AdS/CFT Correspondence in Operator Formalism

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based on the paper:

JHEP 1802 (2018) 019,arXiv:1710.07298 [hep-th]

Introduction

Quantum gravity (QG) on AdS

We often consider theory in a "box"
in quantum mechanics and field theory
in order to avoid IR problems.

What is QG in a box?

A. quantum gravity on
(asymptotic) AdS

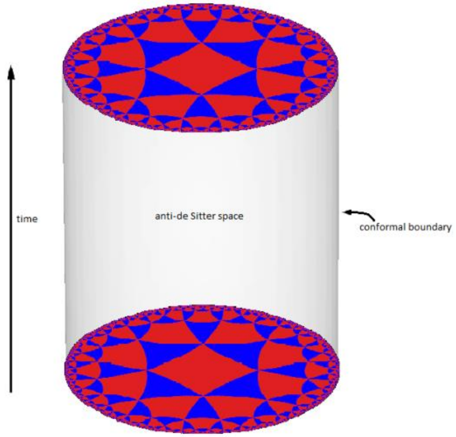

figure from
Wikipedia

Gravity Side:

Difficulties for quantization:

- Perturbatively non-renormalizable
- Un-boundedness of action
- Summations over different topologies

AdS/CFT (conjecture)

Maldacena

quantum gravity on (asymptotic) AdS

= conformal field theory

Highly non-trivial and important!

Dual CFT
(for QG with asymptotic AdS)

- Renormalizable,
- Positive definite action,
- No geometries (for finite N)

CFT could be a definition of QG.

However, **no** proof of AdS/CFT

(usual) formulation of AdS/CFT

GKPW relation
(for partition functions)

boundary condition in AdS
⇕
Source terms in CFT

Another formulation of AdS/CFT

In operator formalism,

equivalence between
Hilbert spaces and Hamiltonians
of gravity on AdS and CFT

Another formulation of AdS/CFT

equivalence between
Hilbert spaces and Hamiltonians

$$\text{Gravity theory on global } AdS_{d+1}$$

$$\Updownarrow$$

$$CFT_d \text{ on } \mathbf{R} \times S^{d-1}$$

What we will determine explicitly:

Low energy spectrum of CFT_d
which is realized as
large N strong coupling gauge theory

- leading order in large N limit

- without assuming SUSY, nor string, D-brane
- without assuming dual gravity, nor AdS
- without assuming GKPW (nor BDHM)

We will study the spectrum under only 3 assumptions (natural for strong coupling large N CFT):

1. low energy spectrum is determined only by conserved symmetry currents

2. large N factorization, which was shown for perturbation theory

3. complete independence of spectrum except symmetry relations

From these assumptions we can determine low energy spectrum of the theory,

together with the conformal symmetry.

Here, low energy means $\mathcal{O}(N^0)(\ll N)$

To do so, we have NOT used any information on the possible gravity dual, or AdS.

We just considered the field theory.

Then, we can show explicitly:

Low energy spectrum of large N CFT_d

⇕ **equivalent!**

Spectrum of free gravity on AdS_{d+1}

The spectrum determine the theory itself.

⇓

From CFT, we can

- construct bulk local fields in AdS
- derive the GKPW relation

Plan

1. **Introduction**
2. **Low energy spectrum of CFT**
3. **Deriving bulk local field and GKPW (will be skipped)**
4. **Conclusion**

Low energy spectrum of CFT

c.f. Balasubramanian-Kraus-Lawrence
Banks-Douglas-Horowitz-Martinec
Fitzpatrick-Kaplan

$$CFT_d \text{ on } \mathbf{R} \times S^{d-1}$$

unit radious

Conformal mapping from the complex plane to the cylinder (for $d = 2$)

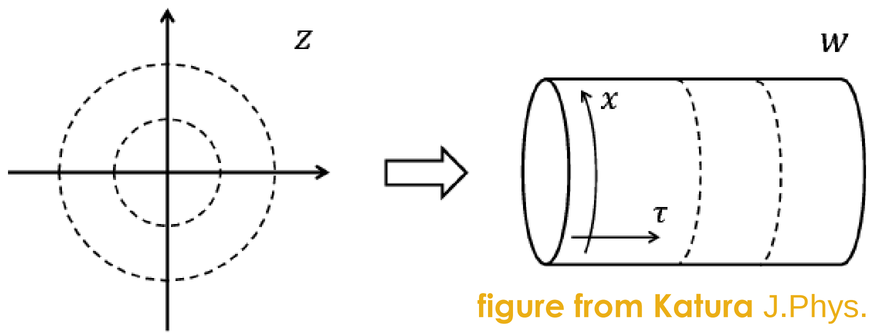

figure from Katura J.Phys. A45 (2012) 115003

CFT_d on $\mathbf{R}^d$ $\quad\Rightarrow\quad$ CFT_d on $\mathbf{R} \times S^{d-1}$

$\hat{P}_\mu = \text{translations} \longrightarrow \hat{P}_\mu = ?$

$\hat{M}_{\mu\nu} = \text{rotations} \longrightarrow \hat{M}_{\mu\nu} = \text{rotations in } S^{d-1}$

$\hat{D} = \text{dilatation} \longrightarrow \hat{D} = \text{translation in } \mathbf{R} = \boxed{\hat{H}}$

$\hat{K}_\mu = \text{special conformal} \longrightarrow \hat{K}_\mu = ?$

$$CFT_d \text{ on } \mathbf{R} \times S^{d-1}$$

unit radious

Conformal algebra

$$\left\{ \begin{aligned} &[\hat{D}, \hat{P}_\mu] = \hat{P}_\mu, \quad [\hat{D}, \hat{K}_\mu] = -\hat{K}_\mu, \\ &[\hat{K}_\mu, \hat{P}_\nu] = 2\delta_{\mu\nu}\hat{D} - 2\hat{M}_{\mu\nu} \\ &[\hat{D}, \hat{M}_{\mu\nu}] = 0, \ldots \end{aligned} \right.$$

Diagonalizing $\hat{H} = \hat{D}$ and "$\hat{M}_{\mu\nu}$",

$\hat{K}$, $\hat{P}$ are "creation" and "annihilation" operators

"Highest weight" represenation

Define primary state, $|\Delta\rangle$, s.t

$$\hat{K}_\mu|\Delta\rangle = 0, \quad \hat{D}|\Delta\rangle = \Delta|\Delta\rangle$$

Then, any state in CFT can be represented as

$$\hat{P}_{\mu_1}\hat{P}_{\mu_2}\cdots\hat{P}_{\mu_l}|\Delta\rangle$$

Primary field $\mathcal{O}_\Delta(x)$

$$\mathcal{O}_\Delta(x=0)|0\rangle = |\Delta\rangle = \hat{\mathcal{O}}_\Delta|0\rangle,$$

where $\hat{\mathcal{O}}_\Delta(x) = \lim_{x\to 0}$ (regular part of $\mathcal{O}_\Delta(x)$)

ex. for $T_{\mu\nu}$ in CFT_2, $\hat{\mathcal{O}}_\Delta = L_{-2}$ or $\tilde{L}_{-2}$

Let us consider **large** N CFT_d

$$\hat{H}(\hat{P}_{\mu_2}\cdots\hat{P}_{\mu_l}|\Delta\rangle) = (\Delta + l)(\hat{P}_{\mu_2}\cdots\hat{P}_{\mu_l}|\Delta\rangle)$$

$\Delta \gg \mathcal{O}(N^0)$ for a generic state
because of the quatum corrections

⇩

Only for symmetry currents,
energy is expected to be $\mathcal{O}(N^0)$

ex. for $T_{\mu\nu}$, $\Delta = d$

analogous to hydrodynamics

Large N factorization

't Hooft

vanishing of connected n-pont func. for $n > 2$

i.e. $\langle \mathcal{O}_1 \mathcal{O}_2 \cdots \mathcal{O}_n \rangle = \sum \langle \mathcal{O}_1 \mathcal{O}_2 \rangle \langle \mathcal{O}_3 \mathcal{O}_4 \rangle \cdots \langle \mathcal{O}_{l-1} \mathcal{O}_l \rangle$

⇕ by Wick theorem

$$[\hat{\mathcal{O}}_{\Delta_a}(x), \hat{\mathcal{O}}_{\Delta_b}(y)] = f(x-y),$$

⇨ $[\hat{\mathcal{O}}_{\Delta_a}, (\hat{\mathcal{O}}_{\Delta_b})^\dagger] = \delta_{ab}$

Polynomial

⇨ Low energy states: $R(\hat{P}^\mu, \hat{\mathcal{O}}_{\Delta_a})|0\rangle$

Complete independence

Furthermore, we assume

complete independene of states, $R(\hat{P}^\mu, \hat{\mathcal{O}}_{\Delta_a})|0\rangle$,

up to the relation $[\hat{\mathcal{O}}_{\Delta_a}, (\hat{\mathcal{O}}_{\Delta_b})^\dagger] = \delta_{ab}$

and symmetry relation, (ex. $\partial_\mu J^\mu = 0$)

because there is no specific scale for $N \to \infty$

and strong coupling.

Nothing special happens, "randomeness" or "chaos".

Large N CFT spectrum

For scalar field, we conclude that large N CFT spectrum is the Fock space spanned by $\prod_{n,l,m}(\hat{a}^\dagger_{nlm})^{\mathcal{N}_{nlm}}|0\rangle$ where

$$\hat{a}^\dagger_{nlm} \equiv c_{nl}\, s^{\mu_1\mu_2\ldots\mu_l}_{(l,m)} P_{\mu_1}P_{\mu_2}\cdots P_{\mu_l}(P^2)^n\hat{\mathcal{O}}_\Delta$$

$$[\hat{a}_{nlm}, \hat{a}^\dagger_{n'l'm'}] = \delta_{n,n'}\delta_{l,l'}\delta_{m,m'}$$

where

- c_{nl} is the normalization constant
- $s^{\mu_1\mu_2\ldots\mu_l}_{(l,m)}$ is a normalized rank l symmetric traceless constant tensor
- P^μ act on an operator such that $P^\mu\hat{\phi} = [\hat{P}^\mu, \hat{\phi}]$

Energy is given by $[\hat{H}, \hat{a}^\dagger_{nlm}] = \Delta + 2n + l$ and $n = 0, 1, 2, \cdots$ and $l = 0, 1, 2, \cdots$

Thus, the low energy limit of the large N CFT is a free theory.

What is this free theory?

Thus,
the low energy limit of the large N CFT is
a free theory.

What is this free theory?

A. free theory on AdS space

Free scalar field in AdS_{d+1}

c.f. Breitenlohner-Freedman

The metric of global AdS_{d+1} $(l_{AdS} = 1)$ is

$$ds^2_{AdS} = -(1+r^2)dt^2 + \frac{1}{1+r^2}dr^2 + r^2 d\Omega^2_{d-1}$$

where $0 \le r < \infty$, $-\infty < t < \infty$ and

$d\Omega^2_{d-1}$ is the metric for round unit sphere S^{d-1}

$$= \frac{1}{\cos^2(\rho)}\left(-dt^2 + d\rho^2 + \sin^2(\rho) d\Omega^2_{d-1}\right)$$

where $r = \tan\rho$, $0 \le \rho < \pi/2$

Boundary of AdS_{d+1} is located at $\rho = \pi/2$

Free scalar field in AdS_{d+1}

The action is

$$S_{scalar} = \int d^{d+1}x\sqrt{-\det(g)}\left(\frac{1}{2}g^{MN}\nabla_M\phi\nabla_N\phi + \frac{m^2}{2}\phi^2\right)$$

The e.o.m. is

$$0 = -g^{MN}\nabla_M\nabla_N\phi + m^2\phi^2.$$

We expand ϕ with spherical harmonics $Y_{lm}(\Omega)$,

$$\phi(t,\rho,\Omega) = \sum_{n,l,m}\left(a^\dagger_{nlm}e^{i\omega_{nl}t} + a_{nlm}e^{-i\omega_{nl}t}\right)\psi_{nlm}(\rho)Y_{lm}(\Omega)$$

Then, normalized solution for the e.o.m. is given as

$$\psi_{nlm}(\rho) = \frac{1}{N_{nl}}\sin^l(\rho)\cos^\Delta(\rho)\,{}_2F_1\left(-n, \Delta+l+n, l+\frac{d}{2}, \sin^2(\rho)\right)$$

$$\omega_{nl} = \Delta + 2n + l$$

Gauss's hyper geometric function

where $\Delta = d/2 \pm \sqrt{m^2 + d^2/4}$

Free scalar field in AdS_{d+1}

Then, quantized field is

$$\hat{\phi}(t,\rho,\Omega) = \sum_{n,l,m} \left(\hat{a}^{\dagger}_{nlm} e^{i\omega_{nl} t} + \hat{a}_{nlm} e^{-i\omega_{nl} t} \right) \psi_{nlm}(\rho) Y_{lm}(\Omega)$$

$$\begin{cases} \psi_{nlm}(\rho) = \dfrac{1}{N_{nl}} \sin^{l}(\rho) \cos^{\Delta}(\rho)\, {}_2F_1\left(-n, \Delta + l + n, l + \dfrac{d}{2}, \sin^2(\rho)\right) \\ \omega_{nl} = \Delta + 2n + l \\ N_{nl} = (-1)^n \sqrt{\dfrac{n!\Gamma(l+\frac{d}{2})^2 \Gamma(\Delta + n + 1 - \frac{d}{2})}{\Gamma(n + l + \frac{d}{2}) \Gamma(\Delta + n + l)}} \end{cases}$$

where $\Delta = d/2 \pm \sqrt{m^2 + d^2/4}$

The commutation relation and Hamiltonian are

$$[\hat{a}_{nlm}, \hat{a}^{\dagger}_{n'l'm'}] = \delta_{n,n'} \delta_{l,l'} \delta_{m,m'} \qquad [\hat{H}, \hat{a}_{nlm}] = -\omega_{nl}$$

Therefore,

(low energy theory of)
large N CFT is equivalent to
a free theory on AdS

under the 3 assumptions.

(We considered the scalar.
We think that this is protected by the SUSY
or it has just accidentally low energy spectrum.)

Plan

1. **Introduction**

2. **Low energy spectrum of CFT**

3. **Deriving bulk local field and GKPW**

4. **Conlcusion**

Deriving bulk local field and GKPW

Local field in bulk

Decompose local operator in bulk
to positive and negative frequency modes as

$$\hat{\phi}(t,\rho,\Omega) = \hat{\phi}^{+}(t,\rho,\Omega) + \hat{\phi}^{-}(t,\rho,\Omega)$$

Then using the map,

$$\hat{a}^{\dagger}_{nlm} \quad \longleftrightarrow \quad c_{nl} s^{\mu_1\mu_2\ldots\mu_l}_{(l,m)} P_{\mu_1} P_{\mu_2} \cdots P_{\mu_l} (P^2)^n \hat{\mathcal{O}}_{\Delta},$$

bulk local operator in CFT description is

$$\begin{aligned} \hat{\phi}^{+}(t=0,\rho,\Omega) &= \sum_{n,l,m} \psi_{nlm}(\rho) Y_{lm}(\Omega) \hat{a}^{\dagger}_{nlm} \\ &= \sum_{n,l,m} \psi_{nlm}(\rho) Y_{lm}(\Omega) c_{nl} s^{\mu_1\mu_2\ldots\mu_l}_{(l,m)} P_{\mu_1} P_{\mu_2} \cdots P_{\mu_l} (P^2)^n \hat{\mathcal{O}}_{\Delta} \end{aligned}$$

Local field in bulk

In particular at $\rho = 0$, only $l = 0$ modes remain:

$$\psi_{nlm}(\rho) = \frac{1}{N_{nl}} \sin^l(\rho) \cos^\Delta(\rho)\, {}_2F_1\left(-n, \Delta + l + n, l + \frac{d}{2}, \sin^2(\rho)\right) \to \frac{1}{N_{n\,0}}$$

Then, bulk local operator at the origin in CFT description is

$$\begin{aligned}\hat{\phi}^+(t = 0, \rho, \Omega) &= \sum_{n,l,m} \psi_{nlm}(\rho) Y_{lm}(\Omega) c_{nl} s^{\mu_1\mu_2\ldots\mu_l}_{(l,m)} P_{\mu_1} P_{\mu_2} \cdots P_{\mu_l} (P^2)^n \hat{\mathcal{O}}_\Delta \\ &\to \sum_{n=0}^{\infty} \frac{1}{N_{n0}} c_{n0} (P^2)^n \hat{\mathcal{O}}_\Delta \\ &= \sqrt{\frac{\Gamma(\Delta)\Gamma(\Delta + 1 - \frac{d}{2})}{\Gamma(d/2)}} \sum_{n=0}^{\infty} \frac{(-1)^n 2^{-2n}}{n!\,\Gamma(\Delta + 1 - d/2 + n)} (P^2)^n \hat{\mathcal{O}}_\Delta\end{aligned}$$

which agrees with the previous results

Miyaji-Numasawa-Shiba-Takayanagi-Watanabe
Verlinde
Nakayam-Ooguri

Bulk local field near boundary

Below, we will show

$$\lim_{\rho\to\pi/2} \frac{\hat{\phi}^+(t=0,\rho,\Omega)}{\cos^\Delta(\rho)} = \sqrt{\frac{\pi}{2}}\sqrt{\frac{\Gamma(\Delta)}{\Gamma(\Delta+1-d/2)\Gamma(d/2)}}\left(e^{P_\mu x^\mu}\hat{\mathcal{O}}_\Delta\right)|_{x^2=1}.$$

First, for the wave function,

$$\psi_{nlm}(\rho) = \frac{1}{N_{nl}}\sin^l(\rho)\cos^\Delta(\rho)\,{}_2F_1\left(-n,\Delta+l+n,l+\frac{d}{2},\sin^2(\rho)\right)$$

at the boundary is evaluated using

$$\frac{c_{nl}}{N_{nl}}\,{}_2F_1\left(-n,\Delta+l+n,l+\frac{d}{2},1\right) = \frac{2^{-2n-l}}{n!}\frac{1}{\Gamma(n+l+d/2)}\sqrt{\frac{\Gamma(d/2)\Gamma(\Delta)}{\Gamma(\Delta+1-d/2)}}.$$

Bulk local field near boundary

Expansion formula of plain wave by spherical harmonics:

$$e^{ik_\mu x^\mu} = (d-2)!! \sum_{l=0}^{\infty} i^l\, j_l^d(kr) \sum_m Y_{lm}^*(\Omega_k) Y_{lm}(\Omega)$$

$$= \sum_{l=0}^{\infty} i^l \sqrt{\frac{\pi}{2}} (kr)^l \sum_{n=0}^{\infty} 2^{-2n-l} \frac{\Gamma(d/2)(ikr)^{2n}}{n!\Gamma(n+l+d/2)} \sum_m Y_{lm}^*(\Omega_k) Y_{lm}(\Omega),$$

where
- j_l^d is hyper spherical Bessel function,
- $r = \sqrt{x^\mu x_\mu}$, $k = \sqrt{k^\mu k_\mu}$,
- Ω and Ω_k are the angular variables for x^μ and k^μ,

Applying these to (with $r = 1$ and $k_\mu = -iP_\mu$,)

$$\hat{\phi}^+(t=0,\rho,\Omega) = \sum_{n,l,m} \psi_{nlm}(\rho) Y_{lm}(\Omega) c_{nl} s_{(l,m)}^{\mu_1\mu_2\ldots\mu_l} P_{\mu_1} P_{\mu_2} \cdots P_{\mu_l} (P^2)^n \hat{\mathcal{O}}_\Delta,$$

we find $$\lim_{\rho\to\pi/2} \frac{\hat{\phi}^+(t=0,\rho,\Omega)}{\cos^\Delta(\rho)} = \sqrt{\frac{\pi}{2}} \sqrt{\frac{\Gamma(\Delta)}{\Gamma(\Delta+1-d/2)\Gamma(d/2)}} \left(e^{P_\mu x^\mu} \hat{\mathcal{O}}_\Delta\right)|_{x^2=1}.$$

Bulk local field near boundary

$$\lim_{\rho\to\pi/2}\frac{\hat{\phi}^+(t=0,\rho,\Omega)}{\cos^\Delta(\rho)} = \sqrt{\frac{\pi}{2}}\sqrt{\frac{\Gamma(\Delta)}{\Gamma(\Delta+1-d/2)\Gamma(d/2)}}e^{P_\mu x^\mu}\hat{\mathcal{O}}_\Delta,$$

$$= \sqrt{\frac{\pi}{2}}\sqrt{\frac{\Gamma(\Delta)}{\Gamma(\Delta+1-d/2)\Gamma(d/2)}}\hat{\mathcal{O}}^+_\Delta(x)|_{x^2=1}.$$

Operator on cylinder $\mathbf{R}\times S^{d-1}$ is given by $\mathcal{O}^{cy}_\Delta(\tau,\Omega)=\mathcal{O}_\Delta(x)e^{\Delta\tau}$

where $\tau=\ln(x^2)/2$

from the operator $\mathcal{O}_\Delta(x)$ which is radially quantized on $\mathbf{R}^d$.

Thus, bulk operator at bounadry is CFT field:

$$\lim_{\rho\to\pi/2}\frac{\hat{\phi}(t,\rho,\Omega)}{\cos^\Delta(\rho)} = \sqrt{\frac{\pi}{2}}\sqrt{\frac{\Gamma(\Delta)}{\Gamma(\Delta+1-d/2)\Gamma(d/2)}}\mathcal{O}^{cy}_\Delta(t,\Omega),$$

BDHM

GKPW relation

GKPW relation is essentially obtained from this BDHM

With a background "non-normalizable" mode

$\delta\phi = (\cos(\rho))^{\Delta^-}\bar{\phi} + \cdots$ with $\Delta^- = d - \Delta$,

$$\delta S = -\int_{\text{boundary}} d^d x \left((\cos(\rho))^{1-d} \delta\phi \, \frac{\partial}{\partial_\rho} \phi \right) \sim \int_{\text{boundary}} d^d x \left(\bar{\phi} \, \mathcal{O}^{cy}_{\Delta} \right) ,$$

This is a GKPW relation

c.f. HKLL

Other topics

- **Conserved current and energy momentum tensor** **c.f. Ishibashi-Wald**
- **2d CFT case**
- **Finite temperature and brick-wall**

Conclusion

- **Spectrum of large N CFT is identical to spectrum of free gravitational theory in AdS under some assumptions which are expected to be valid.**

- **Thus, two theories are equivalent for the low energy region under the assumptions.**

- **Using this equivalence, the bulk local field is constructed and the GKPW relation is derived.**

Fin.