The Canonical Transformation Generated by the Area Operator

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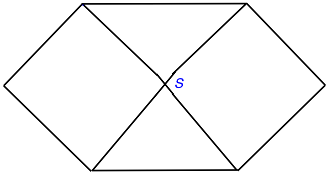
Roughly speaking, any (gauge-invariant) function on phase space can be used to generate a canonical transformation by Poisson brackets. An interesting example is the area A of an extremal surface S such as the bifurcate horizon of a black hole:



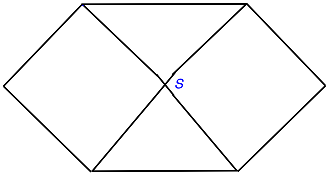
The most relevant paper for my talk is this one (where it is introduced differently and called the "kink transform")

Bousso, Chandrasekharan Rath, and Shabhazi-Moghaddam (BCRS) "Gravity Dual of Connes Cocycle Flow" arxiv:2007.00230

One of the much older papers is

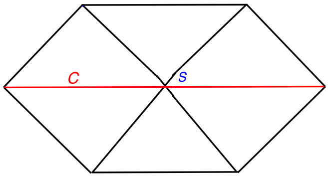
S. Carlip and C. Teitelboim, "The Off-Shell Black Hole," gr-qc/9312002.

A key fact is that the extremal surface S is a well-defined, diffeomorphism invariant notion



so its area $A(S)$ is a well-defined function on phase space, and it make sense to ask what is the symmetry that this function generates by Poisson brackets.

As in BCRS, to learn about the new spacetime that we get if we act with a canonical transformation generated by $A(S)$, one way is to think about what happens on an initial value surface C that passes through S:



The canonical variables are the metric h_{ij} on C and the canonical momentum Π_{ij}. The canonical momentum is expressed in terms of the "extrinsic curvature" of C in the full spacetime M by

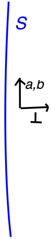
$$\Pi_{ij} = \frac{1}{8\pi G}\sqrt{\det h}\,(K_{ij} - Kh_{ij})\,,$$

where $K = h^{ij}K_{ij}$ is the trace of the extrinsic curvature.

Obviously $A(S)$, since it is only a function of the metric, commutes with h_{ij} everywhere. And since it only depends on the metric on S, it commutes with Π_{ij} everywhere except along S, where we get

$$\Pi_{ab} = 8\pi G \cdot \epsilon \cdot h_{ab}\delta_S, \quad \Pi_{\perp a} = \Pi_{\perp\perp} = 0.$$

Here ϵ is a small parameter which I included because we want to look at the infinitesimal action of a symmetry generator; δ_S is a delta function supported on S; indices a, b are tangent to S and $\perp$ is the normal direction, as I tried to show:

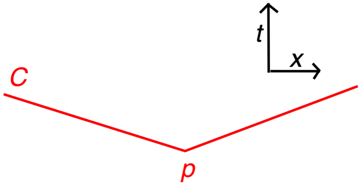


Remembering that $\Pi_{ij} = -\frac{1}{8\pi G}\sqrt{\det h}\,(K_{ij} - Kh_{ij})$, we can turn the formula for Π into a formula for K:

$$K_{\perp\perp} = \epsilon\delta_S, \quad K_{ab} = K_{a\perp} = 0.$$

So now we need to know what it means, in D spacetime dimensions, if the $D-1$-geometry of an initial value surface is smooth but its extrinsic curvature has this kind of delta function on a codimension one surface.

The basic model for this is just an initial value surface in $\mathbb{R}^{1,1}$ (two-dimensional Minkowski space) with a delta function in its intrinsic curvature. If C is an initial value surface with a "kink" or bend at a point p – but otherwise geodesically embedded –



then its extrinsic curvature has a delta function at p

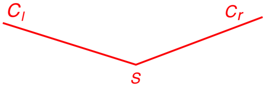
$$K = \epsilon \delta_p,$$

where ϵ is the "boost angle" associated to the kink.

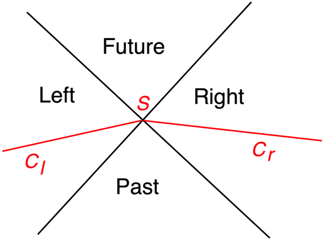
In higher dimensions, an extremal surface S is not just a point p but is of dimension $D-2$. But the local structure is the same: near S, the initial value hypersurface C looks like $S \times \mathbb{R}$ and the full spacetime M looks like $S \times \mathbb{R}^{1,1}$. The effect of the delta function is to put a "kink" in the second factor of $C \cong S \times \mathbb{R}$ while doing nothing to S.

Since we found this by acting with a canonical transformation generated by the area $A(S)$, the "kink transformed" initial value surface C satisfies the Einstein constraint equations (as shown explicitly by BCRS) and therefore it provides valid initial data for a "kink transformed spacetime." I was curious for a more explicit description of the kink-transformed spacetime. For example, what kind of singularity is there on the "horizon"? A more explicit answer to such questions is really all that I have to report today.

The extremal surface S divides the initial value surface C into a "right" C_r and "left" C_l:

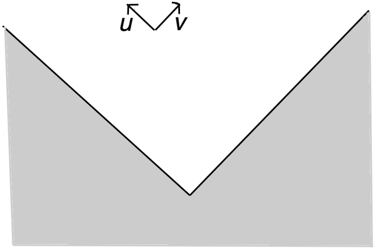


The kink transform does not change C_l or C_r, only the way they are joined. Locally near the extremal surface S, the two "congruences" of orthogonal null geodesic divide the spacetime M into four wedges:



Since the left wedge is the domain of dependence of C_l and the right wedge is the domain of dependence of C_r, the kink transform does not affect either of them. But (as noted by BCRS) it does change the future and past wedges.

To understand how the kink transform changes the future and past wedges, we will first practice in Minkowski space with a scalar field theory, say ϕ^4 theory. It is convenient to use what is called the characteristic initial value problem, in which a solution is predicted in a future wedge given initial data on the past boundary of the wedge:

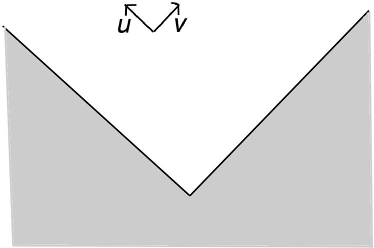


For a scalar the characteristic initial value problem is easy to describe. One just specifies initial data $\phi(u, 0, \vec{y})$ at $v = 0$ and $\phi(0, v, \vec{y})$ at $u = 0$, constrained to agree at $u = v = 0$. To see that this data suffices to determine a solution, note that the equation is

$$\frac{\partial^2 \phi}{\partial u \partial v} - \left(\frac{\partial}{\partial \vec{y}} \right)^2 \phi - \lambda \phi^3 = 0.$$

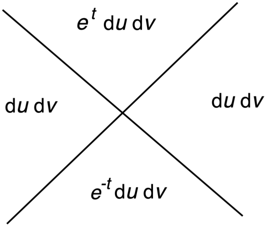
We would like to determine $\partial_v|_{v=0}\, \phi(u, v, \vec{y})$. The equation tells us ∂_u of this, so we integrate in u down to $u = 0$, where we know $\partial_v \phi(0, v, \vec{y})$ for all v. The same works for higher derivatives with respect to v at $v = 0$. Likewise we have enough information to compute derivatives with respect to u at $v = 0$. So we have a well-defined initial value problem.

Now let us go back to our problem:



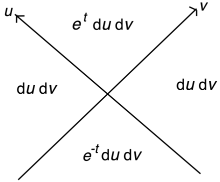
What will we choose for $\phi(u, 0, \vec{y})$ and $\phi(0, v, \vec{y})$? The answer is determined by the fact that the solution should be continuous on the future horizon (or there would be an infinite energy flux). Since the metric in the future wedge is also supposed to the Minkowski metric, juat as in the left and right wedge, it might seem that there is no way for the kink transform to give us a new solution.

Actually, there is. One way to describe it is by using different coordinate systems in different patches with various rescalings of u and v in going from patch to patch. However, there is a simpler way: use the same u and v coordinates everywhere but take different normalizations of the $\mathrm{d}u\mathrm{d}v$ part of the Lorentz metric in different patches:



In this description, we just choose the characteristic initial data in the future wedge (and also in the past wedge) so that ϕ is continuous over all past and future horizons. So we get a solution in which ϕ is continuous on both horizons, and it depends on a parameter t which is the parameter of the kink transform.

But is this a solution in Minkowski space?



Yes, as one can see by replacing u, v with new coordinates u', v' defined by

$$v = \begin{cases} v' & \text{if } v < 0 \\ e^{-t} v' & \text{if } v > 0. \end{cases}$$

$$u = \begin{cases} u' & \text{if } u > 0 \\ e^{t} u' & \text{if} u < 0. \end{cases} \tag{1}$$

The metric is now $\mathrm{d}u'\mathrm{d}v' - (\mathrm{d}\vec{y})^2$ in all four wedges. So we've found the right data for the characteristic initial value problem in the future and past wedges.

From this one can read off the nature of the shock wave along the past and future horizons. ϕ is continuous everywhere, but $\partial_v \phi$ is discontinuous across $v = 0$ and $\partial_u \phi$ is discontinuous across $u = 0$.

A very similar construction is possible for gravity (possibly coupled to matter fields, but I'll omit them). The characteristic initial value problem was first described by Sachs (1962). The metric near any codimension 2 surface S can be put in the form

$$\mathrm{d}s^2 = -e^{2q}\mathrm{d}u\mathrm{d}v + g_{AB}(\mathrm{d}x^A + C^A\mathrm{d}u)(\mathrm{d}x^B + C^B\mathrm{d}u)$$

with some conditions $q = 0$ if u or $v = 0$, $C^A = 0$ if $v = 0$. Good initial data for to find the solution in the future wedge $u, v > 0$ are the following:

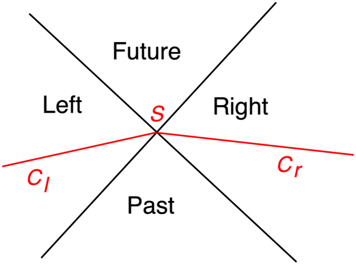
(1) one must be given g_{AB} at $u = v = 0$, and g_{AB} up to a Weyl transformation if $u = 0, v > 0$ or $u > 0, v = 0$

(2) one also needs $\partial_u \det g_{AB}$ and $\partial_v \det g_{AB} = 0$ at $u = v = 0$ for all x^A (Raychaudhuri's equation plus conditions (1) and (2) then determine g_{AB} on the future horizon, $u = 0$ or $v = 0$).

(3) finally one needs $\partial_v C_A$ at $u = v = 0$.

What set of data do we want on the past of the future wedge, and on the future of the past wedge, so as to determine the kink-transformed solution? We want to pick the characteristic initial data so that the metric will be continuous (not smooth!) on all past and future horizons. Since the starting point was that the metric in left and right wedges was given, this actually tells us, up to a coordinate choice, what should be the initial data on the boundaries of the past and future wedges. But if we choose g_{AB} on the future and past horizons to make the metric continuous, then what freedom do we have to introduce a kink parameter?

As in the discussion of the scalar, it is possible to give an answer by using different coordinate systems in different patches, but there is a simpler way. We use the same null coordinates everywhere, and we fix the initial data on the past of the future wedge and the future of the past wedge so that the metric is continuous along all horizons:



However, we depart slightly from Sachs's conventions.

In the metric

$$\mathrm{d}s^2 = -e^{2q}\mathrm{d}u\mathrm{d}v + g_{AB}(\mathrm{d}x^A + C^A\mathrm{d}u)(\mathrm{d}x^B + C^B\mathrm{d}u)$$

instead of saying that $q = 0$ if u or v vanishes, we say that if u or v vanishes on the boundary of the future wedge then

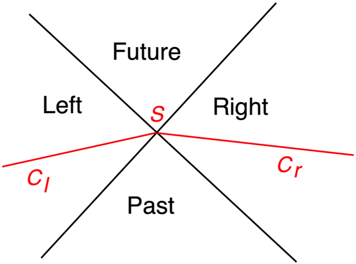
$$e^{2q} = e^t$$

and if u or v vanishes on the boundary of the past wedge, then

$$e^{2q} = e^{-t}.$$

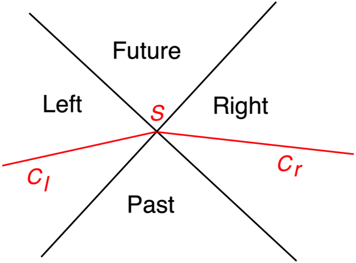
(One also does something similar for $\partial_v C^A\big|_{u=v=0}$.)

This way we get a solution of Einstein's equations in all four wedges



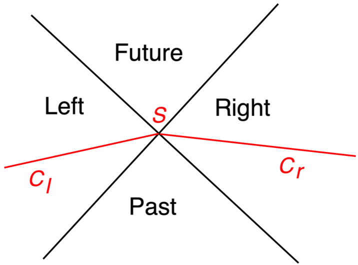
and the metric is continuous across all horizons except for the discontinuity involving g_{uv} (and some similar details involving C^A). This can be undone with the same redefinition of u, v as in the scalar case.

In our starting point, we assumed that the intersection S of the various horizons is an extremal surface:



Otherwise, we couldn't define the kink transform, since we did not have a diffeomorphism invariant function $A(S)$ on the classical phase space to use as the generator of a canonical transformation. The problem is, what is S? If S isn't an extremal surface, we don't have a diffeomorphism way to characterize it and to say which surface we are taking the area of.

But in the approach that I've explained with the characteristic initial value problem, why does S have to be extremal? Actually there is no trouble constructing the solution in the future wedge even if S isn't extremal



and there is no trouble in the past wedge. But when we combine the past and future wedges both, what happens is that if S isn't extremal, then Raychaudhuri's equation (which is part of the Einstein equations) has a delta function at $u = v = 0$ and so Einstein's equations aren't satisfied.

From this point, it is straightforward to understand what sort of singularity the solution has on the past and future horizons. The metric is continuous, but its u derivative is discontinuous at $u = 0$ and its v derivative is discontinuous at $v = 0$. As a result, R_{uAuB} has a delta function at $u = 0$ and R_{vAvB} has a delta function at $v = 0$. So there is a curvature singularity but it is a "null" singularity which does not show up in any curvature invariants.

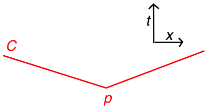
This is all I will say about the classical kink transform. Now let us discuss the situation quantum mechanically. We go back to Minkowski space, without gravity. The kink transform is trying to be a "one-sided boost." The boost operator in the $x-t$ plane can be defined as an integral on the initial value hypersurface $t=0$:

$$K = \int_{t=0} \mathrm{d}x \mathrm{d}\vec{y}\, x\, T_{00}(x, \vec{y}).$$

Naively, if we want to make a boost only on the $x > 0$ half of the hypersurface, we use

$$K_R = \int_{t=0, x>0} \mathrm{d}x\, \mathrm{d}\vec{y}\, x\, T_{00}(x, \vec{y}).$$

Classically this can be viewed as the generator of the kink transform: it boosts half of the initial value surface, producing a kink:

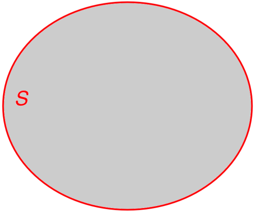


Quantum mechanically, K_R isn't a good operator. The basic way to see this is to ask if $K_R|\Omega\rangle$ is normalizable (where Ω is the vacuum vector). It isn't; a straightforward calculation gives

$$\langle\Omega|K_R^2|\Omega\rangle = \infty$$

due to fluctuations near $x = 0$. Since every state looks like the vacuum at short distances, the same divergent fluctuations are present for every state so $K_R|\Psi\rangle$ is unnormalizable for every Ψ; thus K_R does not make sense as an operator.

Here we considered a kink transform where the surface S is the boundary of the Rindler wedge (which made it possible to write explicit an explicit formal expression for the kink generator K_R) but the story is the same in ordinary quantum field theory for any S in any spacetime:



That is because the would-be kink transform generator near S suffers from the same short distance fluctuations near S that K_R suffers from near $x = 0$ in the Rindler example.

There is, however, a partial substitute for K_R: a *state-dependent* operator that I will call $K_{R,\Psi}$ which has the property that if $\mathcal{O}$ is an operator (or product of operators) in either the left or the right Rindler wedge, then

$$\langle\Psi|e^{\mathrm{i}\alpha K_{R,\Psi}}\mathcal{O}e^{-\mathrm{i}\alpha K_{R,\Psi}}|\Psi\rangle$$

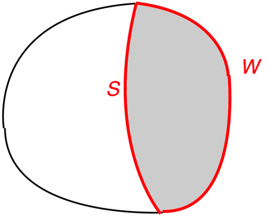
depends on α as one would formally expect if $K_{R,\Psi}$ is a one-sided boost operator:

(1) If $\mathcal{O}$ is in the left Rindler wedge, this correlator is independent of α.

(2) If $\mathcal{O}$ is in the right Rindler wedge, this correlator depends on α exactly as one would expect for a boost generator.

The catch is that the operator that does this depends on the state Ψ. It is the (generator of) the Connes cocycle flow. This partial substitute for the one-sided boost was important in the paper BCRS and in various other papers (such as the proof of the QNEC by Ceyhan and Faulker).

BCRS considered a holographic situation and asked what is the holographic dual of the Connes cocycle flow generator (i.e. the generalization of what in Rindler space is the one-sided boost) in the bulk. In other words in the usual holographic setup,



where we could define the Connes cocycle flow for the boundary region W, they asked what is the bulk dual of this flow. They argued that to leading order as $G \to 0$, the dual is the kink transform, in other words in the classical limit of the bulk theory, the dual is the flow generated by

$$\frac{A}{4G}.$$

Since the Connes cocycle flow generator is a rigorously defined operator in the boundary theory, its bulk dual must be something that also makes sense quantum mechanically. As I've explained, $A/4G$ doesn't quality, but something that does qualify is the operator associated to the generalized entropy

$$\frac{A}{4G} - \log \rho$$

where ρ is a formal density matrix for the bulk state reduced to the entanglement wedge. As we know from the example of Rindler space (where $-\log \rho$ is $2\pi K_R$), $\log \rho$ is also not a well-defined operator quantum mechanically, but the sum is. Roughly, $A/4G$ generates a one-sided boost of the geometry in the sense that I explained. By itself this isn't a good quantum operation. But $-\log \rho$ generates a compensating operation on the bulk quantum fields in the entanglement wedge.

Anyway, what I've explained is the result of my attempt to understand the kink transform more explicitly.