Gravity in three spacetime dimensions

One of the motivations to study gravity in lower dimensions is that it provides simple toy models for classical and quantum gravity that may allow to address key questions and open puzzles in quantum gravity and holography, such as what are the black hole microstates responsible for the Bekenstein–Hawking entropy, or what is the fate of an evaporating black hole and how can it be reconciled with quantum mechanical unitarity, or how general is holography?

In this section, we focus specifically on gravity in three spacetime dimensions. We provide now some additional motivations for this specific dimension.

1 Motivations

The lowest spacetime dimension where black holes and (at least off-shell) gravitational degrees of freedom can exist is three. Black holes alone could exist also in two spacetime dimension, since all we need for them is the concept of a horizon, which in turn needs a lightcone, and for a lightcone we need at least one time and at least one spatial dimension. Indeed, this argument will provide the main motivation for the next section, where we shall focus on gravity in two spacetime dimensions. However, gravitational degrees of freedom cannot possibly exist in two spacetime dimensions, since in the York-decomposition of fluctuations $h_{\mu\nu}$ of the metric around some background $g_{\mu\nu}$,

$$h_{\mu\nu} = h^{\text{TT}}_{\mu\nu} + \nabla_{(\mu}\xi_{\nu)} + \frac{1}{D}\,h\,g_{\mu\nu} \qquad\qquad g^{\mu\nu}h^{\text{TT}}_{\mu\nu} = \nabla^\mu_g h^{\text{TT}}_{\mu\nu} = 0 \tag{1}$$

the transverse-traceless part $h^{\text{TT}}_{\mu\nu}$ vanishes in two spacetime dimensions ($D = 2$).

In three spacetime dimensions $h^{\text{TT}}_{\mu\nu}$ is not necessarily trivial, and even though there are no classical gravitational waves in 3d Einstein gravity, there are two ways in which the transverse-traceless modes can enter our phenomenology: they can contribute to 1-loop effects (even in Einstein gravity) and they can be excited classically in generalization of Einstein gravity, like massive gravity theories.

Another motivation to focus on three spacetime dimensions comes directly from Einstein gravity: the lowest integer dimension where Einstein gravity can be formulated is three. In two spacetime dimensions the Einstein–Hilbert action does not lead to any equations of motion (any metric in two spacetime dimensions obeys $R_{\mu\nu} = \frac{1}{2}\,g_{\mu\nu}R$ for kinematical reasons) and in one dimension there is no notion of intrinsic curvature. Thus, if we want to study specifically Einstein gravity in the lowest possible dimension we have to pick three.

Yet-another motivation comes from the horizon of black holes. In two spacetime dimensions, while there do exist black holes, the geometry transversal to the horizon is trivial, namely a point. By contrast, in three spacetime dimensions the geometry transversal to the horizon is an S^1, so that one can imagine having non-trivial (quantum) structure located on the horizon.

Finally, practicalities of holography also often lead to gravity in three spacetime dimensions. The main point of holography is that the dual quantum field theory, if it exists, is formulated in one lower dimension than the gravity theory. Now, quantum field theories in three or four spacetime dimensions are also often difficult to deal with beyond perturbation theory. By contrast, quantum field theories in two spacetime dimensions often lead to integrable structures and enhanced symmetries that allow a more complete analytic control of the theory and powerful calculational tools. Since progress in theory largely comes from being able to do certain calculations you can expect that the consideration of gravity in three spacetime dimensions (and quantum field theory in two spacetime dimensions) is going to be helpful, at least for conceptual questions.

2 Einstein gravity

The 3d Einstein–Hilbert action

$$I_{\text{EH}} = \frac{1}{16\pi G} \int \mathrm{d}^3x\sqrt{-g}\,(R-2\Lambda) \tag{2}$$

leads to a theory that has no local physical degree of freedom, so a first impulse may be to dismiss the theory as trivial. This local triviality extends to geometry: the Einstein equations determine the Ricci-tensor in terms of the metric, and since there is no Weyl-tensor in three dimensions also the Riemann-tensor is determined in terms of the metric. In fact, all solutions to the Einstein equations

$$R_{\mu\nu} - \frac{1}{2}\,g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0 \tag{3}$$

are locally flat or (A)dS$_3$, depending on the sign of the cosmological constant Λ.

Another, equally naive, line of thought comes to the opposite conclusion, namely that the theory is so complicated that it does not even exist: The Newton constant G has dimension of length, so like in higher spacetime dimensions 3d Einstein gravity is non-renormalizable by power counting.

Both considerations are too naive. Let us start dispelling the triviality argument. While it is true that locally the theory is trivial, solutions can have non-trivial global properties, such as black hole event horizons, and physical "charges" like mass or angular momentum. Moreover, the physical phase space (or its quantum mechanical Hilbert space version) can be non-trivial, despite of the absence of local physical excitations. Soap bubbles provide an analogy: from the bulk perspective soap bubbles are trivial and there is no interesting dynamics, but if you look at the boundary of a soap bubble you get highly non-trivial dynamics (just look at the flowmarks of a soap bubble).

The non-existence result is harder to dismiss completely — we do not know for which values of the coupling constants Einstein gravity exists as a fully fledged quantum theory — but it is at least possible to dismiss the naive argument above. See for instance the top of page 2 in 0706.3359. The main point is that the Riemann tensor (even off-shell) is determined by the Ricci-tensor, which in turn is (on-shell) determined by the metric, so that any possible counterterm or divergence can be absorbed by field redefinitions and a renormalization of the cosmological constant.

So for the time being let us stay agnostic about what we should expect from 3d Einstein gravity, and rather than using naive arguments in one way or another let us focus on actual calculations in the remainder of this section. To get going we use the Cartan formulation, and as a first step we dualize the spin-connection to a vector-like quantity,

$$\omega^a := \frac{1}{2}\,\epsilon^{abc}\,\omega_{bc} \tag{4}$$

which is a unique feature of three spacetime dimensions. This implies that dualized spin-connection and dreibein have the same index structures, a fact that we are going to exploit heavily in the next subsection. Similarly, we dualize the curvature 2-form $R^a = \frac{1}{2}\,\epsilon^{abc}\,R_{bc} = \mathrm{d}\omega^a + \frac{1}{2}\,\epsilon^a{}_{bc}\,\omega^b\wedge\omega^c$.

In terms of Cartan variables (with dualized connection) the Einstein–Hilbert–Palatini action reads

$$I_{\text{EHP}}[e^a,\,\omega^a] = \frac{1}{8\pi G}\int \big(e_a\wedge R^a - \frac{\Lambda}{6}\,\epsilon_{abc}\,e^a\wedge e^b\wedge e^c\big)\,. \tag{5}$$

The field equations establish vanishing torsion and constant curvature.

$$T^a = \mathrm{d}e^a + \epsilon^a{}_{bc}\omega^b\wedge e^c = 0 \qquad\qquad R^a = \mathrm{d}\omega^a + \frac{1}{2}\,\epsilon^a{}_{bc}\,\omega^b\wedge\omega^c = \frac{\Lambda}{2}\,\epsilon^a{}_{bc}\,e^b\wedge e^c \tag{6}$$

3 Chern–Simons formulation

We switch gears for a few moments and consider a 3d gauge theory that at first glance has nothing to do with gravity, namely Chern–Simons theory. The field content consists of a gauge field 1-form A, with some associated (non-abelian) gauge symmetry. This is all you need to know to construct the action from first principles: again, we make a derivative expansion, keeping only the terms with the lowest number of derivatives, while imposing all our required symmetries as constraints.

So let us do this. We know that the action must be some integral over a 3-form, and the only quantities available are the de-Rahm differential d and the gauge connection 1-form $A = A_\mu \, \mathrm{d}x^\mu = T_I A_\mu^I \, \mathrm{d}x^\mu$, where T_I are generators in the Lie-algebra associated with the gauge group (which is one of our inputs that we need to provide). Thus, our first attempt for an action is

$$I[A] = \int_{\mathcal{M}} \langle \alpha_0 \, A \wedge A \wedge A + \alpha_1 \, A \wedge \mathrm{d}A \rangle \tag{7}$$

where $\mathcal{M}$ denotes our 3d manifold on which the theory is defined, α_i are coupling constants, and $\langle\rangle$ denotes the invariant bilinear form associated with our postulated gauge group (more on this below). We included terms with zero and one derivatives. Unless we introduce a Hodge-$*$ (which would require additional structure), there are no higher derivative terms that we could add, since any term with two de-Rahm differentials is zero.

The action (7) certainly has all the correct properties regarding diffeomorphisms (it is an integral over a 3-form), but the equations of motion descending from it

$$2\alpha_1 \big(\mathrm{d}A + \frac{3\alpha_0}{2\alpha_1} \, A \wedge A \big) = 0 \tag{8}$$

are not gauge covariant equations of motion since they depend explicitly on the gauge connection, unless the coupling constants are finetuned.

In our improved second attempt we do such a finetuning, choosing

$$\alpha_0 = \frac{2}{3} \, \alpha_1 \qquad\qquad \alpha_1 := \frac{k}{4\pi} \, . \tag{9}$$

The second equality is just a conventional name and normalization for α_1, but the first equality is crucial. It guarantees that the field equations

$$\mathrm{d}A + A \wedge A = F = 0 \tag{10}$$

are gauge covariant. The quantity F is the non-abelian field strength 2-form.

Thus, we conclude that to lowest order in a derivative expansion the (bulk) action for a gauge field 1-form A in three dimensions is given by

$$\boxed{I_{\mathrm{CS}}[A] = \frac{k}{4\pi} \int_{\mathcal{M}} \langle A \wedge \mathrm{d}A + \frac{2}{3} \, A \wedge A \wedge A \rangle \, .} \tag{11}$$

This action is called "Chern–Simons action".

Under gauge transformations generated by some group element g the connection transforms as

$$A \to g^{-1} \left(\mathrm{d} + A \right) g \tag{12}$$

the field equations (10) are invariant and the Chern–Simons action (11) transforms

$$I_{\mathrm{CS}}[A] \to I_{\mathrm{CS}}[A] - \frac{k}{12\pi} \int_{\mathcal{M}} \langle g^{-1} \, \mathrm{d}g \wedge g^{-1} \, \mathrm{d}g \wedge g^{-1} \, \mathrm{d}g \rangle - \frac{k}{4\pi} \int_{\partial\mathcal{M}} \langle (\mathrm{d}g) g^{-1} \wedge A \rangle \, . \tag{13}$$

For group elements continuously connected with the identity the additive terms in (13) vanish so that not only the field equations but also the Chern–Simons action is invariant under such ("small") gauge transformations.

To come back to gravity let us now pick a specific gauge connection,

$$A = e^a P_a + \omega^a J_a \tag{14}$$

where we used already suggestive notation. The generators P_a, J_a generate the Lie algebra

$$[P_a,\, P_b] = -\Lambda\, \epsilon_{ab}{}^c\, J_c \tag{15}$$

$$[J_a,\, P_b] = \epsilon_{ab}{}^c\, P_c \tag{16}$$

$$[J_a,\, J_b] = \epsilon_{ab}{}^c\, J_c \tag{17}$$

where the indices in the epsilon-symbol are raised with the Minkowski metric η^{ab}. For positive (vanishing) [negative] Λ the Lie algebra above is so$(3,1)$ (iso$(2,1)$) [so$(2,2)$], with invariant bilinear form

$$\langle J_a,\, P_b\rangle = \eta_{ab} \qquad\qquad \langle P_a,\, P_b\rangle = 0 = \langle J_a,\, J_b\rangle\,. \tag{18}$$

The key observation (see Achucarro–Townsend or Witten) is that the Chern–Simons action (11) with the specifications (14)-(18) is equivalent to the Einstein–Hilbert–Palatini action (5) provided we identify $k = 1/(4G)$. The Chern–Simons gauge flatness conditions (10) are equivalent to the Einstein–Hilbert–Palatini equations of motion (6). Note that the Chern–Simons connection (14) linearly combines dreibein and dualized spin-connection, and that the gauge symmetries are local versions of the expected spacetime symmetries: either the de-Sitter group SO$(3,1)$ for positive Λ, or the anti-de Sitter group SO$(2,2)$ for negative Λ or the Poincaré group ISO$(2,1)$ for vanishing Λ.

Thus, Einstein gravity in three dimensions is classically equivalent to a Chern–Simons theory. Since gauge theories are slightly simpler than gravity theories this reformulation is at the heart of many simplifications that we shall encounter in the following.

We clarify now how gauge transformations (12) relate to diffeomorphisms, whose action on the connection is given by

$$\mathcal{L}_\xi A_\alpha = \xi^\mu \partial_\mu A_\alpha + A_\mu \partial_\alpha \xi^\mu\,. \tag{19}$$

The infinitesimal version of (12) with $g = \mathbb{1} + \epsilon$ reads

$$\delta_\epsilon A = \mathrm{d}\epsilon + [A,\, \epsilon]\,. \tag{20}$$

At first glance, diffeomorphisms generated by a vector field ξ^μ cannot have anything to do with gauge transformations generated by a Lie-algebra valued scalar field ϵ. However, we can connect them using the connection,

$$\epsilon = \xi^\mu A_\mu\,. \tag{21}$$

Inserting the ansatz (21) into the gauge variation (20) yields

$$\delta_{\xi^\mu A_\mu} A_\alpha = (\partial_\alpha \xi^\mu) A_\mu + \xi^\mu \partial_\alpha A_\mu + \xi^\mu [A_\alpha,\, A_\mu] = \mathcal{L}_\xi A_\alpha + \xi^\mu F_{\mu\alpha}\,. \tag{22}$$

Thus, gauge variations (20) with gauge parameter (21) are on-shell equivalent with (and off-shell inequivalent to) diffeomorphisms (19), since $F_{\mu\nu} = 0$ according to the equations of motion (10).

Before focusing on AdS$_3$ let us address briefly the variational principle in the presence of a boundary $\partial\mathcal{M}$. Variation of the Chern–Simons action (11) — besides the bulk equations of motion — yields a boundary term, which is not obviously zero. Thus, we have to impose suitable boundary conditions and potentially add suitable boundary terms to the bulk action. We shall discuss this in more detail at a later stage and consider now specifically AdS$_3$.

4 AdS_3 Einstein gravity

As we just discussed, AdS_3 Einstein gravity can be described by a Chern–Simons action (11) with gauge group $\mathrm{SO}(2,2)$. Before proceeding further it is convenient to massage the action a bit, exploiting the algebraic relations $\mathrm{so}(2,2) \simeq \mathrm{so}(2,1) \oplus \mathrm{so}(2,1) \simeq \mathrm{sl}(2,\mathbb{R}) \oplus \mathrm{sl}(2,\mathbb{R})$. This split of the algebra can be implemented by a change of basis,

$$T_a^\pm = \tfrac{1}{2}\big(J_a \pm \ell P_a\big) \tag{23}$$

where $\Lambda = -1/\ell^2$ defines the AdS_3 radius. In terms of the new generators the gauge algebra reads

$$[T_a^\pm,\, T_b^\pm] = \epsilon_{ab}{}^c\, T_c^\pm \qquad\qquad [T_a^+,\, T_b^-] = 0\,. \tag{24}$$

An explicit realization of the generators $T_a^\pm$ is given by

$$T_a^+ = \begin{pmatrix} J_a^+ & 0 \\ 0 & 0 \end{pmatrix} \qquad\qquad T_a^- = \begin{pmatrix} 0 & 0 \\ 0 & J_a^- \end{pmatrix} \tag{25}$$

where $J_a^\pm$ are generators of $\mathrm{sl}(2,\mathbb{R})$ algebras with invariant bilinear form

$$\langle J_a^\pm,\, J_b^\pm\rangle = \pm\tfrac{\ell}{2}\,\eta_{ab}\,. \tag{26}$$

The full Chern–Simons connection in this basis is then given by

$$A = \begin{pmatrix} \big(\omega^a + \frac{1}{\ell}\,e^a\big)J_a^+ & 0 \\ 0 & \big(\omega^a - \frac{1}{\ell}\,e^a\big)J_a^- \end{pmatrix} = \begin{pmatrix} A^{a+}J_a^+ & 0 \\ 0 & A^{a-}J_a^- \end{pmatrix} \tag{27}$$

where we introduced the $\mathrm{sl}(2)$ connections

$$A^{a\pm} = \omega^a \pm \frac{1}{\ell}\,e^a\,. \tag{28}$$

Linearly combining dualized spin-connection and vielbein in this way is only possible in three spacetime dimensions.

The Chern–Simons action (11) thus splits into a sum of two $\mathrm{sl}(2)$ Chern–Simons actions for the connections $A^\pm$. Since it is a bit cumbersome to have different signs for the bilinear form (26) a common trick is to redefine the minus generators $J_a^- \to J_a^+$ and to correct this sign change by taking the difference of two Chern–Simons actions (instead of their sum).

Putting all these ingredients together we end up with the following Chern–Simons action for AdS_3 Einstein gravity

$$I_{\mathrm{AdS}_3} = \frac{k}{4\pi}\int_{\mathcal{M}}\langle A^+\wedge A^+ + \frac{2}{3}\,A^+\wedge A^+\wedge A^+\rangle - \frac{k}{4\pi}\int_{\mathcal{M}}\langle A^-\wedge A^- + \frac{2}{3}\,A^-\wedge A^-\wedge A^-\rangle \tag{29}$$

where $A^\pm = A^{a\pm}L_a$ and L_a are $\mathrm{sl}(2,\,\mathbb{R})$ generators (e.g. $L_a = J_a^+$, but we can and will choose a slightly different basis). The Chern–Simons level is a dimensionless ratio of AdS-radius and Newton constant.

$$\boxed{k = \frac{\ell}{4G}} \tag{30}$$

Above we worked with generators adapted to $\mathrm{so}(2,1)$. In what follows we will work instead with a convenient representation of the $\mathrm{sl}(2,\,\mathbb{R})$ algebra, given by the commutation relations

$$[L_n,\, L_m] = (n-m)\,L_{n+m} \qquad\qquad n,m\in\{-1,0,1\} \tag{31}$$

and with invariant bilinear form

$$\langle L_{+1},\, L_{-1}\rangle = -1 \qquad\qquad \langle L_0,\, L_0\rangle = \frac{1}{2}\,. \tag{32}$$

Finally, let us remind ourselves that the metric is bilinear in the vielbein.

$$\boxed{g_{\mu\nu} = \frac{\ell^2}{2}\,\langle\big(A_\mu^+ - A_\mu^-\big)\big(A_\nu^+ - A_\nu^-\big)\rangle} \tag{33}$$

5 Brown–Henneaux boundary conditions

If the manifold M is topologically a filled cylinder or torus (as it happens to be for AdS_3) it is often convenient to split off the radial dependence from the connection by defining

$$\boxed{A = b^{-1}(\rho)\left(\mathrm{d} + a(x^\mu)\right) b(\rho)} \tag{34}$$

where $b(\rho)$ is some suitably chosen group element and

$$a(x^\mu) = a_\nu(x^\mu)\, \mathrm{d}x^\nu \qquad \mu, \nu \in \{0, 1\} \tag{35}$$

is effectively a boundary connection, in the sense that it has no leg in the radial direction and no dependence on the radius ρ. Decomposing gauge connection

$$A_\rho = b^{-1}\partial_\rho b \qquad\qquad A_\mu = b^{-1}\, a_\mu\, b \tag{36}$$

and gauge curvature F with respect to the radial coordinate and the boundary coordinates shows that two of the three gauge-flatness conditions (10) hold identically

$$F_{\rho\mu} = \partial_\mu A_\rho - \partial_\rho A_\mu + [A_\mu,\, A_\rho] = -\left(\partial_\rho b^{-1}\right) a_\mu b + b^{-1} a_\mu \partial_\rho b + [b^{-1} a_\mu b,\, b^{-1}\partial_\rho b] = 0 \tag{37}$$

whereas the third one reduces to gauge flatness of the boundary connection

$$F_{\mu\nu} = b^{-1}\left(\partial_\mu a_\nu - \partial_\nu a_\mu + [a_\mu,\, a_\nu]\right) b = b^{-1}\, f_{\mu\nu}\, b = 0\,. \tag{38}$$

Brown–Henneaux boundary conditions require connections of the form

$$A^\pm = e^{\mp\rho/\ell\, L_0}\left(\mathrm{d} + a^\pm(x^+,\, x^-)\right) e^{\pm\rho/\ell\, L_0} \tag{39}$$

with the "boundary connection"

$$a^+ = \left(L_{+1} - \mathcal{L}^+(x^+)\, L_{-1}\right)\frac{\mathrm{d}x^+}{\ell} \qquad \Rightarrow \qquad \delta a^+ = -\delta\mathcal{L}^+(x^+)\, L_{-1}\,\frac{\mathrm{d}x^+}{\ell} \tag{40}$$

$$a^- = \left(L_{-1} - \mathcal{L}^-(x^-)\, L_{+1}\right)\frac{\mathrm{d}x^-}{\ell} \qquad \Rightarrow \qquad \delta a^- = -\delta\mathcal{L}^-(x^-)\, L_{+1}\,\frac{\mathrm{d}x^-}{\ell}\,. \tag{41}$$

Note that both connections $A^\pm$ obey the gauge flatness conditions (38) and hence this configuration solves the Chern–Simons field equations (and thus provides solutions of AdS_3 Einstein gravity) for all functions $\mathcal{L}^\pm(x^\pm)$.

To check the claim above we insert the proposed connection (39) into the result for the metric (33) and test whether we get the expected Fefferman–Graham expansion of the metric. In order to proceed we need to evaluate the expressions

$$e^{-\rho/\ell\, L_0} L_{\pm1} e^{\rho/\ell\, L_0} = e^{\pm\rho/\ell} L_{\pm1} \qquad\qquad e^{\rho/\ell\, L_0} L_{\pm1} e^{-\rho/\ell\, L_0} = e^{\mp\rho/\ell} L_{\pm1} \tag{42}$$

using the Baker–Campbell–Hausdorff formula. We can then read off the metric and find

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + \mathrm{d}x^+\,\mathrm{d}x^-\left(e^{2\rho/\ell} + e^{-2\rho/\ell}\mathcal{L}^+(x^+)\mathcal{L}^-(x^-)\right) + \mathcal{L}^+(x^+)\left(\mathrm{d}x^+\right)^2 + \mathcal{L}^-(x^-)\left(\mathrm{d}x^-\right)^2. \tag{43}$$

Defining $\gamma^{(0)}_{\mu\nu} = \eta_{\mu\nu}$ with $\eta_{\pm\mp} = \frac{1}{2}$, $\eta_{\pm\pm} = 0$ and $\gamma^{(2)}_{\pm\pm} = \mathcal{L}^\pm(x^\pm)$, $\gamma^{(2)}_{\pm\mp} = 0$ the metric (43) can be rewritten as

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + \left(e^{2\rho/\ell}\,\gamma^{(0)}_{\mu\nu} + \gamma^{(2)}_{\mu\nu} + \mathcal{O}(e^{-2\rho/\ell})\right)\mathrm{d}x^\mu\,\mathrm{d}x^\nu \tag{44}$$

which is precisely the required Fefferman–Graham expansion. This proves our claim above. The solutions (43) are also known as "Bañados geometries", see hep-th/9901148.

6 Phase space of AdS_3 Einstein gravity (á la BH)

We are finally in a position to make precise statements about the physical phase space of 3d Einstein gravity with various boundary conditions. For concreteness — and also since it is probably the most interesting example for holography — we focus on Brown–Henneaux boundary conditions in AdS_3 Einstein gravity using the Chern–Simons formulation in terms of two $\mathrm{sl}(2,\mathbb{R})$ connections $A^\pm$, discussed in the previous section.

Our first task is to determine the boundary condition-preserving transformations generated by some $\varepsilon^\pm = b_\pm^{-1}\hat\varepsilon^\pm b_\pm$. To simplify the notation we focus on the upper sign equations and drop all $\pm$ and $\mp$ decorations since all formulas are analogous for both signs. We also set the AdS radius to unity, $\ell = 1$, decompose $\hat\varepsilon$ algebraically

$$\hat\varepsilon = \varepsilon(x)\,L_1 + \varepsilon_0(x)\,L_0 + \varepsilon_{-1}(x)\,L_{-1} \tag{45}$$

and solve (20) in order to find all boundary condition-preserving transformations.

$$\begin{aligned}\big(\varepsilon(x)'\,L_1 + \varepsilon_0(x)'\,L_0 + \varepsilon_{-1}(x)'\,L_{-1}\big)\ \mathrm{d}x + [L_1,\,\varepsilon(x)\,L_1 + \varepsilon_0(x)\,L_0 + \varepsilon_{-1}(x)\,L_{-1}]\ \mathrm{d}x\\ -\,[\mathcal{L}(x)\,L_{-1},\,\varepsilon(x)\,L_1 + \varepsilon_0(x)\,L_0 + \varepsilon_{-1}(x)\,L_{-1}]\ \mathrm{d}x \stackrel{!}{=} -\delta\mathcal{L}(x)\,L_{-1}\ \mathrm{d}x\end{aligned} \tag{46}$$

The L_1 component of this equation yields

$$\varepsilon_0(x) = -\varepsilon(x)'\,. \tag{47}$$

The L_0 component obtains

$$\varepsilon_{-1}(x) = \frac{1}{2}\,\varepsilon(x)'' - \mathcal{L}(x)\varepsilon(x)\,. \tag{48}$$

The L_{-1} component does not lead to any restriction of the functions ε_n since $\delta\mathcal{L}$ is an arbitrary function. It is still useful to see the explicit expression, after inserting the results (47) and (48):

$$\boxed{\delta_\varepsilon\mathcal{L} = 2\mathcal{L}\,\varepsilon' + \mathcal{L}'\,\varepsilon - \frac{1}{2}\,\varepsilon'''} \tag{49}$$

The function $\mathcal{L}$ transforms with an infinitesimal Schwarzian derivative.

We have just derived the variation of the canonical boundary charges for Brown–Henneaux boundary conditions,

$$\delta Q[\varepsilon] = \frac{k}{2\pi}\oint_{S^1}\varepsilon\,\delta\mathcal{L}\ \mathrm{d}x\,. \tag{50}$$

Assuming that ε is state-independent and hence has vanishing variation, $\delta\varepsilon = 0$, the charges (50) are integrable in field space and we can drop the δ's on both sides. (This is also why we did ot bother using the symbol $\not\delta$.)

Reinstating the $\pm$-decorations we thus find two non-trivial towers of canonical boundary charges for AdS_3 Einstein gravity with Brown–Henneaux boundary conditions:

$$\boxed{Q^\pm[\varepsilon^\pm(x^\pm)] = \frac{k}{2\pi}\oint_{S^1}\varepsilon^\pm(x^\pm)\,\mathcal{L}^\pm(x^\pm)\ \mathrm{d}x^\pm} \tag{51}$$

Our next task is to establish the canonical realization of the asymptotic symmetries, i.e., to derive the asymptotic symmetry algebra as Poisson bracket algebra generated by the charges (51). This is important since the physical phase space falls into representations of that algebra, so it is valuable information to discover what that algebra is. To this end, we exploit that the brackets of two charges yields the variation of one charge with respect to the parameter of the other charge. This means that for the determination of the asymptotic symmetry algebra, we only need to know the general variation of the charges for any boundary conditions-preserving transformation — which we just have derived! Inserting (49) into (50) together with the results from Lecture 2 establishes

$$\{Q[\varepsilon_1],\, Q[\varepsilon_2]\} = Q[\varepsilon_1'\varepsilon_2 - \varepsilon_2'\varepsilon_1] - \frac{k}{4\pi}\oint_{S^1}\varepsilon_1'''\varepsilon_2\,\mathrm{d}x\,. \tag{52}$$

Intriguingly, the asymptotic symmetry algebra (52) comes with a central extension (the last term).

In order to see more explicitly which algebra this is we introduce Fourier modes for all functions. This is possible due to our assumption of the theory living on a cylinder. We define the Fourier mode generators

$$L_n^\pm := Q[e^{inx^\pm}] \tag{53}$$

and multiply (52) by $-i$ to obtain our final result for the asymptotic symmetry algebra.

$$\boxed{-\,i\{L_n^\pm,\, L_m^\pm\} = (n-m)\,L_{n+m}^\pm + \frac{k}{2}\,n^3\,\delta_{n+m,\,0}} \tag{54}$$

Comparing (54) with the standard form of the Virasoro algebra, we see that upon replacing $-i\{,\} \to [,]$ (which is canonical quantization) the asymptotic symmetry algebra consists of two copies of the Virasoro algebra with central charge

$$\boxed{c = 6k = \frac{3\ell}{2G}\,.} \tag{55}$$

If you are wondering what happened to the term linear in n in the Virasoro central extension: we can recover it from (54) by shifting the zero mode generator $L_0 \to L_0 + k/4 = L_0 + c/24$ (remember that the precise form of a central extension is basis dependent). This shift of the Virasoro zero mode by $c/24$ is well-known in the CFT_2 literature and can be interpreted as the Casimir energy of the cylinder.

This derivation (albeit in the metric formulation) was first performed by Brown and Henneaux in 1986 and proves that **$\mathbf{AdS_3}$ Einstein gravity for Brown–Henneaux boundary conditions is equivalent to a $\mathbf{CFT_2}$**, in the sense that the physical phase space (or upon quantization the physical Hilbert space) falls into representations of two copies of the Virasoro algebra, which is the conformal algebra in two dimensions. In retrospect, this result was a milestone on the road to the AdS/CFT correspondence (this paper currently has more than 2200 citations according to INSPIRE; however, in the first decade after publication it was widely unknown — a classic example of a “sleeper” paper).

The remaining pages contain appendices with supplementary material on BTZ black holes, and on 3d massive and conformal gravity. For additional information, exercises, and references see chapter 7 in my black holes book.

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A BTZ black holes

In the main text, we found an infinite set of solutions to AdS$_3$ Einstein gravity in the Chern–Simons formulation, parametrized by a holomorphic and an anti-holomorphic function. Here we focus on specific solutions obtained when both of these functions are constants. Let us define two new parameters,

$$m = \mathcal{L}^+ + \mathcal{L}^- \qquad\qquad j = \ell\left(\mathcal{L}^+ - \mathcal{L}^-\right) \tag{56}$$

and Schwarzschild-like coordinates ($\varphi \sim \varphi + 2\pi$)

$$t = \frac{x^+ - x^-}{2} \qquad\qquad \varphi = \frac{x^+ + x^-}{2\ell} \qquad\qquad r = f(\rho) \tag{57}$$

where the function $f(\rho)$ is determined such that the prefactor of the $\mathrm{d}\varphi^2$-term in the metric is r^2, yielding

$$\mathrm{d}s^2 = -\,\mathrm{d}t^2\,\frac{r^4 - 2\ell^2 m r^2 + \ell^2 j^2}{\ell^2 r^2} + \frac{\ell^2 r^2\,\mathrm{d}r^2}{r^4 - 2\ell^2 m r^2 + \ell^2 j^2} + r^2\left(\mathrm{d}\varphi + \frac{\ell j}{r^2}\,\mathrm{d}t\right)^2. \tag{58}$$

Introducing as variables the loci of the outer and inner Killing horizon $r_\pm$

$$m = \frac{r_+^2 + r_-^2}{2\ell^2} \qquad\qquad j = -\frac{r_+ r_-}{\ell} \tag{59}$$

the metric (58) can be recast into a suggestive form

$$\boxed{\mathrm{d}s^2_{\text{BTZ}} = -\frac{(r^2 - r_+^2)(r^2 - r_-^2)}{\ell^2 r^2}\,\mathrm{d}t^2 + \frac{\ell^2 r^2\,\mathrm{d}r^2}{(r^2 - r_+^2)(r^2 - r_-^2)} + r^2\left(\mathrm{d}\varphi - \frac{r_+ r_-}{\ell r^2}\,\mathrm{d}t\right)^2} \tag{60}$$

For real $r_+ > r_-$ this geometry is known as BTZ black hole, see hep-th/9204099 and gr-qc/9302012. At the time when it was discovered its existence was a big surprise, since the community did not expect black holes to exist in 3d Einstein gravity.

Here are the top ten properties of BTZ black holes:

- **Asymptotically AdS$_3$.** Since any Bañados geometry is asymptotically AdS$_3$ also BTZ black holes are asymptotically AdS$_3$ solutions.

- **Locally AdS$_3$.** Any solution to the AdS$_3$ Einstein equations is locally AdS$_3$, with $R_{\mu\nu} = -\frac{2}{\ell^2}\,g_{\mu\nu}$ and $R = -\frac{6}{\ell^2}$. Since BTZ is such a solution it must be locally AdS$_3$. Nevertheless, BTZ black holes are not globally equivalent to AdS$_3$ since they exhibit horizons.

- **Event horizon at $r = r_+$.** The locus $r = r_+$ is an event horizon, which can be checked by constructing the Penrose diagram. It is also a Killing horizon for the Killing vector $\partial_t + \Omega\partial_\varphi$, with the angular rotation parameter $\Omega = \frac{r_-}{\ell r_+}$.

- **Inner horizon at $r = r_-$.** Also the locus $r = r_-$ is a Killing horizon, which you can verify with elementary methods.

- **Rotation.** For non-vanishing r_- the horizon rotates (similar to the one of a Kerr black hole) with angular rotation parameter $\Omega = \frac{r_-}{\ell r_+}$. For $r_- = 0$ there is no rotation and no inner horizon, similarly to the Schwarzschild black hole.

- **Orbifolds of AdS$_3$.** An elegant way to understand BTZ black holes is as orbifolds of global AdS$_3$ along certain Killing directions (see gr-qc/9302012). This means that we start with global AdS$_3$ (which is formally BTZ for $r_+^2 = -1$ and $r_- = 0$) and identify points by a discrete subgroup of the isometry group $SO(2,2)$. While orbifolding in this way generates singularities, for BTZ black holes they are hidden behind a horizon as long as $r_+ > r_-$ remain real.

- **Penrose diagram.** See Fig. 1 below

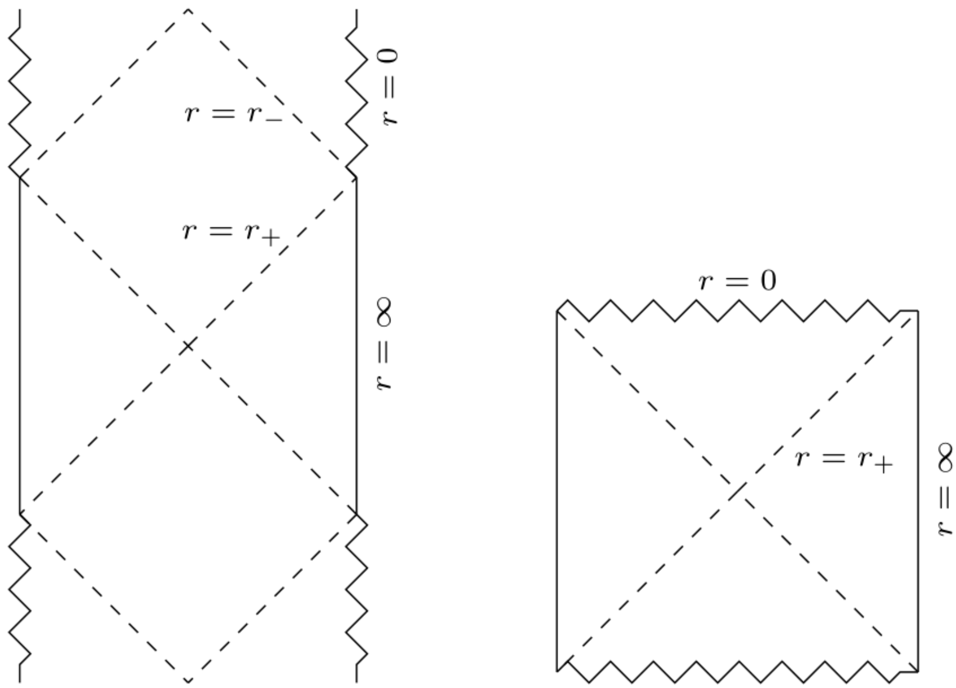

Figure 1: Two-dimensional cut through Penrose diagram of rotating (left) and non-rotating (right) BTZ black hole (Fig. by Hugo Ferreira, used with permission)

- **Hawking temperature.** It is also of interest to consider BTZ black hole thermodynamics. Using standard methods yields temperature and angular velocity:
$$T = \frac{r_+^2 - r_-^2}{2\pi r_+ \ell^2} \qquad\qquad \Omega = \frac{r_-}{\ell r_+} \tag{61}$$
The result for mass M and angular momentum $|J|$ differ both by a factor of $1/(4G)$ from the mass parameter m and angular momentum parameter $|j|$ defined in (59).

- **Bekenstein–Hawking entropy.** The Bekenstein–Hawking entropy of BTZ black holes is
$$\boxed{S = \frac{2\pi r_+}{4G} = 2\pi\sqrt{\frac{cL^+}{6}} + 2\pi\sqrt{\frac{cL^-}{6}}} \tag{62}$$
where c is the Brown–Henneaux central charge
$$c = \frac{3\ell}{2G} = 6k \tag{63}$$
and $L^\pm = \ell\mathcal{L}^\pm/(4G)$. The last equality (62) is known as Cardy formula.

- **Extremal BTZ.** Finally, we may consider the limit $r_- \to r_+ \neq 0$, which requires that either $\mathcal{L}^+$ or $\mathcal{L}^-$ vanishes, but not both. In this limit the two horizons coalesce to a single extremal one, with vanishing surface gravity and hence also vanishing Hawking temperature. Despite of having zero temperature, the extremal BTZ entropy is non-zero and can be macroscopically large.

We shall come back to BTZ black holes on numerous occasions in these lectures, but for now we move on and consider in the next appendix generalizations of AdS$_3$ Einstein gravity that also feature BTZ black holes as part of their spectra.

B Massive gravity theories and conformal gravity

There is a whole zoo of gravity theories beyond Einstein gravity (see 1105.3735 for a review of massive gravity). A large class of them introduces higher curvature corrections to the action, which potentially is dangerous since higher curvature terms tend to generate ghosts. In three spacetime dimension there is a unique[1] possibility, namely to add a gravitational Chern–Simons action I_{gCS} for the Christoffel connection to the Einstein Hilbert action. We focus on this possibility, introduced by Deser, Jackiw and Templeton, dubbed **topologically massive gravity** (TMG).

The gravitational Chern–Simons term

$$I_{\text{gCS}}[g] = \frac{k_g}{4\pi} \int_{\mathcal{M}} \big(\Gamma \wedge \mathrm{d}\Gamma + \frac{2}{3}\,\Gamma\wedge\Gamma\wedge\Gamma\big) \tag{64}$$

is a three-derivative action, since each Christoffel connection Γ features one derivative of the metric. Note that you are not supposed to vary the action (64) with respect to the connection Γ, but rather with respect to the metric g.

There is again an Einstein–Hilbert–Palatini-like formulation (cf. Baekler, Mielke and Hehl), where you replace Γ by the spin-connection ω, so that effectively you have a Chern–Simons action for the spin-connection. The first order form of the TMG action

$$I_{\text{TMG}}[e^a,\,\omega^a,\,\lambda^a] = I_{\text{EHP}}[e^a,\,\omega^a] + \frac{k_g}{2\pi}\int_{\mathcal{M}} \big(\omega_a \wedge \mathrm{d}\omega^a + \frac{1}{3}\,\epsilon_{abc}\,\omega^a\wedge\omega^b\wedge\omega^c + \lambda_a\wedge T^a\big) \tag{65}$$

contains the Einstein–Hilbert–Palatini action $I_{\text{EHP}}[e^a,\,\omega^a]$ as defined in (5). The 1-form λ_a plays the role of a Lagrange-multiplier ensuring vanishing torsion on-shell.

Since TMG has one additional derivative as compared to Einstein gravity, but no additional symmetries, we need to specify more initial data to solve the field equations. This means that we should expect one additional local physical degree of freedom as compared to Einstein gravity. Since Einstein gravity has zero, we expect that TMG propagates one physical degree of freedom. This expectation turns out to be correct, as a canonical analysis reveals (see for instance 0806.4185). The degree of freedom corresponds to a massive graviton with mass proportional to $1/k_g$, hence the name of TMG ("topological" refers to the Chern–Simons term, "massive" to the mass of the graviton and "gravity" to a theory of the metric).

Varying the action (65) leads to three equations of motion: varying with respect to the Lagrange-multiplier λ establishes vanishing torsion, which can be solved for the spin-connection; varying with respect to the spin-connection ω relates the curvature 2-form R_a linearly to the 2-form $\epsilon_{abc}\,\lambda^b\wedge e^c$, which can be solved for the Lagrange-multiplier; finally, varying with respect to the dreibein e yields some first order equation for the Lagrange-multiplier. Inserting the solution for the Lagrange multiplier (which has one derivative of the spin connection and hence two derivatives of the metric) shows that λ is proportional to the so-called Schouten 1-form. Translating all these statements into metric formulation establishes third order partial differential equations ($1/\mu := 8Gk_g$; the mass of the graviton is μ)

$$\text{TMG:} \qquad R_{\alpha\beta} - \frac{1}{2}\,g_{\alpha\beta}R + \Lambda g_{\alpha\beta} + \frac{1}{\mu}\,C_{\alpha\beta} = 0 \tag{66}$$

that feature the so-called Cotton tensor (see e.g. gr-qc/0309008)

$$C_{\alpha\beta} = \varepsilon_\alpha{}^{\mu\nu}\,\nabla_\mu\big(R_{\nu\beta} - \frac{1}{4}\,g_{\nu\beta}R\big) = C_{\beta\alpha}\,. \tag{67}$$

An interesting special case (studied holographically in 1110.5644 for AdS and in 1208.1658 for flat space) arises in the limit $\mu\to 0$, $G\to\infty$, keeping finite k_g. This theory is called **conformal gravity** and, like Einstein gravity, has no local physical degree of freedom due to an extra gauge symmetry, namely Weyl rescalings. In this case the equations of motion (66) demand Cotton-flatness, which means that all solutions of conformal gravity are locally conformally flat, since in three dimensions conformal flatness is equivalent to vanishing Cotton tensor.

[1] A similar action can be introduced in seven and eleven dimensions.