, x^-)\right) e^{\pm\rho/\ell\,L_0} \tag{3}$$

with the "boundary connection" ($\mathcal{L}^\pm(x^\pm)$ are state-dependent functions)

$$a^+ = \left(L_{+1} - \mathcal{L}^+(x^+)\,L_{-1}\right)\frac{\mathrm{d}x^+}{\ell} \qquad \Rightarrow \qquad \delta a^+ = -\delta\mathcal{L}^+(x^+)\,L_{-1}\,\frac{\mathrm{d}x^+}{\ell} \tag{4}$$

$$a^- = \left(L_{-1} - \mathcal{L}^-(x^-)\,L_{+1}\right)\frac{\mathrm{d}x^-}{\ell} \qquad \Rightarrow \qquad \delta a^- = -\delta\mathcal{L}^-(x^-)\,L_{+1}\,\frac{\mathrm{d}x^-}{\ell}\,. \tag{5}$$

In the metric formulation, this generates a Fefferman–Graham expansion

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + \left(e^{2\rho/\ell}\,\gamma^{(0)}_{\mu\nu} + \gamma^{(2)}_{\mu\nu} + \mathcal{O}(e^{-2\rho/\ell})\right)\mathrm{d}x^\mu\,\mathrm{d}x^\nu \tag{6}$$

with $\gamma^{(0)}_{\mu\nu} = \eta_{\mu\nu}$ where $\eta_{\pm\mp} = \frac{1}{2}$, $\eta_{\pm\pm} = 0$, and $\gamma^{(2)}_{\pm\pm} = \mathcal{L}^\pm(x^\pm)$, $\gamma^{(2)}_{\pm\mp} = 0$, $\delta\gamma^{(2)}_{\pm\pm} \neq 0$.

In what follows, we set $\ell = 1$. The BH boundary conditions are preserved, i.e., $\delta_\varepsilon A = \mathrm{d}\varepsilon + [A, \varepsilon] = \mathcal{O}(\delta A)$, by gauge transformations $\varepsilon^\pm = e^{\mp\rho\,L_0}\,\hat\varepsilon\,e^{\pm\rho\,L_0}$ with

$$\hat\varepsilon^\pm = \varepsilon^\pm(x)\,L_{\pm1} - \varepsilon^{\pm\,\prime}(x)\,L_0 + \left(\tfrac{1}{2}\,\varepsilon^{\pm\,\prime\prime}(x) - \mathcal{L}^\pm(x)\,\varepsilon^\pm(x)\right)L_{\mp1} + \dots \tag{7}$$

where the ellipsis denotes subleading terms. The associated boundary charges

$$Q^\pm[\varepsilon^\pm] = \frac{k}{2\pi}\oint_{S^1}\varepsilon^\pm(x^\pm)\,\mathcal{L}^\pm(x^\pm)\,\mathrm{d}x^\pm \tag{8}$$

depend on the state-dependent functions $\mathcal{L}^\pm(x^\pm)$. (EOMs dictate $\partial_\mp\mathcal{L}^\pm = 0$.)

## 2 $\text{AdS}_3$ Einstein gravity á la BH is a $\text{CFT}_2$ if it exists

The BH boundary charges (8) are integrable in field space (since we assumed $\varepsilon^\pm(x)$ to be state-independent!), finite for $\rho\to\infty$ (in fact, independent from $\rho$!), non-zero for infinitely many physical states, zero for all gauge trafos that fall off so quickly that $\varepsilon^\pm(x^\pm)=0$, and conserved in time as a connsequence of the EOMs (in this context a.k.a. "holographic Ward identities").

Let us elaborate on the last point. If we had not assumed $\partial_\mp\mathcal{L}^\pm=0$ already in our ansatz (3), the EOMs would have enforced these conditions. If we rewrite the lightcone coordinates in terms of time $t$ and angle $\varphi$, $x^\pm=\varphi\pm t$ then we see that we can trade time-derivatives for (plus or minus) angle-derivatives when acting on functions that depend only on one chiral combination, either $x^+$ or $x^-$, but not both. Therefore, the time derivative of the charges

$$\partial_t Q^\pm[\varepsilon^\pm]\propto\partial_t\oint_{S^1}\varepsilon^\pm(x^\pm)\,\mathcal{L}^\pm(x^\pm)\,\mathrm{d}x^\pm\propto\oint_{S^1}\partial_\varphi\big(\varepsilon^\pm(x^\pm)\,\mathcal{L}^\pm(x^\pm)\big)\,\mathrm{d}\varphi=0 \tag{9}$$

vanishes, as long as $\varepsilon^\pm(x^\pm)$ and $\mathcal{L}^\pm(x^\pm)$ are globally defined on the $S^1$.

Since the state-dependent functions transform with an infinitesimal Schwarzian,

$$\delta_{\varepsilon^\pm}\mathcal{L}^\pm=\varepsilon^\pm\mathcal{L}^{\pm\,\prime}+2\varepsilon^{\pm\,\prime}\mathcal{L}^\pm-\frac{1}{2}\,\varepsilon^{\pm\,\prime\prime\prime} \tag{10}$$

the canonical realization of the asymptotic symmetry algebra

$$\delta_{\varepsilon_1^\pm}Q[\varepsilon_2^\pm]=\{Q[\varepsilon_1^\pm],\,Q[\varepsilon_2^\pm]\}=Q[\varepsilon_1^{\pm\,\prime}\varepsilon_2^\pm-\varepsilon_2^{\pm\,\prime}\varepsilon_1^\pm]-\frac{k}{4\pi}\oint_{S^1}\varepsilon_1^{\pm\,\prime\prime\prime}\,\varepsilon_2^\pm\,\mathrm{d}x^\pm \tag{11}$$

has a non-trivial central extension proportional to the Chern–Simons level $k$.

In Fourier modes for the generators, $L_n^\pm:=Q[e^{inx^\pm}]+\frac{k}{4}\,\delta_{n,0}$, the mode version of the asymptotic symmetry algebra (11)[1]

$$\boxed{-i\big\{L_n^\pm,\,L_m^\pm\big\}=(n-m)\,L_{n+m}^\pm+\frac{k}{2}\,(n^3-n)\,\delta_{n+m,\,0}} \tag{12}$$

is recognized as two copies of the Virasoro algebra with central charges

$$\boxed{c=6k=\frac{3\ell}{2G}} \tag{13}$$

that tend to infinity in the classical limit on the gravity side, $G\to0$.

In the quantum theory we replace $-i\{,\}$ by commutators. What we have shown above (and what BH have shown in 1986) is that the physical phase space (and in the quantum theory the physical Hilbert space) of $\text{AdS}_3$ Einstein gravity (with BH boundary conditions) must fall into representations of two copies of the Virasoro algebra. This is the defining property of a $\text{CFT}_2$. In this sense, BH have proved $\text{AdS}_3/\text{CFT}_2$ more than a decade before people knew about AdS/CFT. (However, they did not show that $\text{AdS}_3$ Einstein gravity exists as a consistent quantum theory. It might not exist. Nor is it clear which $\text{CFT}_2$ precisely is supposed to be dual to the gravity theory. See 0706.3359 for an attempt.)

A caveat before moving on to some explicit checks/implications of $\text{AdS}_3/\text{CFT}_2$: most of the checks below are "only" checks of the asymptotic symmetries, i.e., they relate gravity observables to $\text{CFT}_2$ observables without the need (and without the capability) of precisely identifying the $\text{CFT}_2$ beyond its values of the central charges. For such an identification, most likely we need a UV completion and have to go beyond the (super)gravity approximation (see, e.g., 1812.01007 for such a completion within string theory). Formulated positively, all the checks we are going to discuss are completely universal and do only depend on the existence of a UV completion but not on its details.

[1] The constant shifts of the zero modes $L_0^\pm$ by the Casimir energy $-\frac{c}{24}$ guarantees that the Virasoro algebra (12) has an sl$(2,\mathbb{R}\oplus$ SL$(2,\mathbb{R})$-invariant subalgebra generated by $L_1^\pm$, $L_0^\pm$, $L_{-1}^\pm$.

# 3 Elementary checks of $AdS_3/CFT_2$

Let us start with some elementary checks that do not require a lot of calculation. As always, symmeries are a good place to start.

## 3.1 Match of global symmetries

The defining property of a $CFT_2$ is conformal symmetry: the physical Hilbert space must fall into representations of the conformal algebra, which in two dimensions consists of two copies of the Virasoro algebra. For AdS/CFT to have any chance to be true it must be the case that the physical Hilbert space (or in the classical approximation the physical phase space) falls into representations of two copies of the Virasoro algebra. In the previous sections we have proved that this is true for $AdS_3$ Einstein gravity with BH boundary conditions. Starting from a QFT with Poincaré plus scale symmetries it is even possible to give a slick derivation of the AdS line element as follows. Suppose we have the daring idea to use energy $E$ as additional coordinate, for instance to geometrize renormalization group flow of a $D$-dimensional QFT. The most general line-element in $D+1$ dimensions compatible with Poincaré symmetries is then given by

$$\mathrm{d}s^2 = f_1(E)\,\mathrm{d}E^2 + f_2(E)\,\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \qquad \mu,\nu = 0..(D-1) \tag{14}$$

with two unknown scalar functions $f_i(E)$. Suppose further that our QFT has scale symmetry

$$x^\mu \to \lambda x^\mu \qquad\qquad E \to E\lambda^{-1}\,. \tag{15}$$

Then the most general line-element (14) compatible with the scale symmetry (15) is given by

$$\mathrm{d}s^2 = \ell^2\left(\frac{\mathrm{d}E^2}{E^2} + E^2\,\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu\right) \tag{16}$$

where we introduced some arbitrary but fixed length scale $\ell$ to have the correct units. The metric (16) is Poincaré patch AdS in $D+1$ dimensions, with AdS radius $\ell$ and the asymptotic boundary at $E \to \infty$.

## 3.2 Absence of gravitational/Lorentz anomaly

Any $CFT_2$ is characterized, among other things, by the values of the two central charges. In the absence of gravitational or Lorentz anomalies, the left and right central charges must be equal in magnitude. In the previous sections, we have proved this is true for $AdS_3$ Einstein gravity with BH boundary conditions.

## 3.3 Compatibility with non-triviality and unitarity

All unitary CFTs that are non-trivial must have strictly positive central charges. In the previous sections, we have proved his is true for $AdS_3$ Einstein gravity with BH boundary conditions as long as Newton's constant is positive.

## 3.4 Heuristics from supergravity limit

From the way the string theory construction á la Maldacena works, it is clear that AdS/CFT is a duality of strong/weak type, meaning that strongly coupled CFTs are mapped to weakly coupled gravity theories. Heuristically, we expect that the supergravity limit, which is very simple, should produce CFTs that are very complicated. More concretely, in the classical limit of vanishing Newton constant, $G \to 0$, the CFT central charge is expected to diverge, $c \to \infty$. In the previous section we have proved his is true for $AdS_3$ Einstein gravity with BH boundary conditions.

# 4 Thermodynamical check of $AdS_3/CFT_2$ a.k.a. Cardyology

## 4.1 Bekenstein–Hawking entropy of BTZ black holes

Thermal states in a $CFT_2$ are dual to BTZ black holes, with metric ($\varphi \sim \varphi + 2\pi$)

$$\mathrm{d}s^2_{\text{BTZ}} = -\frac{(r^2-r_+^2)(r^2-r_-^2)}{\ell^2 r^2}\,\mathrm{d}t^2 + \frac{\ell^2 r^2\,\mathrm{d}r^2}{(r^2-r_+^2)(r^2-r_-^2)} + r^2\left(\mathrm{d}\varphi - \frac{r_+ r_-}{\ell r^2}\,\mathrm{d}t\right)^2 \tag{17}$$

Their Bekenstein–Hawking entropy

$$S = \frac{2\pi\,r_+}{4G} = 2\pi\sqrt{\frac{c\,L}{6}} + 2\pi\sqrt{\frac{c\,\bar{L}}{6}} \tag{18}$$

with $L = \ell\,\mathcal{L}^+/(4G)$, $\bar{L} = \ell\,\mathcal{L}^-/(4G)$ is of Cardy-type.

## 4.2 Cardy entropy of states dual to BTZ black holes

As reviewed in appendix A, the last expression for the entropy is a generic result for thermal states in $CFT_2$, assuming the existence of a gap in the spectrum of states. Therefore, it is worthwhile to verify whether or not the gravity state space naturally leads to such a gap.

BTZ black holes and global $AdS_3$ are examples of zero-mode solutions, i.e., states where the functions $\mathcal{L}^\pm(x^\pm)$ are both constant. The translation between gravity- and CFT-notation is

$$\frac{r_+^2 + r_-^2}{2\ell^2} = m = \mathcal{L}^+ + \mathcal{L}^- \qquad\qquad -\frac{r_+ r_-}{\ell^2} = \frac{j}{\ell} = \mathcal{L}^+ - \mathcal{L}^- \tag{19}$$

where $m$ is the mass parameter and $j$ the angular momentum parameter (the corresponding charges are multiplied with the Chern–Simons level $k$).

On the gravity side we exclude naked singularities, including naked conical defects, implying $\mathcal{L}^\pm \geq 0$. Thus, starting from the value zero we have a continuous spectrum of states. So far, so bad. However, this continuous spectrum of states describes just the black hole sector of our theory (and the massless limit to so-called Poincaré patch AdS, $\mathcal{L}^\pm = 0$). To find/exclude a gap we still need to identify the gravity solution dual to the $CFT_2$ vacuum. From the CFT perspective this must be an $sl(2,\mathbb{R})\oplus sl(2,\mathbb{R})$-invariant ground state, with a mass that is gapped by $-\frac{c}{12}$ from the rest of the spectrum. On the gravity side, the ground state is global $AdS_3$, which indeed has the correct six Killing symmetries. It is given by the metric (17) with $r_+^2 = -\ell^2$, $r_- = 0$, corresponding to a mass gap of $-\frac{\ell}{8G}$. Using the BH result (13) for the central charge reveals that we have the correct mass gap, $-\frac{c}{12} = -\frac{\ell}{8G}$.

## 4.3 Semiclassical corrections to the entropy

The classical matching (17) between Bekenstein–Hawking entropy and Cardy entropy can be generalized to a semi-classical matching, by taking into account 1-loop corrections on both sides of the correspondence. Generically, this leads to results for the entropy of the form

$$S = S_0 - q\,\ln S_0 + \mathcal{O}(1) \tag{20}$$

where $S_0$ is the Bekenstein–Hawking entropy/leading order Cardy entropy, and $q$ is an ensemble-dependent number than can be calculated on both sides of the $AdS_3/CFT_2$ correspondence, using the saddle-point approximation. In a mixed ensemble (where the mass is fixed but the angular momentum is allowed to vary), the result turns out to be $q = \frac{3}{2}$. See, e.g., section 5 in 1205.0971. The key idea for this last check is that $\ln S_0$ is still a large number for $S_0 \to \infty$, even though it is much smaller than $S_0$. Therefore, the result for $q$ must be universal and cannot depend on details of the UV completion (as long as the UV completion is compatible with semiclassical Einstein gravity).

# 5 Stress tensor correlation functions

The main class of observables in a QFT are correlation functions of gauge invariant operators. Thus, the most important entry in the AdS/CFT dictionary is how to relate such correlation functions with corresponding observables on the gravity side. The conjectured relation by Gubser–Klebanov–Polaykov and Witten is

$$\boxed{\Big\langle \exp\Big(\int j(x)\,\mathcal{O}(x)\Big)\Big\rangle_{\text{CFT}} = Z_{\text{gravity}}\Big[\phi(x,z)|_{z\to 0} = j(x)\Big]\,.} \tag{21}$$

The left hand side is evaluated in the CFT. Here $\mathcal{O}(x)$ is some gauge invariant operator whose source is given by $j(x)$. The expression on the left hand side is nothing but the generating functional of correlation functions, and you get arbitrary $n$-point functions by taking $n$ functional derivatives with respect to the sources and then setting the sources to zero. If you know quantum field theory all these statements must be familiar to you; if not, you should acquire this knowledge by studying quantum field theory, which has applications all over physics and beyond.

The right hand side is evaluated in quantum gravity; in the supergravity approximation this reduces to an evaluation in classical gravity. In that limit the quantity $Z_{\text{gravity}}$ is the classical partition function evaluated with boundary conditions for the field $\phi$ given by the function $j(x)$ (which coincides with the source on the CFT side), where the limit $z \to 0$ denotes approaching the asymptotically AdS boundary (while $x$ are the boundary coordinates). The field $\phi$ must be the one corresponding to the gauge invariant operator $\mathcal{O}$.

Which operators exist in a given CFT depends very much on the details of the CFT. However, all of them have at least one operator, namely the stress energy tensor. The natural guess for the field on the gravity side corresponding to the CFT stress tensor is the metric, since it also exists universally in any (reasonable) theory of gravity. Thus, for $n$-point correlation functions of the CFT stress tensor the GKPW prescription in the supergravity approximation reads

$$\begin{aligned}&\langle T_{\mu_1\nu_1}(x_1) T_{\mu_2\nu_2}(x_2)\dots T_{\mu_n\nu_n}(x_n)\rangle^{\text{connected}}_{\text{CFT}}\\ &= \frac{\delta^n}{\delta\gamma^{(0)}_{\mu_1\nu_1}(x_1)\,\delta\gamma^{(0)}_{\mu_2\nu_2}(x_2)\,\dots\delta\gamma^{(0)}_{\mu_n\nu_n}(x_n)}\,\Gamma_{\text{gravity}}\Big[g_{\mu\nu}(x,z)|_{z\to 0} = \gamma^{(0)}_{\mu\nu}(x)\Big]\Big|_{\text{EOM}}\end{aligned} \tag{22}$$

where $\Gamma_{\text{gravity}}$ is the classical gravity action and the subscript EOM means going on-shell, which is equivalent to switching off the sources.

Thus, we have the surprising claim that, say, the $42^{\text{nd}}$ functional derivative of the (holographically renormalized) Einstein–Hilbert AdS action with respect to the metric reproduces the 42-point correlation function of the stress tensor in a CFT.

In a $\text{CFT}_2$ all such correlation functions can be derived using the recursion relation

$$\langle T^1 T^2 \dots T^n\rangle = \sum_{i=2}^{n}\left(\frac{2}{z_{1i}^2} + \frac{1}{z_{1i}}\,\partial_{z_i}\right)\langle T^2\dots T^n\rangle \tag{23}$$

by Belavin, Polyakov and Zamolodchikov. Here, we introduced the abbreviations $z_{ij} := z_i - z_j$ and $T^i := T(z_i) := T_{zz}(z_i)$, where $z$ and $\bar{z}$ are coordinates on the plane with metric $g_{zz} = 0 = g_{\bar{z}\bar{z}}$ and $g_{z\bar{z}} = 1$ and $T_{zz}(z)$ is the holomorphic flux component of the stress tensor. We focus on the holomorphic sector, but the anti-holomorphic sector is analogous, just swapping barred and unbarred quantities.

To get the recursion (23) started, we need the 2-point function,

$$\langle T^1 T^2\rangle = \frac{c}{2z_{12}^4} \tag{24}$$

which is uniquely determined from conformal invariance and the central charge $c$.

To holographically calculate all the stress-tensor correlation functions, it is therefore sufficient to verify that the holographic 2-point function is given by (24) and that the BPZ-recursion relations (23) hold on the gravity side. We shall do this using the Chern–Simons formulation of $\text{AdS}_3$ Einstein gravity. (See lecture 3.)

A convenient trick to calculate the 2-point function is to calculate instead the 1-point function

$$\langle T^1\rangle_\mu = \langle T^1\rangle + \epsilon\,\langle T^1 T^2\rangle + \mathcal{O}(\epsilon^2) \tag{25}$$

of a CFT whose action (the undeformed action is $\Gamma_0$)

$$\Gamma_\mu = \Gamma_0 - \int \mathrm{d}^2 z\,\mu(z,\,\bar z)\,T(z) \tag{26}$$

is deformed by a localized source $\mu(z,\,\bar z) = \epsilon\,\delta^{(2)}(z - z_2,\,\bar z - \bar z_2)$ for the stress tensor.

On the gravity side, the $\mu$-deformed version of BH boundary conditions is given by connections $A = b^{-1}(\mathrm{d}+a)b$ with the group element $b = e^{\rho\,L_0}$ and the boundary connection

$$a_z = L_+ + \frac{\mathcal{L}}{k}\,L_- \qquad\qquad a_{\bar z} = -\mu\,L_+ + \dots \tag{27}$$

where $k = \ell/(4G)$ is the Chern–Simons level and $L_\pm, L_0$ are the standard $\mathrm{sl}(2,\mathbb{R})$ generators (see lecture 3 for more on these notations and conventions.) The state-dependent function $\mathcal{L}$ is allowed to vary but the source $\mu$ is kept fixed. The ellipsis denotes the remaining algebraic components of $a_{\bar z}$, which are determined completely by the EOM. Moreover, the EOM lead to the holographic Ward identities

$$-\,\bar\partial\mathcal{L} = \mu\partial\mathcal{L} + 2(\partial\mu)\mathcal{L} + \frac{k}{2}\,\partial^3\mu\,. \tag{28}$$

To holographically calculate the 2-point function (24), we localize the source as above and expand the state-dependent function as $\mathcal{L}(z) = \mathcal{L}^{(0)}(z) + \epsilon\,\mathcal{L}^{(1)}(z) + \mathcal{O}(\epsilon^2)$ where $\mathcal{L}^{(0)}$ is the background value, for which we take the result for Poincaré patch $\mathrm{AdS}_3$, $\mathcal{L}^{(0)} = 0$ (since we are interested in comparing with a $\mathrm{CFT}_2$ defined on a plane this is the correct choice.) Inserting this expansion into the holographic Ward identities (28) and neglecting terms of higher order in $\epsilon$ yields

$$\bar\partial\mathcal{L}^{(1)} = -\frac{k}{2}\,\partial^3\,\delta^{(2)}(z - z_2,\,\bar z - \bar z_2)\,. \tag{29}$$

The inhomogeneous linear PDE (29) is solved using the Green function of the flat space Laplacean $\partial\bar\partial$, given by $G(z_{12},\,\bar z_{12}) = \ln(z_{12}\bar z_{12})$, establishing

$$\mathcal{L}^{(1)} = -\frac{k}{2}\,\partial_{z_1}^4\,G(z_{12},\,\bar z_{12}) = \frac{3k}{z_{12}^4}\,. \tag{30}$$

The result for the holographic 2-point function (30) coincides with the $\mathrm{CFT}_2$ result for the 2-point function (24), provided the central charge is given by $c = 6k$, which is precisely the result obtained by BH (13).

The holographic $n$-point functions can be obtained similarly, by localizing the source at $n-1$ points, $\mu(z,\,\bar z) = \sum_{i=2}^n \epsilon_i\,\delta^{(2)}(z - z_i,\,\bar z - \bar z_i) := \sum_{i=2}^n \epsilon_i\,\delta_i$. On the CFT side, this yields the desired term $\langle T^1 T^2 \dots T^n\rangle$ mutiplied by the multi-linear factor $\prod_{i=2}^n \epsilon_i$, i.e., each of the $\epsilon_i$ appears exactly once. On the gravity side, repeating the analysis above for this new form of the source, keeping track of the multi-linear factor $\prod_{i=2}^n \epsilon_i$ in the EOM, and using again the Green function for the Laplacean on the plane yields a recursion relation

$$\mathcal{L}^{(n-1)}(z_1) = \sum_{i=2}^n \epsilon_i\left(\frac{2}{z_{1i}^2} + \frac{1}{z_{1i}}\,\partial_{z_i}\right)\mathcal{L}^{(n-2)}(z_i)\,. \tag{31}$$

This result is the gravity version of the BPZ recursion relation (23). The sum of terms in this expression proportional to $\prod_{i=2}^n \epsilon_i$ yields the holographic $n$-point stress tensor correlator.

For more details, references, and a generalization for $\mathrm{CFT}_2$ defined on a cylinder see the first 2.2 pages of 1507.05620.

## 6 Virasoro descendants of the vacuum

The CFT$_2$ vacuum $|0\rangle$ obeys the highest weight conditions $L_n|0\rangle = 0$, $\forall n \geq -1$, where $L_n$ are the Virasoro generators. Generic descendants of the vacuum,

$$|n_1,\, n_2,\, \dots,\, n_m\rangle := L_{-n_1} L_{-n_2} \dots L_{-n_m}|0\rangle \qquad n_i \geq 2\, \forall i = 1..m \tag{32}$$

can be organized by their $L_0$ eigenvalue, called the level of the descendant. It is a well-known result[2] that the number of descendants at level $N$ coincides with the $N^{\text{th}}$ coefficient in the Taylor expansion around $q = 0$ of the generating function

$$\prod_{n=2}^{\infty} \frac{1}{1-q^n} = 1 + q^2 + q^3 + 2q^4 + 2q^5 + 4q^6 + 4q^7 + 7q^8 + \mathcal{O}(q^9)\,. \tag{33}$$

The leading order 1 refers to the vacuum $|0\rangle$, the next-to-leading order $q^2$ to the level-2 vacuum descendant $L_{-2}|0\rangle$, etc. For instance, the 4 descendants at level 7 are $L_{-7}|0\rangle$, $L_{-5}L_{-2}|0\rangle$, $L_{-4}L_{-3}|0\rangle$, $L_{-3}L_{-2}^2|0\rangle$.

On the gravity side, the vacuum state corresponds to global AdS$_3$ and Virasoro excitations correspond to boundary gravitons above the global AdS$_3$ vacuum that are generated through 1-loop effects. Thus, we should calculate the 1-loop partition function on the gravity side and verify whether or not it reproduces the generating function (33) (times its ant-holomorphic counter part).

The (Euclidean) 1-loop torus partition function for fluctuations above global (Euclidean) AdS$_3$ is given by

$$Z^{(1)} = Z_{\text{gh}} \times \int \mathcal{D}h_{\mu\nu}^{\text{TT}}\, \mathcal{D}\tilde{h}\, \exp\bigg(-\kappa \int \mathrm{d}^3x\sqrt{\bar{g}}\, h^{\mu\nu}\mathcal{G}(h)_{\mu\nu}\bigg) \tag{34}$$

where we split the metric into the AdS$_3$ background plus fluctuations, $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, linearized in $h_{\mu\nu}$, and decomposed the latter as

$$h_{\mu\nu} = h_{\mu\nu}^{\text{TT}} + \frac{1}{3}\,\tilde{h}\,\bar{g}_{\mu\nu} + \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu \tag{35}$$

where TT stands for “transverse-traceless”, i.e., $\bar{\nabla}^\mu h_{\mu\nu}^{\text{TT}} = 0 = \bar{g}^{\mu\nu}h_{\mu\nu}^{\text{TT}}$. The quantity $\mathcal{G}(h)_{\mu\nu}$ is the linearized Einstein tensor (including the contribution from the cosmological constant). The value of the coupling constant $\kappa$ is irrelevant for us, as long as $\kappa$ is large enough to make the semi-classical approximation valid. We separated already the gauge part generated by the $\xi_\mu$-contributions to $h_{\mu\nu}$ and denoted the associated ghost contribution as $Z_{\text{gh}}$. We work with unit AdS radius, $\ell = 1$.

Using the path integral measure (for more on this see, e.g., 1007.5189)

$$1 = \int \mathcal{D}h_{\mu\nu}\, e^{-\int\langle h,\, h\rangle} = \int \mathcal{D}\xi_\mu\, e^{-\int\langle \xi,\, \xi\rangle} = \int \mathcal{D}\sigma\, e^{-\int\langle \sigma,\, \sigma\rangle} \tag{36}$$

with the ultra-local inner products

$$\langle h,\, h'\rangle = \int \mathrm{d}^2x\sqrt{\bar{g}}\, h^{\mu\nu}h'_{\mu\nu} \qquad \langle \xi,\, \xi'\rangle = \int \mathrm{d}^2x\sqrt{\bar{g}}\, \xi^\mu\xi'_\mu \qquad \langle \sigma,\, \sigma'\rangle = \int \mathrm{d}^2x\sqrt{\bar{g}}\, \sigma\sigma'$$

allows to calculate the ghost partition function defined as

$$\mathcal{D}h_{\mu\nu} = Z_{\text{gh}} \times \mathcal{D}h_{\mu\nu}^{\text{TT}}\, \mathcal{D}\xi_\mu \mathcal{D}\tilde{h} \tag{37}$$

after the convenient split $\xi_\mu = \xi_\mu^{\text{T}} + \bar{\nabla}_\mu\sigma$ with $\bar{\nabla}^\mu\xi_\mu^{\text{T}} = 0$ yielding $Z_{\text{gh}} = J_2/J_1$ where

$$\mathcal{D}\xi_\mu = J_1\, \mathcal{D}\xi_\mu^{\text{T}}\, \mathcal{D}\sigma \qquad\qquad \mathcal{D}h_{\mu\nu} = J_2\, \mathcal{D}h_{\mu\nu}^{\text{TT}}\, \mathcal{D}\tilde{h}\, \mathcal{D}\xi_\mu^{\text{T}}\, \mathcal{D}\sigma\,. \tag{38}$$

[2]The generating function (33) is related to partitions of integers. You should be able to figure out why this is the correct combinatorics from the definition of the descendants and the level. For details on the combinatorics of integer partitions see section 2 in these lecture notes by Wilf.

The quantity $J_1$ follows from the chain of identities

$$1=\int \mathcal{D}\xi_\mu^{\mathrm{T}}\mathcal{D}\sigma J_1 e^{-\langle\xi,\xi\rangle}=\int \mathcal{D}\xi_\mu^{\mathrm{T}}\mathcal{D}\sigma J_1\, e^{-\int \mathrm{d}^3x\sqrt{\bar g}\left(\xi_\mu^{\mathrm{T}}\xi^{\mu\,\mathrm{T}}-\sigma\bar\nabla^2\sigma\right)}=J_1\big[\det(-\bar\nabla^2)_0\big]^{-1/2} \tag{39}$$

where in the last step we used the standard QFT result about path integrals of Gaussians. The subscript 0 indicates that the functional determinant is evaluated for a spin-0 field. Similarly, one can calculate $J_2$, obtaining

$$J_2=\big[\det(-\bar\nabla^2)_0\det(-\bar\nabla^2+3)_0\det(-\bar\nabla^2+2)_1^{\mathrm{T}}\big]^{1/2} \tag{40}$$

Thus, the ghost partition function is a product of functional determinants.

$$Z_{\mathrm{gh}}=\frac{J_2}{J_1}=\big[\det(-\bar\nabla^2+3)_0\det(-\bar\nabla^2+2)_1^{\mathrm{T}}\big]^{1/2} \tag{41}$$

We do now the same for the remaining part in the partition function (34), using results for the linearized Einstein tensor when acting on transverse-traceless or scalar modes (see appendix B) and obtain

$$Z^{(1)}=Z_{\mathrm{gh}}\times\big[\det(-\bar\nabla^2+3)_0\det(-\bar\nabla^2-2)_2^{\mathrm{TT}}\big]^{-1/2}=\sqrt{\frac{\det(-\bar\nabla^2+2)_1^{\mathrm{T}}}{\det(-\bar\nabla^2-2)_2^{\mathrm{TT}}}}\,. \tag{42}$$

The remaining task is to evaluate the functional determinants in (42), which can be done using heat kernel techniques, see e.g. the user manual by Vassilevich.

$$-\ln\det\big(-\bar\nabla^2-2\big)_2^{\mathrm{TT}}=\int\limits_0^\infty\frac{\mathrm{d}t}{t}\,K^{(2)}(t)\,e^{2t} \tag{43}$$

The heat kernel of the operator $(-\bar\nabla^2)_2^{\mathrm{TT}}$ reads (see, e.g., 0804.1773 or 0911.5085)

$$K^{(2)}(t)=\sum_{n=1}^\infty\frac{\tau_2\,\cos(2n\tau_1)}{\sqrt{4\pi t}|\sin(n\tau/2)|^2}\,e^{-\frac{n^2\tau_2^2}{4t}}\,e^{-3t} \tag{44}$$

where $\tau=\tau_1+i\,\tau_2$ and for later convenience we define the modular parameter $q:=e^{i\tau}$. The quantities $2\pi\tau_1$ and $2\pi\tau_2$ correspond physically to the angular potential $\theta$ and inverse temperature $\beta$, respectively, i.e., to the chemical potentials with which we evaluate the partition function. Similarly, the vector heat kernel is given by

$$K^{(1)}(t)=\sum_{n=1}^\infty\frac{\tau_2\,\cos(n\tau_1)}{\sqrt{4\pi t}|\sin(n\tau/2)|^2}\,e^{-\frac{n^2\tau_2^2}{4t}}\,e^{-2t}\,. \tag{45}$$

Putting all results together, we find that the logarithm of the 1-loop partition function (42) evaluates to

$$\ln Z^{(1)}=\sum_{n=1}^\infty\frac{\cos(2n\tau_1)e^{-n\tau_2}-\cos(n\tau_1)e^{-2n\tau_2}}{2n|\sin(n\tau/2)|^2}=\sum_{n=1}^\infty\frac{1}{n}\Big(\frac{q^{2n}}{1-q^n}+\frac{\bar q^{2n}}{1-\bar q^n}\Big)\,. \tag{46}$$

Exponentiating this result and using the power series of $\ln(1-q^n)$ finally establishes

$$\boxed{Z^{(1)}[q,\,\bar q]=\prod_{n=2}^\infty\frac{1}{|1-q^n|^2}} \tag{47}$$

in full agreement with the $\mathrm{CFT}_2$ result.[3]

[3] On the CFT side we are considering not just the $L_n$-descendants of the vacuum discussed in (33) but also the anti-holomorphic counterpart. So the CFT result for the generating function is $Z^{\mathrm{CFT}}[q,\,\bar q]=\prod_{n=2}^\infty\frac{1}{1-q^n}\cdot\frac{1}{1-\bar q^n}=\prod_{n=2}^\infty\frac{1}{|1-q^n|^2}$.

# 7 Entanglement entropy

Consider a bipartite quantum system with a direct product Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ and consider a general state described by some density matrix $\rho$ (normalized such that $\mathrm{tr}\rho = 1$). Then define the reduced density matrix as the partial trace

$$\rho_A = \mathrm{tr}_B \rho \tag{48}$$

where all degrees of freedom associated with $\mathcal{H}_B$ are traced out. Entanglement entropy (EE) is then defined as the van Neumann entropy of the reduced density matrix.

$$S_A := -\mathrm{tr}\big(\rho_A \ln \rho_A\big) \tag{49}$$

In a QFT context we can define spatial entangling regions and calculate (or least define) EE with respect to such regions. Note that EE will always be infinite in a QFT due to UV divergences. Thus, we need to regularize them by introducing some UV cutoff.

For a $\mathrm{CFT}_2$ defined on a plane, the only (connected) entangling region available is some interval of length $L$. EE should then depend in some way on the interval $L$ (in units of the UV cutoff $a$). As shown by Holzhey, Larsen and Wilczek, EE is universally given by

$$\text{EE for planar CFT}_2 \text{ at zero temperature :} \qquad S_A = \frac{c}{3} \ln \frac{L}{a} + \text{const.} \tag{50}$$

and depends linearly on the central charge $c$.

The proposal of Ryu and Takayanagi (RT) hep-th/0603001 is to calculate EE holographically by the following recipe. For any entangling region in the CFT take a minimal surface $\gamma_A$ attached to the boundary defining the entangling region $A$. Its area gives EE. The RT-formula

$$\boxed{\text{holographic EE:} \qquad S_{\mathrm{RT}} = \frac{\mathrm{area}(\gamma_A)}{4G}} \tag{51}$$

resembles the Bekenstein–Hawking formula, but note that the latter is a thermal entropy (not EE) and involves the area of the event horizon of a black hole (not a minimal area hanging from some asymptotically AdS boundary).

Originally, RT was checked only for time-independent situations. In time-dependent situations Hubeny, Rangamani and Takayanagi (HRT) generalized the proposal 0705.0016 with the result that minimal surfaces are replaced by extremal surfaces. The (H)RT proposal applies to any spacetime dimension. In the $\mathrm{AdS}_3/\mathrm{CFT}_2$ context it simplifies to calculating the length of geodesics, which is a rather straightforward calculation. See figure 1.

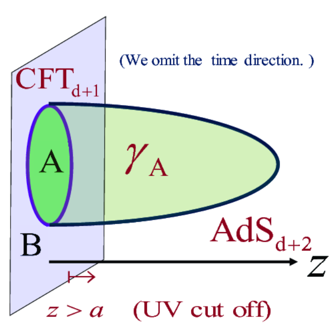


Figure 1: RT prescription for holographic EE

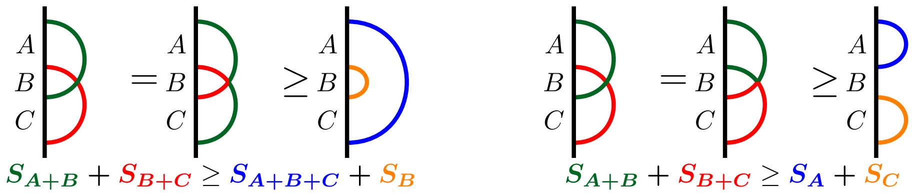


Figure 2: Graphic proofs of strong subadditivity inequalities using holographic EE

One of the neat aspects of the RT proposal (apart from its simplicity) is that it allows to prove straightforwardly the strong subadditivity inequalities, see Fig. 2, just from knowing that EE corresponds to the area of minimal surfaces (see `0704.3719`).

Let us now verify that the RT proposal reproduces the CFT$_2$ result (50). The dual geometry to the CFT$_2$ vacuum on the plane is given by Poincaré patch AdS.

$$\mathrm{d}s^2 = \frac{\ell^2}{z^2}\big(-\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}z^2\big) \tag{52}$$

We apply now the RT prescription (51) to this case for an entangling region of size $L$, i.e., the endpoints of the geodesic are ($z_L = 0$, $x_L = -L/2$) and ($z_R = 0$, $x_R = L/2$). Since $\int^0 \mathrm{d}z/z = \infty$ the length of the geodesic diverges, which recovers the expected UV divergence of the CFT result (50) in the limit $a \to 0$. To introduce the analogue of the UV cutoff on the gravity side we anchor the geodesics not at $z = 0$ but instead at $z = \varepsilon$, with some small but finite cutoff $\varepsilon$. EE is thus given by

$$S_A = \frac{1}{4G}\int \mathrm{d}s = \frac{\ell}{2G}\int\limits_{\varepsilon}^{z^{\max}} \frac{\mathrm{d}z}{z}\,\sqrt{x'^{\,2}+1} = \frac{\ell}{2G}\int\limits_{L/2-\mathcal{O}(\varepsilon)}^{0} \frac{\mathrm{d}x}{z}\,\sqrt{1+\dot z^2} = \frac{\ell}{2G}\int\limits_{L/2-\mathcal{O}(\varepsilon)}^{0}\mathrm{d}x\,\mathcal{L}(z,\,\dot z) \tag{53}$$

where $z^{\max}$ is the maximal value of $z$, i.e., the point where the geodesic turns around back towards the asymptotic boundary. Prime denotes $z$-derivatives and dot $x$-derivatives. We choose the parametrization in terms of $x$.

There is a Noether charge due to invariance under $x$-translations

$$Q = \mathcal{L} - \dot z\,\frac{\partial\mathcal{L}}{\partial\dot z} = \frac{\ell}{z}\,\frac{1}{\sqrt{1+\dot z^2}} \tag{54}$$

which is related to the maximal $z$-value (where $\dot z_{\max} = 0$) through

$$Q = \frac{\ell}{z_{\max}}\,. \tag{55}$$

We can also relate it to the interval length.

$$L/2 - \mathcal{O}(\varepsilon) = \int\limits_{0}^{L/2-\mathcal{O}(\varepsilon)}\mathrm{d}x = \int\limits_{z_{\max}}^{\varepsilon}\frac{\mathrm{d}z}{\dot z} = z_{\max}\sqrt{1-\varepsilon^2} = z_{\max} - \mathcal{O}(\varepsilon^2) \tag{56}$$

The length integral (53) then simplifies to

$$S_A = \frac{\ell}{2G}\int\limits_{\varepsilon/z_{\max}}^{1}\frac{\mathrm{d}y}{y}\,\frac{1}{\sqrt{y^2-1}} = \ln\frac{z_{\max}}{\varepsilon} + \mathcal{O}(\varepsilon^2\ln\varepsilon)\,. \tag{57}$$

Labelling the UV cutoff as $\varepsilon \propto a$ and using the relation (56) the final result for holographic entanglement entropy

$$S_A = \frac{\ell}{2G}\,\ln\frac{L}{a} + \text{const.} = \frac{c}{3}\,\ln\frac{L}{a} + \text{const.} \tag{58}$$

reproduces precisely the CFT$_2$ result (50) for any length $L$ and central charge $c$.

The results above generalize to all Bañados geometries and their dual states in the CFT$_2$, see appendix C.

# 8 Great, but what happens if we deviate from BH?

If you saw them for the first time you should be suitably impressed (and perhaps overwhelmed – if so, check also the appendices and quoted literature) by the calculations and checks of $\mathrm{AdS}_3/\mathrm{CFT}_2$ on the previous ten pages.

In this final section, we step back and reconsider our starting point, the BH boundary conditions summarized in section 1, as a segue to the next set of lectures.

In the Chern–Simons formulation, after we split off the group element that depends on the radial coordinate $\rho$, we were left with a boundary connection $a$ that had only legs in the boundary directions and depended only on the boundary coordinates. So far so good. To get BH boundary conditions we then enforced a fixed (but non-zero) highest weight component (the $L_{+1}$ term was there but had a fixed numerical prefactor), a vanishing Cartan subalgebra component (no $L_0$ term), and an arbitrary (state-dependent) lowest weight component (the $L_{-1}$ term).

Why did we make this choices? The honest answer is, to reproduce known results in the metric formulation, namely the seminal BH results.

But how can we be sure this is the right choice? The short answer is we can't. There are other legitimate choices, and it depends on the physics context which boundary conditions you should use. There is no mathematical way to prove that you must enforce BH boundary conditions. It took the community a while to figure this out.

A first step was done by Compère, Song and Strominger (CSS) in 1303.2662, who imposed boundary conditions different from BH and found as asymptotic symmetry algebra a single Virasoro algebra and a $\hat{u}(1)$ current algebra. With all possible central terms switched on, the mode version of this algebra reads ($n, m \in \mathbb{Z}$)

$$[L_n,\, L_m] = (n-m)\, L_{n+m} + \frac{c}{12}\left(n^3-n\right)\delta_{n+m,\,0} \tag{59}$$

$$[L_n,\, J_m] = -m\, J_{n+m} + i\kappa\left(n^2-n\right)\delta_{n+m,\,0} \tag{60}$$

$$[J_n,\, J_m] = k\, n\, \delta_{n+m,\,0}\,. \tag{61}$$

Imposing CSS boundary conditions in $\mathrm{AdS}_3$ Einstein gravity yields non-zero $c$ and $k$ but vanishing $\kappa$. While the symmetry algebra above clearly is different from the $\mathrm{CFT}_2$ symmetry algebra, they still have something in common: in both cases we have two towers of charges.

Soon after CSS, two additional sets of boundary conditions were found, both of which have four towers of charges, in contrast to BH or CSS. The proposal in 1303.3296 led to two copies of the CSS algebra, so it contains the $\mathrm{CFT}_2$ symmetries but enlarges them by two $\hat{u}(1)$ current algebras. The proposal in 1304.4252 replaced one Virasoro algebra by an $\widehat{\mathrm{sl}}(2)$ current algebra, yielding

$$[L_n,\, L_m] = (n-m)\, L_{n+m} + \frac{c}{12}\left(n^3-n\right)\delta_{n+m,\,0} \tag{62}$$

$$[L_n,\, T^a_m] = -m\, T^a_{n+m} \tag{63}$$

$$[T^a_n,\, T^b_m] = (a-b) T^{a+b}_{n+m} - k\, \kappa_{ab}\, n\, \delta_{n+m,\,0} \tag{64}$$

as asymptotic symmetry algebra, where $a, b \in \{-1, 0, 1\}$ while $n, m \in \mathbb{Z}$.

A few years later, inspired by near horizon considerations (see future lectures and 1603.04824), we found boundary conditions that lead to two $\hat{u}(1)$ current algebras. Confusingly, earlier work in 1511.08687 appeared to impose the same boundary conditions but arrived at a different set of (non-abelian) asymptotic symmetries, without central extension.

This led to some natural questions: what is the most general set of boundary conditions, with the largest number of towers of charges? how is it possible to apparently have the same boundary conditions but end up with different asymptotic symmetries? is there some way to reduce this babylonian confusion of boundary conditions by relating them in some way? We shall address all these questions in the next lectures.

## A Cardy formula in one page

Consider a Euclidean CFT$_2$ on a torus with coordinates $\sigma \sim \sigma + 2\pi$ and $\tau \sim \tau + \beta$, where $\beta = 1/T$ is inverse temperature. The Euclidean partition function

$$Z[\beta] = \mathrm{tr} e^{-\beta H} = e^{-\beta F} \tag{65}$$

yields the free energy $F$. In the low-temperature limit, $\beta \to \infty$, the trace is dominated by the lowest-lying state, the vacuum, which has a free energy given by the Casimir energy $F_0 = -c/12$. (We are using here that the vacuum state is gapped from the rest of the spectrum.)

$$\lim_{\beta\to\infty} Z[\beta] \to e^{\beta c/12}\big|_{\beta\to\infty} = Z_0 = e^{-\beta F_0}\big|_{\beta\to\infty} \tag{66}$$

The entropy of the vacuum state vanishes, $S_0 = -\frac{\partial F_0}{\partial T} = (1-\beta\partial_\beta)\ln Z_0 = 0$.

We exploit now a duality, known in the literature as "S-duality", "low-high temperature duality", "Kramers–Wannier duality" or "Tauberian theorem". Namely, we swap the two cycles $\sigma \to \tilde{\tau}$, $\tau \to \tilde{\sigma}$ and then make a dilatation $\{\tilde{\tau},\, \tilde{\sigma}\} \to 2\pi/\beta\{\hat{\tau},\, \hat{\sigma}\}$ so that the exchanged and rescaled cycles now have the periodicities $\hat{\sigma} \sim \hat{\sigma} + 2\pi$ and $\hat{\tau} \sim \hat{\tau} + 4\pi^2/\beta$. This geometric duality (a special case of modular invariance of the torus) implies

$$Z[\beta] = Z[4\pi^2/\beta] = Z[\hat{\beta}] \qquad \text{with the dual temperature } \hat{\beta} = 4\pi^2/\beta \tag{67}$$

and thus shows a relation between low- and high-temperature limits of the partition function that allows to extract the high-temperature limit as

$$\lim_{\beta\to 0} Z[\beta] = \lim_{\hat{\beta}\to\infty} Z[\hat{\beta}] \to e^{\hat{\beta}c/12} = e^{c\pi^2/(3\beta)} = e^{-\beta F} \tag{68}$$

yielding the 2d (Stefan–Boltzmann-)free energy

$$F = -\frac{\pi^2 c T^2}{3} \tag{69}$$

and the (Cardy-)entropy

$$S = \frac{2\pi^2 c T}{3}\,. \tag{70}$$

Introducing energy $E = F + TS = \frac{1}{2}TS$ allows to re-express entropy as

$$S = 2\pi\sqrt{\frac{cE}{3}}\,. \tag{71}$$

Taking into account rotation, and allowing for different central charges $c$, $\bar{c}$ and left-/right-chiral temperatures $T$, $\bar{T}$, the entropy formula (70) generalizes to

$$S = \frac{\pi^2 c T}{3} + \frac{\pi^2 \bar{c}\bar{T}}{3} \tag{72}$$

or, equivalently:

$$\boxed{S = 2\pi\sqrt{\frac{cL_0}{6}} + 2\pi\sqrt{\frac{\bar{c}\bar{L}_0}{6}}} \tag{73}$$

The last equation provides a common form of the Cardy formula. The quantities $L_0$, $\bar{L}_0$ refer to the eigenvalues of the Virasoro zero-mode generators (on the cylinder) evaluated for the (high-temperature-)state whose entropy is being computed. Often the Cardy formula is expressed in terms of Virasoro zero-mode eigenvalues with respect to the plane, which leads to shifts by the Casimir energy, $L_0 \to L_0 - c/24$, $\bar{L}_0 \to \bar{L}_0 - \bar{c}/24$ in (73).

For more details, refs., and an estimate of the subleading terms see 1904.06359.

## B Linearized fluctuations around global $\mathrm{AdS}_3$

The first step is to calculate the linearized Einstein tensor around global $\mathrm{AdS}_3$, in presence of negative cosmological constant (with unit AdS radius, to reduce clutter). I assume you are familiar with linearizing Einstein's equations (if not, have a look at appendix A of my black holes book and then read section 4.4 therein.)

Indices are raised and lowered with the global $\mathrm{AdS}_3$ background metric, which in standard coordinates and Euclidean signature is given by

$$\bar{g}_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \mathrm{d}\rho^2 + \cosh^2\!\rho\;\mathrm{d}t^2 + \sinh^2\!\rho\;\mathrm{d}\varphi^2\,. \tag{74}$$

Euclidean time $t$ and angular coordinate $\varphi$ obey the periodicity properties

$$(t,\,\varphi) \simeq (t,\,\varphi+2\pi) \simeq (t+\beta,\,\varphi+\theta) \tag{75}$$

where physically $\beta$ is interpreted as inverse temperature and $\theta$ as angular potential. Topologically, Euclidean $\mathrm{AdS}_3$ is a filled torus with modular parameter $2\pi\tau = \theta + i\beta$.

On such a background, the linearized Einstein tensor (including the negative cosmological constant term) is given by

$$\mathcal{G}(h)_{\mu\nu} = \frac{1}{2}\big(-\bar{\nabla}^2 h_{\mu\nu} - \bar{\nabla}_\mu\bar{\nabla}_\nu h + \bar{\nabla}_\nu\bar{\nabla}^\sigma h_{\sigma\mu} + \bar{\nabla}_\mu\bar{\nabla}^\sigma h_{\sigma\nu} - 2h_{\mu\nu} \\ - \bar{g}_{\mu\nu}(\bar{\nabla}_\sigma\bar{\nabla}_\tau h^{\sigma\tau} - \bar{\nabla}^2 h)\big) \tag{76}$$

where $h := \bar{g}^{\mu\nu}h_{\mu\nu}$.

We consider now separately the three terms in the decomposition (35). For the transverse-traceless part we use transversality $\bar{\nabla}^\mu h_{\mu\nu} = 0$ and tracelessness, $\bar{g}^{\mu\nu}h^{\mathrm{TT}}_{\mu\nu} = 0$. The only subtlety is that in some expressions we need to swap the covariant derivatives to be able to exploit transversality, which leads to Riemann-tensor terms. However, since the background is maximally symmeric all these Riemann-tensor terms simplify considerably using the identity $\bar{R}_{\alpha\beta\gamma\delta} = \bar{g}_{\alpha\delta}\bar{g}_{\beta\gamma} - \bar{g}_{\alpha\gamma}\bar{g}_{\beta\delta}$. In the end, you should find

$$\mathcal{G}(h^{\mathrm{TT}})_{\mu\nu} = 0 \qquad \leftrightarrow \qquad (-\bar{\nabla}^2 - 2)\,h^{\mathrm{TT}}_{\mu\nu} = 0\,. \tag{77}$$

For the gauge part $\bar{\nabla}_\mu\xi_\nu + \bar{\nabla}_\nu\xi_\mu$ you can either do a straightforward but boring brute-force calculation, or you realize that linearized EOM are tensor equations and thus must be trivially annihilated by gauge modes. Thus, $\mathcal{G}(\nabla\xi)_{\mu\nu} = 0$ holds identically.

Finally, for the trace part we obtain

$$\mathcal{G}(\tilde{h}\bar{g})_{\mu\nu} = 0 \qquad \leftrightarrow \qquad \bar{g}_{\mu\nu}(-\bar{\nabla}^2 + 2)\,\tilde{h} + \bar{\nabla}_\mu\bar{\nabla}_\nu\tilde{h} = 0\,. \tag{78}$$

Contracting this expression from the left with $\tilde{h}\bar{g}^{\mu\nu}$ yields

$$\tilde{h}\bar{g}^{\mu\nu}\,\mathcal{G}(\tilde{h}\bar{g})_{\mu\nu} \propto -3\tilde{h}\bar{\nabla}^2\tilde{h} + 6\tilde{h}^2 + \tilde{h}\bar{\nabla}^2\tilde{h} \propto -\tilde{h}\bar{\nabla}^2\tilde{h} + 3\tilde{h}^2\,. \tag{79}$$

In the quadratic action appearing in (34) all terms involving gauge modes cancel, and also all bilinear terms cancel except for the purely quadratic ones,

$$h^{\mu\nu}\mathcal{G}(h)_{\mu\nu} \propto h^{\mu\nu\,\mathrm{TT}}(-\bar{\nabla}^2 - 2)\,h^{\mathrm{TT}}_{\mu\nu} + \#\,\tilde{h}(-\bar{\nabla}^2 + 3)\tilde{h} \tag{80}$$

where # denotes some (known, but irrelevant) numerical coefficient. In particular, the mixed terms between $h^{\mathrm{TT}}_{\mu\nu}$ and $\bar{g}_{\mu\nu}\tilde{h}$ cancel, because of tracelesness of $h^{\mathrm{TT}}_{\mu\nu}$. Therefore, the decomposition (35) of the linearized modes persists to quadratic order, and we can separately path integrate the transverse-traceless modes, the gauge modes, and the trace modes, as done in the main text.

# C Holographic EE for Bañados geometries

In the main text, we derived holographically EE for $\mathrm{CFT}_2$ states dual to Poincaré patch $\mathrm{AdS}_3$. Using the fact that any Bañados geometry (vacuum solutions of $\mathrm{AdS}_3$ Einstein gravity) can be locally mapped to Poincaré patch $\mathrm{AdS}_3$ the result can be generalized to EE for $\mathrm{CFT}_2$ states dual to arbitrary Bañados geometries, including thermal $\mathrm{AdS}_3$, global $\mathrm{AdS}_3$, BTZ black holes and their Virasoro descendants.

The Bañados geometries are labelled by a holomorphic and an antiholomorphic function, $\mathcal{L}^\pm(x^\pm)$, see Eq. (43) of lecture 3. The final result of these calculations for holographic EE yields (see, e.g., section 2 in 1901.04499)

$$S_A = \frac{c}{6}\,\ln\left(\frac{\ell^+(x_1^+,\,x_2^+)\ell^-(x_1^-,\,x_2^-)}{a^2}\right) + \text{const.} \tag{81}$$

where $a$ is again a UV cutoff, $c$ is the central charge, $x_1^\pm$ and $x_2^\pm$ are the two endpoints defining the entangling region and the functions $\ell^\pm$ are bilinears of other functions $\psi^\pm_{1,2}$.

$$\ell^\pm(x_1^\pm, x_2^\pm) = \psi_1^\pm(x_1^\pm)\psi_2^\pm(x_2^\pm) - \psi_2^\pm(x_1^\pm)\psi_1^\pm(x_2^\pm) \tag{82}$$

The functions $\psi^\pm_{1,2}$ are two independent solutions to Hill's equation

$$\psi^{\pm\prime\prime} - \mathcal{L}^\pm\,\psi^\pm = 0 \tag{83}$$

with unit Wronskian, $\psi_2^\pm\psi_1^{\pm\,\prime} - \psi_1^\pm\psi_2^{\pm\,\prime} = \pm 1$.

As a sanity check, let us recover first from above the Poincaré patch result (58). In that case $\mathcal{L}^\pm = 0$ and the normalized solutions to Hill's equation read $\psi_1^+ = x^+$, $\psi_2^+ = 1 = \psi_1^-$ and $\psi_2^- = x^-$. For a constant time slice we have $|x_1^+ - x_2^+| = |x_1^- - x_2^-| = L$ and thus the general result (81) yields

$$\text{Poincaré}: \qquad S_A = \frac{c}{6}\,\ln\left(\frac{|x_1^+ - x_2^+||x_1^- - x_2^-|}{a^2}\right) + \text{const.} = \frac{c}{6}\,\ln\left(\frac{L^2}{a^2}\right) + \text{const.} \tag{84}$$

which coincides precisely with (58).

For BTZ black holes we have constant $\mathcal{L}^\pm \geq 0$ and the appropriate solutions to Hill's equation read

$$\psi_1^\pm = \frac{1}{\sqrt{2\sqrt{\mathcal{L}^\pm}}}\,e^{\sqrt{\mathcal{L}^\pm}x^\pm} \qquad\qquad \psi_2^\pm = \frac{1}{\sqrt{2\sqrt{\mathcal{L}^\pm}}}\,e^{-\sqrt{\mathcal{L}^\pm}x^\pm}\,. \tag{85}$$

Assuming again an equal time entangling region of length $L$ inserting (85) into (81)-(82) yields (we drop from now on the trivial additive constant to EE)

$$\text{BTZ}: \qquad S_A = \frac{c}{6}\,\ln\left(\frac{\sinh(\sqrt{\mathcal{L}^+}L)\sinh(\sqrt{\mathcal{L}^-}L)}{\sqrt{\mathcal{L}^+\mathcal{L}^-}a^2}\right). \tag{86}$$

The simpler case of non-rotating BTZ black holes, $\mathcal{L}^+ = \mathcal{L}^- = \pi^2T^2$ (with $T$ being the Hawking temperature, see chapter 4 or Black Holes II), yields

$$\text{non-rotating BTZ}: \qquad S_A = \frac{c}{3}\,\ln\left(\frac{\sinh\left(\pi T\,L\right)}{\pi T\,a}\right) \tag{87}$$

which coincides precisely with the EE for thermal states in a $\mathrm{CFT}_2$ at temperature $T$, see 0905.4013. The small temperature limit $T \to 0$ reproduces the Poincaré patch result (58), as expected, while the high temperature limit yields a volume law

$$\lim_{T\to\infty} S_A = \frac{c}{3}\,\pi T\,L + \dots \tag{88}$$

This volume law is a well-known QFT result, see, e.g. this summary by Calabrese and Cardy.