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OIST Exercises on Asymptotic symmetries

(3.1) **Transformation of boundary fields**

Show that the Schwarzian transformation behavior

$$\delta_\xi L = \xi L' + 2\xi' L - \frac{1}{2}\,\xi'''$$

is compatible with the twisted Sugawara construction

$$L = \frac{1}{4}\,(\Phi')^2 + \frac{1}{2}\,\Phi''$$

provided Φ transforms analogously to entanglement entropy in a CFT_2, i.e., like an anomalous scalar field. Moreover, show that $X' = e^{-\Phi}$ transforms like a non-anomalous scalar field and $Y = -\frac{1}{2}\Phi'$ like an anomalous vector field under ξ.

(3.2) **Orbifolds of $\mathbf{AdS_3}$**

Take AdS_3 (with AdS-radius ℓ) in embedding coordinates $\mathrm{d}s_4^2 = -\,\mathrm{d}u^2 - \mathrm{d}v^2 + \mathrm{d}x^2 + \mathrm{d}y^2$, $-u^2 - v^2 + x^2 + y^2 = -\ell^2$, and consider identifications of AdS_3 (a.k.a. orbifolds) by $e^{2\pi\xi}$ with the Killing vector ($r_+ > 0$) $\xi = \frac{r_+}{\ell}\,(x\partial_u u + u\partial_x)$. Derive the conditions for the identification to be free from closed time-like curves.

How does the Killing vector ξ and the metric look like in the coordinates t, r, φ defined by

$$u = \ell\frac{r}{r_+}\,\cosh\left(\frac{r+}{\ell}\,\varphi\right) \qquad x = \ell\frac{r}{r_+}\,\sinh\left(\frac{r+}{\ell}\,\varphi\right)$$

$$y = \ell\frac{\sqrt{r^2 - r_+^2)}}{r_+}\,\cosh\left(\frac{r+}{\ell^2}\,t\right) \qquad v = \ell\frac{\sqrt{r^2 - r_+^2)}}{r_+}\,\sinh\left(\frac{r+}{\ell^2}\,t\right)$$

for $r > r_+ > 0$? BONUS QUESTION: What does the identification above imply for the angular coordinate φ?

(3.3) **BTZ black hole ergo region**

Do BTZ black holes, given by the three-dimensional metric ($r_+ \geq r_- \geq 0$; $r_\pm \in \mathbb{R}^+$; $t, r \in \mathbb{R}$; $\varphi \sim \varphi + 2\pi$)

$$\mathrm{d}s^2_{\text{BTZ}} = -\frac{(r^2 - r_+^2)(r^2 - r_-^2)}{\ell^2 r^2} + \frac{\ell^2 r^2\ \mathrm{d}r^2}{(r^2 - r_+^2)(r^2 - r_-^2)} + r^2\Big(\,\mathrm{d}\varphi - \frac{r_+ r_-}{\ell r^2}\ \mathrm{d}t\Big)^2$$

have an ergo region? Are you sure?

Hints:

- Remember that entanglement entropy in a CFT_2 transforms as

$$\delta_\xi S = \xi S' - \#\,\xi'$$

with some number # that depends on the central charge. This is what is meant by "anomalous scalar field" transformation behavior. Determine this number for Φ so that everything works out. If the number vanishes, $\# = 0$, we have instead a non-anomalous scalar field. This should happen for X (possibly up to an irrelevant sign). Finally, an anomalous vector field transforms as

$$\delta_\xi V = \xi V' + \xi' V + \#\,\xi''$$

where # is another number. This should happen for Y. Note that the fields in this exercise, Φ, X, Y, are the boundary fields that appear in a Gauss decomposition of the Brown–Henneaux connection.

- Closed timelike curves exist if and only if the Killing vector ξ is timelike. The first part of this exercise is rather short, but you are encouraged to go further by reading section 3.2 in `gr-qc/9302012`. The second part is straightforward — just apply the coordinate transformation to calculate the Killing vector and the metric. If you have the result for the Killing vector the bonus question can be answered without any further calculation.

- There are two ways you could define an ergo region, hence the follow-up question of how sure you are: 1. with respect to the asymptotic time variable t, checking for zero of the norm of the Killing vector ∂_t; 2. with respect to an arbitrary Killing vector $\partial_t + A\,\partial_\varphi$ (where A is some constant chosen such that this Killing vector remains timelike everyhwere outside the event horizon). The first case is straightforward and leads to a short answer to the first question (never mind answering the second question). The second case is more interesting and leads to an even shorter answer (by one letter).