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OIST Exercises on Asymptotic symmetries

(4.1) **Boundary gravitons on global AdS$_3$**
Consider linearized fluctuations, $g_{\mu\nu} = g^{\text{AdS}}_{\mu\nu} + \psi_{\mu\nu}$, around global AdS,

$$\mathrm{d}s^2_{\text{AdS}} = \mathrm{d}\rho^2 - \cosh^2\rho\,\mathrm{d}t^2 + \sinh^2\rho\,\mathrm{d}\varphi^2 \qquad \varphi \sim \varphi + 2\pi\,.$$

Find all normalizable left-moving linearized fluctuations $\psi_{\mu\nu}$ that obey the SL(2)×SL(2) primary conditions $(L_1^\pm h)_{\mu\nu} = 0$ where $L_n^\pm$ are the six Killing vectors of global AdS$_3$

$$\begin{aligned} L_0^\pm &= i\partial_\pm \\ L_{-1}^\pm &= ie^{-ix^\pm}\Big(\coth(2\rho)\,\partial_\pm - \frac{1}{\sinh(2\rho)}\,\partial_\mp + \frac{i}{2}\,\partial_\rho\Big) \\ L_1^\pm &= ie^{ix^\pm}\Big(\coth(2\rho)\,\partial_\pm - \frac{1}{\sinh(2\rho)}\,\partial_\mp - \frac{i}{2}\,\partial_\rho\Big) \end{aligned}$$

with $x^\pm = t \pm \varphi$. By the attribute "left-moving" we mean $(L_0^-\psi)_{\mu\nu} = 0$ and $(L_0^+\psi)_{\mu\nu} = h^+\psi_{\mu\nu}$, where the weight has to be positive, $h^+ > 0$, for the mode to be called "normalizable".

(4.2) **Soft hairy boundary conditions**
Instead of Brown–Henneaux assume in one chiral sector the boundary conditions ($L_{\pm1}$, L_0 are sl$(2,\mathbb{R})$ generators with standard conventions)

$$A = b^{-1}\big(\,\mathrm{d}{+}a\big)b \qquad a = \big(\,\mathrm{d}t + \mathcal{J}(t,\,\varphi)\,\mathrm{d}\varphi\big)\,L_0 \qquad \delta a = \delta\mathcal{J}(t,\,\varphi)\,\mathrm{d}\varphi\,L_0$$

with the state-independent group element $b = e^{r/(2\ell)(L_1-L_{-1})}$. (The other sector is analogous, up to sign changes, and does not need to be considered here.) Derive the canonical boundary charges and their asymptotic symmetry algebra. Using Fourier modes $J_n = \frac{k}{4\pi}\oint \mathrm{d}\varphi e^{in\varphi}\mathcal{J}$ find one generator J_m that commutes with all other J_n.

(4.3) **Flat space limit of three-dimensional gravity**
Start with the asymptotic symmetry algebra of AdS$_3$ Einstein gravity with Brown–Henneaux boundary conditions (here written as commutator algebra with L_0 shifted suitably)

$$[L_n^\pm,\,L_m^\pm] = (n-m)\,L^\pm_{n+m} + \frac{\ell}{8G}\,\big(n^3-n\big)\,\delta_{n+m,\,0} \qquad\qquad [L_n^+,\,L_m^-] = 0$$

and make a change of basis

$$L_n := L_n^+ - L_{-n}^- \qquad\qquad M_n := \frac{1}{\ell}\,\big(L_n^+ + L_{-n}^-\big)\,.$$

The generators L_n, M_n still generate two copies of Virasoro for finite AdS radius ℓ, albeit in an unusual basis. Take the flat space limit $\ell \to \infty$, which leads to an İnönü–Wigner contraction of the algebra, and write down the commutation relations after you have taken this limit. The resulting algebra is the flat space limit of Virasoro.

Hints:

- Work in a gauge where $\psi_{\mu-} = 0$ and exploit that ψ solving the linearized Einstein equations implies $(C_2^+ + C_2^- + 2)\psi = 0$, where $C_2^\pm = \frac{1}{2}\left(L_1^\pm L_{-1}^\pm + L_{-1}^\pm L_1^\pm\right) - (L_0^\pm)^2$ is the quadratic Casimir. Applying the ancient wisdom of Fourier transforming when you do not know what else to do you can start with the separation ansatz

$$\psi_{\mu\nu}(h^+, h^-) = e^{-ih^+x^+ - ih^-x^-} \begin{pmatrix} F_{++}(\rho) & 0 & F_{+\rho}(\rho) \\ 0 & 0 & 0 \\ F_{+\rho}(\rho) & 0 & F_{\rho\rho}(\rho) \end{pmatrix}_{\mu\nu}$$

  so that you work with $L_0^\pm$ eigenmodes, $L_0^\pm \psi = h^\pm \psi$, and have implemented already the required gauge conditions $\psi_{\mu-} = 0$. The left-moving condition sets one of the weights to zero, $h^- = 0$. The Einstein equations (using the quadratic Casimir) fix the (by normalizability positive!) other weight, $h^+ = 2$. The remaining steps are to solve the Killing equations corresponding to the two primary conditions, using the ansatz above. Note that some of the equations linearly combine to algebraic relations between the three functions $F_{\mu\nu}(\rho)$. One of the $++$ component equations allows to immediately determine $F_{++} \propto \tanh^2 \rho$ by simple integration. In the end this procedure yields a unique result for ψ, up to an overall factor.

  Note: the attribute "boundary gravitons" is justified since we know that in the bulk there are no physical degrees of freedom, i.e., no gravitational waves can propagate through the bulk; however, at the boundary some of the pure gauge excitations can become physical.

- Derive the boundary conditions preserving gauge transofrmations $\varepsilon = b^{-1}\hat{\varepsilon}b$; you should find $\hat{\varepsilon} = \eta(\varphi)L_0$. Then derive how the state-dependent function $\mathcal{J}$ transforms under such gauge transformations; you should find $\delta\mathcal{J} = \partial_\varphi \eta$. These results, together with the general results about charges and asymptotic symmetries in Chern–Simons, yield the canonical boundary charges and their asymptotic symmetry algebra. As a bonus you can prove charge conservation by showing that the EOM imply $\partial_t \mathcal{J} = 0$. For the last question just have a look at the algebra $[J_n, J_m] =?$ and check if there is any value for m for which the right hand side always vanishes.

  Note: you can compare with the results in `1611.09783`*. These boundary conditions were inspired by near horizon physics. The term 'soft hair' was coined by Hawking, Perry and Strominger in 2016, see* `1601.00921`.

- Keep initially all terms containing the AdS radius ℓ when writing the algebra entirely in terms of the new generators L_n and M_n. Then take the limit $\ell \to \infty$. The L_n and M_n are respectively known as 'superrotations' and 'supertranslations' (no relation to supersymmetry).

  Note: The resulting algebra is known as (centrally extended) BMS_3, which is the asymptotic symmetry algebra of flat space Einstein gravity in 3D. See `gr-qc/0610130` *for a derivation of the symmetries and* `1208.1658` *for a holographic proposal based on them.*