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OIST Exercises on Asymptotic symmetries

(1.1) **Variational principle for Euclidean AdS$_3$**
Show that the (Euclidean) action

$$\Gamma = -\frac{1}{16\pi G}\int \mathrm{d}^3x\sqrt{g}\left(R+\frac{2}{\ell^2}\right) - \frac{1}{8\pi G}\int \mathrm{d}^2x\sqrt{\gamma}\left(\alpha K+\frac{\beta}{\ell}\right)$$

with the boundary conditions

$$g_{rr} = \frac{\ell^2}{r^2} + \mathcal{O}(1/r^4) \qquad g_{rt} = \mathcal{O}(1/r^3)$$
$$g_{tt} = \frac{r^2}{\ell^2} + \mathcal{O}(1) \qquad g_{r\varphi} = \mathcal{O}(1/r^3)$$
$$g_{\varphi\varphi} = r^2 + \mathcal{O}(1) \qquad g_{t\varphi} = \mathcal{O}(1) + O(1/r)$$

for the metric has a well-defined variational principle only if $2\alpha = 1-\beta$.

(1.2) **Asymptotic Killing vectors in asymptotically AdS$_3$**
Derive the asymptotic Killing vectors as well as their Lie-bracket algebra for asymptotically AdS$_3$ boundary conditions

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + \Big(e^{2\rho/\ell}\gamma^{(0)}_{\mu\nu} + \gamma^{(2)}_{\mu\nu} + \mathcal{O}\big(e^{-2\rho/\ell}\big)\Big)\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu$$

with $\gamma^{(0)}_{\mu\nu} = \eta_{\mu\nu}$ and $\delta\gamma^{(2)}_{\mu\nu} \neq 0$. (Use lightcone gauge $\eta_{+-} = 1$, $\eta_{\pm\pm} = 0$.)

(1.3) **Anomalous transformation of boundary stress tensor**
Take some asymptotic AdS$_3$ line-element (you can think of a BTZ BH),

$$\mathrm{d}s^2 = \mathrm{d}\rho^2 + 4L(x^+)\,(\mathrm{d}x^+)^2 + 4\bar{L}(x^-)\,(\mathrm{d}x^-)^2 - e^{2\rho}\,\mathrm{d}x^+\,\mathrm{d}x^- + \mathcal{O}(e^{-2\rho})$$

and determine the Lie-variation of L and $\bar{L}$ generated by a vector field $\xi = \xi^+\partial_+ + \xi^-\partial_- + \xi^\rho\partial_\rho$ with

$$\xi^+ = \varepsilon^+(x^+) + \frac{1}{2}\,e^{-2\rho}\,\partial_-^2\varepsilon^-(x^-) + \mathcal{O}(e^{-4\rho})$$
$$\xi^- = \varepsilon^-(x^-) + \frac{1}{2}\,e^{-2\rho}\,\partial_+^2\varepsilon^+(x^+) + \mathcal{O}(e^{-4\rho})$$
$$\xi^\rho = -\frac{1}{2}\,\big(\partial_+\varepsilon^+(x^+) + \partial_-\varepsilon^-(x^-)\big) + \mathcal{O}(e^{-2\rho})$$

that preserves the asymptoic AdS$_3$ line-element above.

Hints/comments:

- Recall that "well-defined variational principle" is synonymous with "the first variation of the full action vanishes on-shell (including all boundary contributions) for all variations that preserve the specified boundary conditions". Solving this exercise is lengthy, but useful — not just to build your character, but to recollect some of the required tools. If you get stuck you find guidance in Eqs. (1)-(14) of `1402.3687`. For comparison, here are the results for normal vector n_μ, induced volume form $\sqrt{\gamma}$ and trace of extrinsic curvature K:

$$n_\mu = \delta^r_\mu \frac{\ell}{r} + \mathcal{O}(1/r^3)$$

$$\sqrt{\gamma} = \frac{r^2}{\ell} + \mathcal{O}(1)$$

$$K = \frac{2}{\ell} + \mathcal{O}(1/r^2)$$

- Solve the asymptotic Killing equations, starting with the components that are fixed ($g_{\rho\rho}$) or vanish ($g_{\rho\mu}$) to determine various constraints on the asymptotic Killing vectors (keeping only the leading order at large ρ). You should find the result

$$\xi = \varepsilon^+(x^+)\partial_+ + \varepsilon^-(x^-)\partial_- - \frac{\ell}{2}\left(\partial_+\varepsilon^+(x^+) + \partial_-\varepsilon^-(x^-)\right)\partial_\rho + \mathcal{O}\big(e^{-2\rho/\ell}\big)$$

Their Lie-bracket algebra follows straightforwardly. In case you want to introduce Fourier modes $\varepsilon_n^\pm$ you should find two Witt algebras,

$$[\varepsilon_n^\pm,\, \varepsilon_m^\pm] = (n-m)\,\varepsilon_{n+m}^\pm\,.$$

- To obtain the variation δL note that $4\delta L = \delta g_{++}$, where δg is the Lie variation of the metric along the vector field ξ. The final result that you find should be of the form

$$\delta L = \varepsilon^+\partial_+ L + 2L\partial_+\varepsilon^+ + \#\,\partial_+^3\varepsilon^+$$

where # is some number (that you should determine). Analogous considerations apply to the other chirality $\bar{L}$. [Note: the first two terms in the transformation law of L are the expected transformation law for a conformal primary of conformal dimension 2; the last term is an anomalous contribution and matches with the central term in the Virasoro algebra, the symmetry algebra of a CFT_2.]