1 Bonus section on Cartan formulation

So far we worked in the metric (or second order) formalism. In this section, we introduce the alternative Cartan formulation, which pragmatically offers many advantages over the metric formulation: simpler calculation of curvature and a covariant first order formalism to formulate gravity actions. At a more fundamental level the Cartan formulation naturally suggests slight modifications of General Relativity in the presence of spinning matter, essentially by allowing for dynamical torsion. We shall mostly disregard this last aspect and focus on the formalism itself and its main applications.

1.1 Differential forms

Antisymmetric tensors with p lower indices are also known as p-forms, and it is common to write them in index free notation.

$$\Omega_p = \frac{1}{p!}\,\Omega_{\mu_1\mu_2\dots\mu_p}\;\mathrm{d}x^{\mu_1}\wedge\mathrm{d}x^{\mu_2}\wedge\dots\wedge\mathrm{d}x^{\mu_p} \tag{1}$$

The wedge product

$$\mathrm{d}x^1\wedge\mathrm{d}x^2 := \mathrm{d}x^1\otimes\mathrm{d}x^2 - \mathrm{d}x^2\otimes\mathrm{d}x^1 \tag{2}$$

ensures total anti-symmetry. The wedge product between a p and a q form, $\Omega_p\wedge\Omega_q$, yields a $p+q$ form. Note that $p\leq D$, where D is the spacetime dimensions, since for $p>D$ necessarily at least one index appears twice and hence the corresponding p-form is identically zero due to anti-symmetry. The top-form (where $p=D$) is also known as "volume form"

$$\Omega_D = \frac{1}{D!}\,\Omega_{\mu_1\mu_2\dots\mu_D}\,\tilde\epsilon^{\mu_1\mu_2\dots\mu_D}\;\mathrm{d}x^D = \sqrt{-g}\,\frac{1}{D!}\,\Omega_{\mu_1\mu_2\dots\mu_D}\,\epsilon^{\mu_1\mu_2\dots\mu_D}\;\mathrm{d}x^D \tag{3}$$

where $\tilde\epsilon^{\mu_1\mu_2\dots\mu_D}$ is the totally antisymmetric Levi–Civitá symbol (with values ±1 or 0) and $\epsilon^{\mu_1\mu_2\dots\mu_D}$ is the corresponding tensor. Thus, p-forms can be integrated over p-dimensional manifolds,

$$\int_{M_p}\Omega_p = \text{diff-invariant expression} \tag{4}$$

which makes particularly top-forms useful as candidates for Lagrangians.

A specific 1-form of interest is the de-Rahm differential or exterior derivative

$$\mathrm{d} = \partial_\mu\;\mathrm{d}x^\mu\,. \tag{5}$$

It is common notation to suppress the wedge-product of $p+1$ forms generated when d acts on a p-form. For instance, when we take a scalar ϕ (or 0-form) and act on it with the exterior derivative we get a 1-form, $\mathrm{d}\phi$. Acting on this 1-form with another exterior derivative yields zero, $\mathrm{d}\wedge\mathrm{d}\phi = \mathrm{d}^2\phi = 0$. By antisymmetry this statement is true when d^2 acts on anything,

$$\mathrm{d}^2 = 0 \tag{6}$$

which expresses various formulas that you know well from basic courses, like curl grad $=0$ or div curl $=0$. The exterior derivative obeys the Leibnitz rule,

$$\mathrm{d}\big(\Omega_p\wedge\Omega_q\big) = \big(\mathrm{d}\Omega_p\big)\wedge\Omega_q + (-1)^p\,\Omega_p\wedge\mathrm{d}\Omega_q\,. \tag{7}$$

The Poincaré lemma states that any closed p-form (with $p\geq1$), $\mathrm{d}\Omega_p=0$, is exact in a star-shaped region, $\Omega_p = \mathrm{d}\hat\Omega_{p-1}$.

To given one physics example and to introduce one additional notation, defining the gauge connection 1-form, $A = A_\mu \, \mathrm{d}x^\mu$, the associated abelian field strength $F = \mathrm{d}A$, and the current 1-form $j = j_\mu \, \mathrm{d}x^\mu$ Maxwell's equations can be written as

$$\mathrm{d}F = 0 \qquad * \, \mathrm{d}{*F} = j \tag{8}$$

where the Hodge-star $*$ converts a p-form into a $(D-p)$-form,

$$* \, \Omega_p = \hat{\Omega}_{D-p} = \frac{1}{p!(D-p)!} \, \Omega_{\nu_1 \ldots \nu_p} \, \epsilon_{\mu_1 \ldots \mu_{D-p}}{}^{\nu_1 \ldots \nu_p} \, \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_{D-p}} \, . \tag{9}$$

Note that the appearance of an ϵ-tensor with mixed indices means that we need a metric to raise the required amount of indices; by contrast, p-forms and their wedge-products do not require the introduction of a metric. The Hodge-star is involutive, meaning that $* * \Omega_p = \pm \Omega_p$, where the sign depends on the form degree p, the spacetime dimension D and the signature of the metric.

While there is a lot more that could be said about differential forms, de-Rahm complexes, cohomology, Poincaré duality etc. we shall move on and focus on the few 1- and 2-forms we care about most in gravity.

1.2 Vielbein

According to Einstein's equivalence principle, locally we can always go into a freely falling elevator frame, where the metric is given by the Minkowski metric η_{ab} instead of the curved metric $g_{\mu\nu}$, but what about tangent space? If we have some vector v^μ it could be useful to have a device that converts its "curved" (Greek) index into a "flat" (Latin) one or vice verse, so that locally

$$v^a = e_\mu^a \, v^\mu \, . \tag{10}$$

The quantity e_μ^a depends on the spacetime coordinates and is known as "vielbein" (in dimensions $1, 2, 3, 4, \ldots, 11$ also known as "einbein", "zweibein", "dreibein", "vierbein", ..., "elfbein", respectively).[1] The "curved" indices are often called "holonomic" and the "flat" ones "anholonomic". They are raised and lowered with the respective metrics η_{ab} and $g_{\mu\nu}$. Note that according to the discussion in the previous subsection the vielbein is a 1-form with an additional anholonomic index, $e^a = e_\mu^a \, \mathrm{d}x^\mu$.

For the definition (10) to make sense we require the respective norms of the vectors to be equal,

$$g_{\mu\nu} v^\mu v^\nu = v^\mu v_\mu \overset{!}{=} v^a v_a = v^a v^b \eta_{ab} = \eta_{ab} e_\mu^a e_\nu^b v^\mu v^\nu \tag{11}$$

which allows to read of a crucial property of the vielbein, namely that its bilinear yields the metric.

$$\eta_{ab} e_\mu^a e_\nu^b = g_{\mu\nu} \tag{12}$$

The same logic also work in the other direction,

$$g^{\mu\nu} e_\mu^a e_\nu^b = \eta^{ab} \, . \tag{13}$$

As a consequence of the relation (12) the volume form is given by

$$\sqrt{|g|} \; \mathrm{d}^D x = |\det e_\mu^a| \; \mathrm{d}^D x = \pm e^{a_1} \wedge e^{a_2} \wedge \cdots \wedge e^{a_D} \, \epsilon_{a_1 a_2 \ldots a_D} \tag{14}$$

where the sign in the last equality depends on our choice of orientation.

[1] Some authors like to discriminate between vielbein with upper (vector) and lower (1-form) holonomic indices, calling 1-forms instead "monad", "dyad", "triad", "tetrad", ... "endekad" in dimensions $1, 2, 3, 4, \ldots, 11$. The highest number I have seen in physics literature is a vierhundertsechsundneunzigbein (I do not know its Greek version). We shall refer to e_μ^a as "vielbein" regardless of the index positions, since the index position itself tells us already whether we are in tangent space or cotangent space, so there is no need to clutter further our nomenclature.

One advantage of converting to anholonomic indices is that we can use Minkowski tensor calculus. For instance, the anholonomic components of a vector transform in the usual way under local Lorentz transformations generated by $\Lambda^a{}_b$,

$$v'^a = \Lambda^a{}_b \, v^b \,. \tag{15}$$

Let us stress the word "local" here — we can in principle perform a different Lorentz transformation here and now than at Jupiter in a million years, and our metric $g_{\mu\nu}$ will not care about it since it is not affected by these local Lorentz transformations. This means we must treat local Lorentz transformations as gauge redundancy of our theory. A different way to see that there has to be some redundancy in the vielbein description is due to simple algebraic counting. The number of independent metric components in D spacetime-dimensions is $D(D+1)/2$, while the number of independent vielbein components in the same dimension is D^2. This means there must be $D(D-1)/2$ redundant variables in the vielbein description; note that $D(D-1)/2$ is precisely the number of Lorentz generators in D dimensions.

1.3 Spin connection

Whenever we have a gauge symmetry (remember electrodynamics) we can naturally define a gauge connection, here called "Lorentz connection" or "spin connection" and denoted by $\omega^a{}_{b\mu}$, and an associated covariant derivative $D^a{}_{b\mu}$. Since both these quantities are essentially 1-forms, e.g. $\omega^a{}_b = \omega^a{}_{b\,\mu}\;\mathrm{d}x^\mu$, we use again the form notation to suppress Greek indices and define the Lorentz covariant derivative as

$$D^a{}_b = \delta^b_a \;\mathrm{d} + \omega^a{}_b \,. \tag{16}$$

Under local Lorentz transformations $\omega^a{}_b$ transforms like any non-abelian connection does,

$$\omega'^a{}_b = \left(\Lambda^{-1} D\Lambda\right)^a{}_b = \Lambda_c{}^a \,\mathrm{d}\Lambda^c{}_b + \Lambda_c{}^a \omega^c{}_d \Lambda^d{}_b \tag{17}$$

while the vielbein transforms as a Lorentz vector

$$e'^a = \Lambda^a{}_b \, e^b \,. \tag{18}$$

We can also define the full covariant derivative (i.e., Lorentz-covariant plus spacetime covariant) denoted by $\mathcal{D}$, which vanishes when acting on the vielbein (this statement is also known as "vielbein postulate").

$$\left(\mathcal{D}_\mu e\right)_a^{\;\nu} = \nabla_\mu e_a^\nu + \omega_a{}^c{}_\mu e_c^\nu = \partial_\mu e_a^\nu + \Gamma^\nu{}_{\mu\lambda} e_a^\lambda + \omega_a{}^c{}_\mu e_c^\nu = 0 \tag{19}$$

This postulate can be derived from the relation of the connection Γ to the Lorentz-covariant derivative acting on the vielbein

$$\Gamma^\lambda{}_{\mu\nu} = e_a^\lambda D^a{}_{b\mu} e_\nu^b \tag{20}$$

which in turn follows from the definition of the covariant derivative acting on vectors, $\nabla_\mu v^\nu = \partial_\mu v^\nu + \Gamma^\nu{}_{\mu\lambda} v^\lambda$, together with the vielbein definition (10) and the requirement $\nabla_\mu v^\nu = (D^a{}_{b\mu} v^b) e_a^\nu$. Analogously we have

$$\left(\mathcal{D}_\mu e\right)_\nu^{\;a} = 0 \tag{21}$$

which can be used to derive metricity, $\nabla_\lambda g_{\mu\nu} = 0$, provided the spin-connection is antisymmetric in the anholonomic indices, $\omega_{ab} = -\omega_{ba}$. We always assume metricity and hence the spin-connection is antisymmetric in the anholonomic indices throughout these lectures.

1.4 Torsion and curvature 2-form

We are now interested in Lorentz-covariant derivatives of Lorentz-covariant quantities. A 2-form T^a, known as "torsion 2-form", is obtained when acting with the Lorentz-covariant derivative on the vielbein,

$$T^a := D^a{}_b e^b = \mathrm{d}e^a + \omega^a{}_b \wedge e^b \,. \tag{22}$$

Similarly, a 2-form $R^a{}_b$, known as "curvature 2-form", is obtained when acting with the Lorentz-covariant derivative on the spin-connection,

$$R^a{}_b := D^a{}_c \omega^c{}_b = \mathrm{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \,. \tag{23}$$

The two definitions above are also known as first and second Cartan structure equations.

Comparing the anti-symmetric part of the Γ-connection in (20) with the torsion components (22) establishes the identity

$$e_a^\lambda \, T^a_{\mu\nu} = \Gamma^\lambda{}_{[\mu\nu]} \tag{24}$$

which shows that absence of torsion implies symmetry of the Γ-connection in the lower index-pair. Note, however, that torsion also can contribute to the symmetric part of the Γ-connection (this contribution is sometimes called "contorsion", which is the difference between the symmetric part of the Γ-connection and the Christoffel connection).

In the absence of torsion (and assuming metricity, as we always do) contorsion vanishes and the Γ-connection is identical with the Christoffel connection. Thus, if we want to recover Riemannian geometry we need to impose anti-symmetry of the spin-connection and vanishing of torsion. In this case the components of the curvature 2-form determine the Riemann tensor components.

$$R^\alpha{}_{\beta\mu\nu} = R^a{}_{b\,\mu\nu} e^\alpha_a e^{b\,\beta} \tag{25}$$

1.5 Bianchi identities

It is of interest to see what happens when repeatedly acting with the Lorentz-covariant derivative on itself or on the torsion- and curvature 2-forms. The relation

$$\left(D^2\right)^a{}_b = D^a{}_c D^c{}_b = R^a{}_b \tag{26}$$

is known as Bianchi's first identity and implies in particular

$$(DT)^a = \left(D^2\right)^a{}_b e^b = R^a{}_b \wedge e^b \,. \tag{27}$$

The relation

$$\left(D^3\right)^a{}_b = (DR)^a{}_b = \mathrm{d}R^a{}_b + \omega^a{}_c \wedge R^c{}_b + \omega^b{}_c \wedge R^{ac} = 0 \tag{28}$$

is known as Bianchi's second identity or Jacobi identity and shows that we do not get any new tensor structure when repeatedly acting with Lorentz-covariant derivatives on vielbein and connection, besides torsion and curvature.

Thus, the most general gravity action in the Cartan formulation must be constructed as volume form depending exclusively on vielbein, torsion and curvature (possibly with some Hodge-star). In the remaining subsections we discuss two explicit examples, Einstein–Hilbert–Palatini in vacuum and with fermionic matter.

1.6 Einstein–Hilbert–Palatini action

Let us construct actions using again ideas of effective field theories, i.e., making a derivative expansion. We want the Lagrange density to be a Lorentz-invariant volume form, so there is a unique term with no derivatives, namely the cosmological constant term $\Lambda\, e^{a_1} \wedge e^{a_2} \wedge \dots \wedge e^{a_D}\, \epsilon_{a_1 a_2 \dots a_D}$. Note that the anholonmoic ϵ-tensor equals to the Levi-Civitá symbol, i.e., its values are ± 1 or 0.

If we disregard torsion (which is a choice many people consider as reasonable, though it is by no means obvious) then there is a unique term with one derivative, namely $R^{a_1 a_2} \wedge e^{a_3} \wedge \dots e^{a_D}\, \epsilon_{a_1 a_2 a_3 \dots a_D}$. This is essentially the Einstein–Hilbert term. Further terms, e.g. containing quadratic terms in $R^a{}_b$, have higher derivatives and on general grounds should play no role at low energies as compared to the quantum gravity cutoff scale.

Let us consider for concreteness Einstein gravity in three spacetime dimensions in this formalisms. The bulk action

$$\boxed{I_{\text{EHP}}[e^a,\, \omega^a{}_b] = \frac{1}{16\pi G} \int \epsilon_{abc}\, R^{ab} \wedge e^c} \tag{29}$$

is known as Einstein–Hilbert–Palatini action. If we did regard the spin-connection as a variable dependent on the vielbein by demanding vanishing torsion (22) then the action (29) is equivalent to the Einstein–Hilbert action. However, a key aspect of Einstein–Hilbert–Palatini is that the connection is varied independently from the vielbein. The equations of motion read

$$\delta e^c : \qquad \epsilon_{abc}\, R^{ab} = 0 \tag{30a}$$

$$\delta\omega^{ab} = \epsilon^{abe}\, \delta\omega_e : \qquad T^e = 0 \tag{30b}$$

where in the second line we exploited that in three spacetime dimension any anti-symmetric tensor can be dualized to a vector using the ϵ-tensor.

Thus, even if we do not impose vanishing torsion as a constraint, the field equations (30) imply vanishing torsion and vanishing Einstein tensor. Therefore, the bulk theory defined by the Einstein–Hilbert–Palatini action (29) is equivalent to ordinary Einstein gravity. This feature works not only in three spacetime dimensions, but in any dimension, in particular in four spacetime dimensions, where this type of action was constructed originally.

1.7 Dynamical torsion

While the vacuum theories in metric or Cartan formulation can be equivalent, as we just have demonstrated, adding matter to the system breaks this equivalence and theories formulated in Cartan variables can differ physically from theories formulated in the metric formulation. This is so, because spinning matter couples to the connection so that the right hand side of the torsion constraint acquires a source.

For example, spin-$\frac{1}{2}$ fermions are described by a generalized Dirac action

$$I_\psi[\psi,\, e^a,\, \omega^a{}_b] = \int \bar\psi\, e^a \gamma_a \wedge * D\psi + \text{h.c.} \tag{31}$$

where $\bar\psi := \psi^\dagger \gamma^0$, the covariant derivative of a spinor is defined as

$$D\psi := \mathrm{d}\psi + \frac{1}{8}\, \omega^{ab} [\gamma_a,\, \gamma_b] \psi \tag{32}$$

and γ^a obey the Minkowski space anticommutation relations $\{\gamma_a,\, \gamma_b\} = 2\eta_{ab}\, \mathbb{1}$.

The key point here is that the Dirac action (31) with (32) explicitly depends on the spin-connection and hence not only the Einstein equations (30a) receive a source term (the energy-momentum tensor), but also the torsion equation (30b). Dynamical torsion is thus sourced by spinning matter, which may be the reason why we have not observed it yet, as spins tend to average to zero macroscopically.

1.8 Two-dimensional example with Cartan variables

Let us implement the Cartan formalism in a simple example. Starting with a two-dimensional metric in Eddington–Finkelstein gauge

$$\mathrm{d}s^2 = -2\,\mathrm{d}u\,\mathrm{d}r - K(u,r)\,\mathrm{d}u^2 \tag{33}$$

let us translate everything into Cartan variables.

A possible choice for the zweibein is

$$e^+ = \mathrm{d}u \qquad e^- = -\,\mathrm{d}r - \frac{1}{2}\,K(u,r)\,\mathrm{d}u \tag{34}$$

where we used anholonomic light-cone coordinates, $\eta_{\pm\mp} = 1$, $\eta_{\pm\pm} = 0$. (Note that $e^{\pm}_\mu = e_{\mu\mp}$.) The defining property (12) shows that we recover the metric (33):

$$g_{uu} = 2e^+_u e^-_u = -K(r) \qquad g_{ur} = e^+_u e^-_r + e^-_u e^+_r = -1 \qquad g_{rr} = 2e^+_r e^-_r = 0 \tag{35}$$

The inverse of (35) is given by

$$e^u_+ = 1 \qquad e^r_+ = -\frac{1}{2}\,K(u,r) \qquad e^u_- = 0 \qquad e^r_- = -1\,. \tag{36}$$

Since $[e_+,\,e_-] = [\partial_u - \frac{1}{2}\,K(u,r)\,\partial_r,\,-\partial_r] = -\frac{1}{2}\,\partial_r K(u,r)\,\partial_r = \frac{1}{2}\,\partial_r K(u,r)\,e_-$ the so-called anolonomoy coefficients are non-zero. From this result we expect the spin-connection to be proportional to $\partial_r K(u,r)$. Let us verify this now.

Setting torsion (22) to zero and dualizing the spin-connection with the flat ϵ, i.e., $\omega^a{}_b = \epsilon^a{}_b\,\omega$ (with $\epsilon^\pm{}_\pm = \pm 1$) yields

$$\mathrm{d}e^\pm + \epsilon^\pm{}_\pm\,\omega\wedge e^\pm = 0 \tag{37}$$

which can be solved for the spin-connection

$$\omega = -\frac{1}{2}\,\partial_r K(u,r)\,\mathrm{d}u = -\frac{1}{2}\,\partial_r K(u,r)\,e^+\,. \tag{38}$$

The curvature 2-form (23) is abelian in two dimensions,

$$R^\pm{}_\pm = \epsilon^\pm{}_\pm\,\mathrm{d}\omega\,. \tag{39}$$

Its components are given by

$$R^\pm{}_{\pm\,ur} = -R^\pm{}_{\pm\,ru} = \pm\big(\partial_u\omega_r - \partial_r\omega_u\big) = \mp\partial_r\omega_u = \pm\frac{1}{2}\,\partial_r^2 K(u,r) \tag{40}$$

which implies that the Riemann-tensor is proportional to the second r-derivative of the function $K(u,r)$ in the metric.

Let us now calculate Ricci-tensor. Its defining relation

$$R_{\mu\nu} = R^a{}_{b\alpha\nu} e^b_\mu e^\alpha_a \tag{41}$$

yields

$$R_{uu} = R^\pm{}_{\pm ru} e^\pm_u e^r_\pm = \frac{1}{2}\,K(u,r)\,\partial_r^2 K(u,r) \tag{42}$$

$$R_{ur} = R^\pm{}_{\pm ru} e^\pm_r e^r_\pm = \frac{1}{2}\,\partial_r^2 K(u,r) = R^\pm{}_{\pm ur} e^\pm_u e^u_\pm = R_{ru} \tag{43}$$

$$R_{rr} = R^\pm{}_{\pm ur} e^\pm_r e^u_\pm = 0 \tag{44}$$

implying that the Ricci-tensor is proportional to the metric

$$R_{\mu\nu} = -\frac{1}{2}\,\partial_r^2 K(u,r)\,g_{\mu\nu} = \frac{1}{2}\,g_{\mu\nu}\,R\,. \tag{45}$$

The Ricci scalar follows either from contracting the Ricci-tensor or from the last equality in (45) or directly from contracting the curvature 2-form (40).

$$R = R^\pm{}_{\pm ur} e^{\pm r} e^u_\pm + R^\pm{}_{\pm ru} e^{\pm u} e^r_\pm = -\partial_r^2 K(u,r) \tag{46}$$

For the special case of a metric (33) with Killing vector ∂_u, where the function $K(r)$ essentially is promoted to the Killing norm, the Ricci-scalar is proportional to the second derivative of the Killing norm.