PERIMETER INSTITUTE FOR THEORETICAL PHYSICS

SYK-like tensor quantum mechanics with $\mathrm{Sp}(N)$ symmetry

Sylvain Carrozza

Workshop on Holographic Tensors, Okinawa, Nov. 2^{nd} 2018

Collaboration with Victor Pozsgay: arXiv:1809.07753

Also based on work with: D. Benedetti, R. Gurau, and M. Kolanowski.

1. Large N limit of irreducible tensors

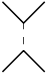
2. $\mathrm{Sp}(N)$ SYK-like model – large N

3. $\mathrm{Sp}(N)$ SYK-like model – small N

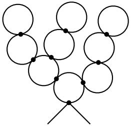
Three generic large N limits

Vector ϕ_a

$\frac{\lambda}{N}(\phi_a\phi_a)^2$



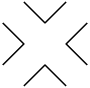
Bubble diagrams



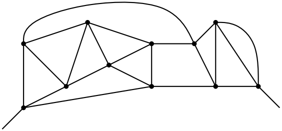
Easy

Matrix M_{ab}

$\frac{\lambda}{N} M_{ab} M_{bc} M_{cd} M_{da}$



Planar diagrams

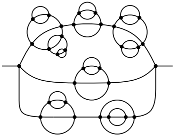


Hard

Tensor T_{abc}

$\frac{\lambda}{N^{3/2}} T_{aeb} T_{bfc} T_{ced} T_{dfa}$

Melon diagrams



Tractable

– A. Einstein (1936)

[...] we must also give up, on principle, the space-time continuum. It is conceivable that human ingenuity will some day find methods which will make it possible to proceed along such a path.

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Tensor models and random geometry:

in 2018, somehow feels like an attempt to cook a fancy meal with *one* ingredient

Main challenge: diversify our diet.

Large N tensor QFT

SYK-like models → melons can be a feature in standard local theories

[Witten '16; Klebanov, Tarnopolsky '16; ...]

- **strongly coupled physics** by analytical means
- **quantum gravity** through a different route: near AdS_2 / near CFT_1

Long-term goal

Explore the landscape of large N tensor QFTs in higher dimension.

Large N tensor QFT

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First question

How robust is the melonic limit ?

Large N tensor QFT

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First question

How robust is the melonic limit ?

- since 2010: (un)colored tensor models i.e. **no** symmetry on the indices, in arbitrary rank [Gurau, Bonzom, Rivasseau,...]
- 2017: conjecture and numerical evidence for a rank-3 symmetric traceless tensor [Klebanov, Tarnopolsky]
- 2018: rigorous proof for any irreducible rank-3 tensor [Gurau '17; Benedetti, SC, Gurau, Kolanowski '17; SC '18; SC, Poszgay '18]

Tensor quantum mechanics

Main features of **Gurau-Witten** and **Klebanov-Tarnopolsky** models:

1. SYK-like properties:
   - solvable at large N and strong-coupling;
   - emergent reparametrization symmetry;
   - same patter of symmetry breaking as in AdS_2 JT gravity;
   - quantum chaos...
2. extra IR modes e.g. $\mathrm{O}(N)^3$ NLSM
3. large number of states $\rightarrow$ not easy to study numerically. Example: KT model

N	Number of singlets
2	2
4	36
6	595 354 780

Questions:

- freedom in choice of symmetry group ?
  $\rightarrow$ unitary groups $\mathrm{U}(N)$, $\mathrm{O}(N)$ and $\mathrm{Sp}(N)$
- somewhat fewer states if we restrict to irreducible tensors ?

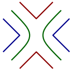
[Gurau, Rivasseau, Bonzom, Riello, ... 10s ; SC, Tanasa '15]

- Statistical model for $T_{i_1 i_2 i_3}$ transforming under $\mathrm{O}(N)^3$ as

$$T_{i_1 i_2 i_3} \to O^{(1)}_{i_1 j_1} O^{(2)}_{i_2 j_2} O^{(3)}_{i_3 j_3} \, T_{j_1 j_2 j_3}$$

- Invariant action:

$$S(T) = \frac{1}{2} T_{i_1 i_2 i_3} T_{i_1 i_2 i_3} + \frac{\lambda_1}{N^{3/2}} T_{i_6 i_2 i_3} T_{i_1 i_4 i_3} T_{i_6 i_4 i_5} T_{i_1 i_2 i_5} + \frac{\lambda_2}{N^2} T_{i_6 i_2 i_3} T_{i_1 i_2 i_3} T_{i_6 i_4 i_5} T_{i_1 i_4 i_5} + \dots$$

$= \frac{1}{2}$  $+ \frac{\lambda_1}{N^{3/2}}$  $+ \frac{\lambda_2}{N^2}$  $+ \dots$

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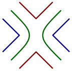
$= \frac{1}{2}$  $+ \frac{\lambda_1}{N^{3/2}}$  $+ \frac{\lambda_2}{N^2}$  $+ \dots$

- **Large N expansion** indexed by a non-negative **degree** ω

$$\mathcal{F}_N := \ln \int [dT] e^{-S(T)} = \sum_{\omega \in \mathbb{N}/2} N^{3-\omega} \mathcal{F}_\omega$$

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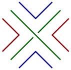
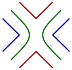
$= \frac{1}{2}$  $+ \frac{\lambda_1}{N^{3/2}}$  $+ \frac{\lambda_2}{N^2}$  $+ \dots$

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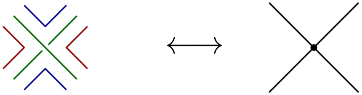
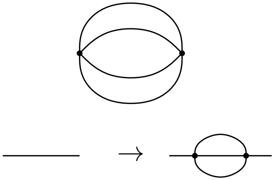
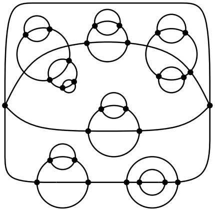
- In the rest of the talk, restrict to a single interaction:



Melon diagrams

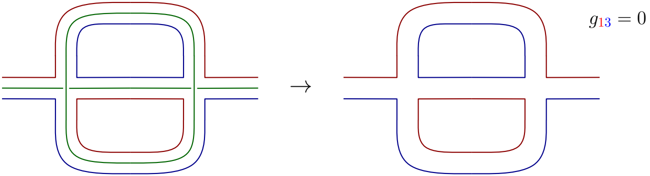
G leading order $\Leftrightarrow$ $\omega = 0$ $\Leftrightarrow$ G is a **melon diagram**





Idea of proof: melons are "super-planar" i.e. they have **planar jackets**

$$\omega := 3 + \frac{3}{2}V - F = g_{13} + g_{12} + g_{23} \in \frac{\mathbb{N}}{2}$$



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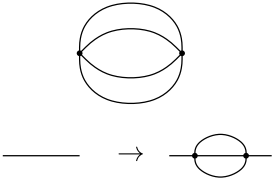

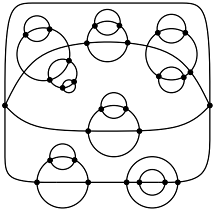

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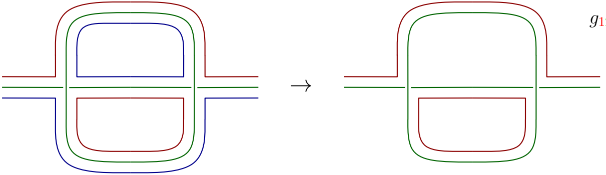

$g_{12} = 0$

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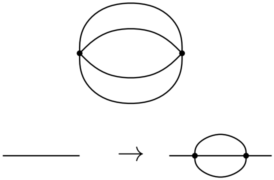

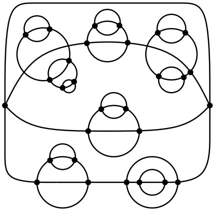

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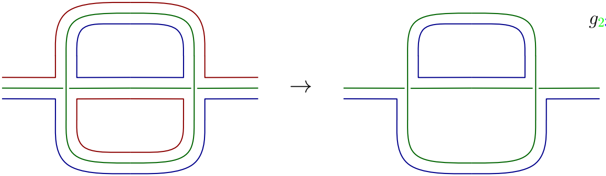
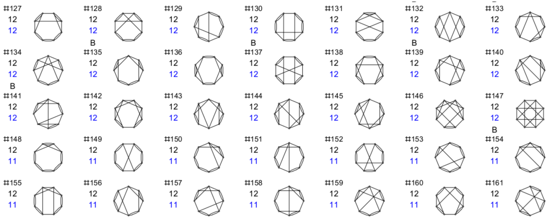


$g_{23} = 0$

(Anti)symmetrized tensors

- Conjecture and numerical evidence for $\mathrm{O}(N)$ symmetric traceless tensor models:



...explicit numerical check of all diagrams up to order λ^8...

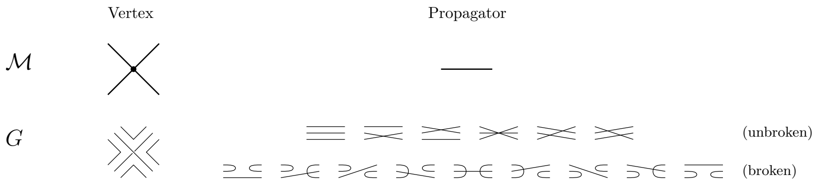
...cancellations but no obvious pattern.

[Klebanov, Tarnopolsky, JHEP '17]

- Full rigorous proof for arbitrary irreducible rank-3 tensors:

1. Simplified model with two symmetric tensors [Gurau '17]
2. $\mathrm{O}(N)$ symmetric traceless or antisymmetric [Benedetti, SC, Gurau, Kolanowski '17]
3. $\mathrm{O}(N)$ mixed symmetric traceless [SC '18]
4. $\mathrm{Sp}(N)$ irreducible [SC, Pozsgay '18]

Synthetic explanation of the cancellations: irreducibility of the representation.



- Perturbative expansion of the **free energy**:

$$\mathcal{F}_N(\lambda) = \sum_{\text{connected maps } \mathcal{M}} \frac{\lambda^{V(\mathcal{M})}}{s(\mathcal{M})} A(\mathcal{M})$$

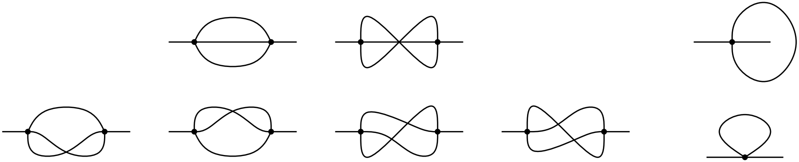
- The amplitude of a map decomposes into up to $15^{E(\mathcal{M})}$ **stranded graphs** G:

$$A(\mathcal{M}) = \sum_{G} \frac{\epsilon(G)}{R(G)} N^{-\omega(G)}$$

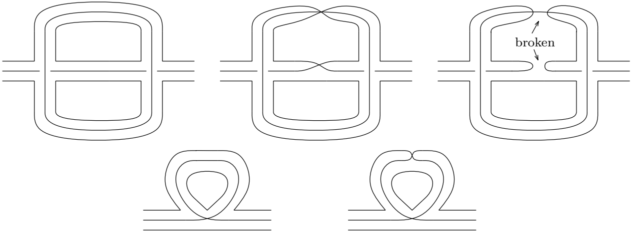
$$\omega(G) = 3 + \frac{3}{2} V(G) + B(G) - F(G)$$

$V = \#\{\text{vertices}\}$, $B = \#\{\text{broken edges}\}$, $F = \#\{\text{faces}\}$

- Maps $\mathcal{M}$: (also called "graphs on surfaces", "embedded graphs", "ribbon graphs")



- Stranded graphs (or simply *graphs*) G:

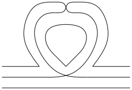
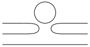


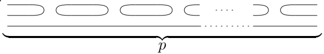
Bad tadpoles

Natural conjecture:

For any stranded graph G, $\omega(G) \geq 0$.

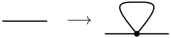
× Not true ! × Counter-example: chain of "bad tadpoles"

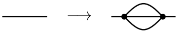
 $\sim \frac{1}{N^{3/2}}$  $\sim \frac{1}{N^{1/2}}$ 

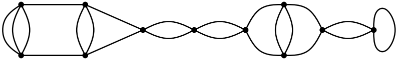
$\left(\frac{1}{N^{1/2}}\right)^p$ 

$\sim \frac{N^{p/2}}{N}$ 

More generally, ω is unbounded from below in the family of **melon-tadpoles**:



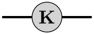


e.g. 

[Benedetti, SC, Gurau, Kolanowski '17]

1. Prove that:

The **melon-tadpole 2-point function K** verifies:



$$= K(\lambda, N) \;\text{———}\; = \left(K_0(\lambda) + \frac{K_1(\lambda)}{\sqrt{N}} + \cdots \right) \;\text{———}$$

2. Define a new perturbative expansion in terms of maps $\mathcal{M}$ with no melon-tadpole:

$$\mathcal{F}_N(\lambda) = \sum_{\substack{\text{connected } \mathcal{M} \\ \text{no melon} \\ \text{no tadpole}}} \frac{\lambda^{V(\mathcal{M})}}{s(\mathcal{M})} K(\lambda, N)^{2V(\mathcal{M})} A(\mathcal{M})$$

3. Prove that:

∀ stranded graph G without melon-tadpole

$$\omega(G) \geq 0$$

$$\Rightarrow \qquad \mathcal{F}_N = \sum_{\omega \in \mathbb{N}/2} N^{3-\omega} \mathcal{F}_\omega$$

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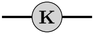

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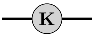

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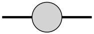
- *Any* 2-point map is proportional to the propagator:

 $\propto$

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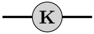

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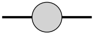 $\propto$

Proof: Schur's lemma. □

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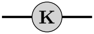

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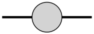

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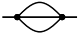
- Example: symmetric traceless tensors



$$= \frac{N^6 + 15N^5 + 64N^4 - 84N^3 - 800N^2 + 384N + 1536}{6^2 N^3 (N+2)^3} \mathbf{P} \sim \frac{1}{36} \mathbf{P}$$

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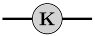

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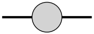
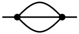


$$\propto \quad \text{——}$$

Proof: Schur's lemma. □

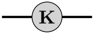
- Example: symmetric traceless tensors



$$= \frac{N^6 + 15N^5 + 64N^4 - 84N^3 - 800N^2 + 384N + 1536}{6^2 N^3 (N+2)^3} \mathbf{P} \sim \frac{1}{36} \mathbf{P}$$

$$= \frac{N^2 + 2N - 8}{2N^{3/2}(N+2)} \mathbf{P} \sim \frac{1}{2N^{1/2}} \mathbf{P}$$

The melon-tadpole 2-point function **K** verifies:

 $= \quad K(\lambda, N) \quad$ —— $\quad = \left(K_0(\lambda) + \frac{K_1(\lambda)}{\sqrt{N}} + \cdots \right)$ ——

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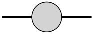 $\propto$ ——

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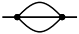 $= \dfrac{N^6 + 15N^5 + 64N^4 - 84N^3 - 800N^2 + 384N + 1536}{6^2 N^3 (N+2)^3} \mathbf{P} \sim \dfrac{1}{36} \mathbf{P}$

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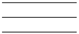
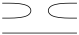
- Explanation of tadpole cancellations: irreducibility of the representation

 $= \dfrac{1}{N^{1/2}} \Big(a$  $+ \cdots + b$  $+ \cdots \Big) + \cdots \propto \mathbf{P}$

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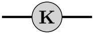

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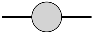

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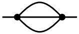

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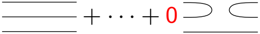

1. Large N limit of irreducible tensors
2. $\mathrm{Sp}(N)$ SYK-like model – large N
3. $\mathrm{Sp}(N)$ SYK-like model – small N

- $\mathrm{O}(N)^3$ Tensor quantum mechanics of N^3 **Majorana fermions**:[Klebanov, Tarnopolsky '16]

$$S = \int dt \left(\frac{\mathrm{i}}{2} \psi_{i_1 i_2 i_3} \partial_t \psi_{i_1 i_2 i_3} + \frac{\lambda}{4 N^{3/2}} \psi_{i_1 i_2 i_3} \psi_{i_4 i_5 i_3} \psi_{i_4 i_2 i_6} \psi_{i_1 i_5 i_6} \right)$$

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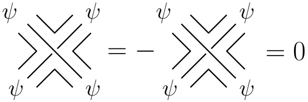
- If one reduces the $\mathcal{S}_3$ symmetry i.e. ψ_{abc} *symmetric*, *antisymmetric* or *mixed*:



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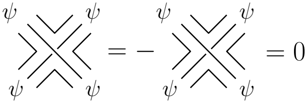
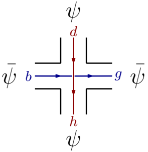


- To avoid this problem: $\mathrm{Sp}(N)$ version

$$\epsilon_{bg} \epsilon_{dh} \bar{\psi}_{abc} \bar{\psi}_{fge} \psi_{ade} \psi_{fhc}$$



with ϵ a skew-symmetric matrix: $\epsilon^\top = -\epsilon = \epsilon^{-1}$

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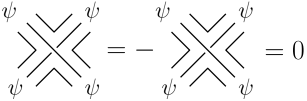

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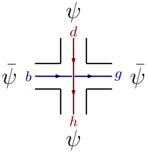

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	$\mathrm{O}(N)$ irreducible	$\mathrm{Sp}(N)$ irreducible
Bosonic	$\neq 0$	$= 0$
Fermionic	$= 0$	$\neq 0$

Irreducible $\mathrm{Sp}(N)$ representation

- $\mathrm{Sp}(N) := \mathrm{U}(2N) \cap \mathrm{Sp}(2N, \mathbb{C})$ preserves: $\bar{\psi}_a \delta_{ab} \psi_b$ and $\psi_a \epsilon_{ab} \psi_b$

$$U \in \mathrm{Sp}(N)\,, \qquad \psi_{a_1 a_2 a_3} \to U_{a_1 b_1} U_{a_2 b_2} U_{a_3 b_3} \psi_{b_1 b_2 b_3}$$

- Two commuting operations: **permutations** of indices and **ϵ-traces**

$$\psi_{a_1 a_2 a_3} \to \psi_{a_{\sigma(1)} a_{\sigma(2)} a_{\sigma(3)}} \qquad \text{and} \qquad \psi_{a_1 a_2 a_3} \to \psi_{a_1 bc} \epsilon_{bc}\,,\ \psi_{b a_2 c} \epsilon_{bc}\,,\ T_{bca_3} \epsilon_{bc}$$

- Three irreducible tensor representations:
  - completely *symmetric* (S)

$$n^{(S)} = \frac{2N}{3} \left(2N^2 + 3N + 1\right)$$

  - completely *antisymmetric traceless* (A)

$$n^{(A)} = \frac{2N}{3} \left(2N^2 - 3N - 2\right)$$

  - *mixed traceless* (M)

1	2
3	

$$n^{(M)} = \frac{8N}{3} \left(N^2 - 1\right)$$

$$S[\bar{\psi},\psi] = \int \mathrm{d}t \, \left(i\bar{\psi}_{abc}\partial_t\psi_{abc} - \frac{g}{2}\epsilon_{bg}\epsilon_{dh}\bar{\psi}_{abc}\bar{\psi}_{fge}\psi_{ade}\psi_{fhc} \right) \qquad g = \frac{\lambda}{N^{3/2}}$$

- $U(1)$ symmetry $\rightarrow$ 1 conserved charge:

$$\mathcal{Q} = \frac{1}{2}[\bar{\psi}_{abc}, \psi_{abc}]$$

- $\mathrm{Sp}(N)$ symmetry $\rightarrow$ $N(2N+1)$ conserved charges:

$$\begin{aligned}
\hat{I}_{k,l} &:= \left[i\left(\bar{\psi}_{(2k-1)bc}\psi_{(2l-1)bc} + \bar{\psi}_{(2k)bc}\psi_{(2l)bc}\right) + \mathrm{c.c}\right] \quad + \quad (1\to 2\to 3)\,, \\
\hat{\Sigma}^{(1)}_{k,l} &:= \left[\left(\bar{\psi}_{(2k-1)bc}\psi_{(2l)bc} + \bar{\psi}_{(2k)bc}\psi_{(2l-1)bc}\right) + \mathrm{c.c}\right] \quad + \quad (1\to 2\to 3)\,, \\
\hat{\Sigma}^{(2)}_{k,l} &:= \left[i\left(\bar{\psi}_{(2k-1)bc}\psi_{(2l)bc} - \bar{\psi}_{(2k)bc}\psi_{(2l-1)bc}\right) + \mathrm{c.c}\right] \quad + \quad (1\to 2\to 3)\,, \\
\hat{\Sigma}^{(3)}_{k,l} &:= \left[\left(\bar{\psi}_{(2k-1)bc}\psi_{(2l-1)bc} - \bar{\psi}_{(2k)bc}\psi_{(2l)bc}\right) + \mathrm{c.c}\right] \quad + \quad (1\to 2\to 3)\,, \qquad 1\leq k<l\leq N\,, \\
\hat{\Sigma}^{(1)}_{m} &:= \left(\bar{\psi}_{(2m-1)bc}\psi_{(2m)bc} + \bar{\psi}_{(2m)bc}\psi_{(2m-1)bc}\right) \quad + \quad (1\to 2\to 3)\,, \\
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\end{aligned}$$

$$\mathcal{C}_2 := \sum_{1\leq k<l\leq N}\left((\hat{I}_{k,l})^2 + \sum_{p=1}^{3}(\hat{\Sigma}^{(p)}_{k,l})^2 \right) + 2\sum_{1\leq m\leq N}\sum_{p=1}^{3}(\hat{\Sigma}^{(p)}_{m})^2$$

Two-point function

$$\langle T(\bar{\psi}_{abc}(t)\psi_{a'b'c'}(t'))\rangle = G(t,t')\,\mathbf{P}_{abc,a'b'c'}$$

- Large N limit $\rightarrow$ closed Schwinger-Dyson equation

$$G = G_0 + \lambda_{\text{eff}}^2\, G_0 * G^3 * G$$

- Further simplification at strong coupling

$$G * G^3 = \frac{-1}{\lambda_{\text{eff}}^2}$$

- Emergent conformal invariance: reparametrization $t \mapsto f(t)$

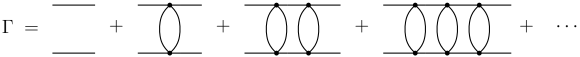
$$G(t_1,t_2) \mapsto |f'(t_1)f'(t_2)|^{1/4}\, G(f(t_1), f(t_2))$$

- Conformal solution:

$$G(t_1,t_2) = -\left(\frac{1}{4\pi\lambda_{\text{eff}}^2}\right)^{1/4} \frac{\operatorname{sgn}(t_1-t_2)}{|t_1-t_2|^{1/2}}$$

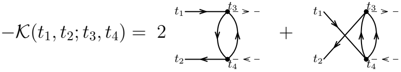
Four-point function

$$\langle \bar{\psi}_{abc}(t_1)\psi_{abc}(t_2)\bar{\psi}_{def}(t_3)\psi_{def}(t_4)\rangle = n^2 G(t_1,t_2)G(t_3,t_4) + n\Gamma(t_1,t_2,t_3,t_4) + \ldots$$



- Ladder operator:

$$\mathcal{K}(t_1,t_2;t_3,t_4) = -\lambda_{\text{eff}}^2 \left[2G(t_1,t_3)G(t_2,t_4) - G(t_1,t_4)G(t_2,t_3)\right] G(t_3,t_4)^2$$



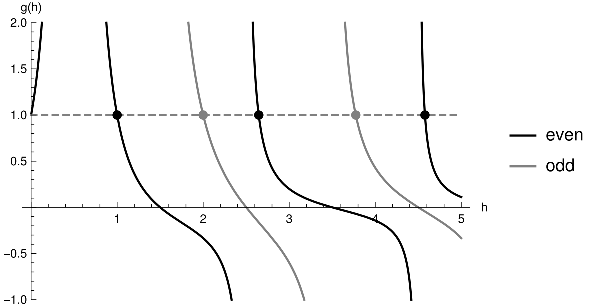
- IR conformal spectrum from Bethe-Salpeter equation:

$$v_k(t_0,t_1,t_2) = g(h_k)\int \mathrm{d}t_3\mathrm{d}t_4\, \mathcal{K}(t_1,t_2;t_3,t_4)v_k(t_0,t_3,t_4) \quad \text{with} \quad g(h_k) = 1$$

with

$$v_k(t_0,t_1,t_2) := \langle \mathcal{O}_2^k(t_0)\psi_{abc}(t_1)\bar{\psi}_{abc}(t_2)\rangle$$

Conformal dimensions



$$h_0 = 1\,, \quad h_1 = 2\,, \quad h_2 \approx 2.65\,, \qquad h_3 \approx 3.77\,, \qquad \text{etc.}$$

→ identical to complex SYK model and $\mathrm{SU}(N) \times O(N) \times \mathrm{SU}(N)$ tensor model

Main difference: $\mathrm{Sp}(N)$ pseudo-Goldstone modes in the IR.

Enumeration of singlets

1$^{\mathrm{st}}$ ingredient: general character formula

$$I_N := \#\{\mathrm{singlets}\} = \int_{\mathrm{Sp}(N)} \mathrm{d}U \, \chi(U)$$

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1[st] ingredient: general character formula

$$I_N := \#\{\text{singlets}\} = \int_{\mathrm{Sp}(N)} \mathrm{d}U\, \chi(U)$$

- For a fermionic Fock representation $\wedge(\rho)$:

$$\chi(U) = \chi_\wedge(\rho)(U) = \sum_{k=0}^{n} \mathrm{Tr}[\wedge^k(\rho(U))] = \det\left[1 + \rho(U)\right]$$

and therefore

$$I_N = \int_{\mathrm{Sp}(N)} \mathrm{d}U \det\left[1 + \rho(U)\right] = \int_{\mathrm{Sp}(N)} \mathrm{d}U \exp\left(-\sum_{k=1}^{+\infty} \frac{(-1)^k}{k} \chi_\rho(U^k)\right)$$

- In our case $\rho = S, A$ or M:

$$\chi_S(U) = \frac{1}{6}\mathrm{tr}(U)^3 + \frac{1}{2}\mathrm{tr}(U^2)\mathrm{tr}(U) + \frac{1}{3}\mathrm{tr}(U^3)$$

$$\chi_A(U) = \frac{1}{6}\mathrm{tr}(U)^3 - \frac{1}{2}\mathrm{tr}(U^2)\mathrm{tr}(U) + \frac{1}{3}\mathrm{tr}(U^3) - \mathrm{tr}(U)$$

$$\chi_M(U) = \frac{1}{3}\mathrm{tr}(U)^3 - \frac{1}{3}\mathrm{tr}(U^3) - \mathrm{tr}(U)\,.$$

Enumeration of singlets

2$^{\mathrm{nd}}$ ingredient: integration formula for a class function f on $\mathrm{Sp}(N)$:

$$\int_{\mathrm{Sp}(N)} f(U)\mathrm{d}U = \int_{[-\pi,\pi]^N} f(\theta_1, ..., \theta_N)\, \mathrm{d}\mu(\theta_1,\, \ldots,\, \theta_N)\,,$$

where

$$\mathrm{d}\mu(\theta_1,\, \ldots,\, \theta_N) := \frac{2^{N^2}}{N!(2\pi)^N} \prod_{i=1}^{N} \sin^2\theta_i \prod_{1\leq i<j\leq N} \left(\cos\theta_i - \cos\theta_j\right)^2 \,.$$

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$\Rightarrow$ explicit integral formula for I_N

e.g.

$$I_N^{(S)} = 2^{\frac{2N}{3}\left(2N^2+3N+1\right)} \int_{[-\pi,\pi]^N} \mathrm{d}\mu(\theta_1, \ldots, \theta_N) \left(\prod_{k=1}^{N} \cos\frac{3\theta_k}{2}\right)^2 \left(\prod_{k=1}^{N} \cos\frac{\theta_k}{2}\right)^{2N}$$
$$\times \left(\prod_{1\leq k<l\leq N} \cos\frac{2\theta_k+\theta_l}{2} \cos\frac{\theta_k+2\theta_l}{2} \cos\frac{2\theta_k-\theta_l}{2} \cos\frac{\theta_k-2\theta_l}{2}\right)^2$$
$$\times \left(\prod_{1\leq k<l<m\leq N} \cos\frac{\theta_k+\theta_l+\theta_m}{2} \cos\frac{\theta_k+\theta_l-\theta_m}{2} \cos\frac{\theta_k-\theta_l+\theta_m}{2} \cos\frac{\theta_k-\theta_l-\theta_m}{2}\right)^2$$

Enumeration of singlets

The number of states grows extremely quickly in tensor models

N	Fock	Singlets
1	2^4	3
2	2^{20}	39
3	2^{56}	170640
4	2^{120}	$\sim 10^{14}$

Symmetric

N	Fock	Singlets
1	–	–
2	–	–
3	2^{14}	8
4	2^{48}	370

Antisymmetric traceless

N	Fock	Singlets
1	–	–
2	2^{16}	18
3	2^{64}	169826605
4	2^{160}	$\sim 10^{26}$

Mixed traceless

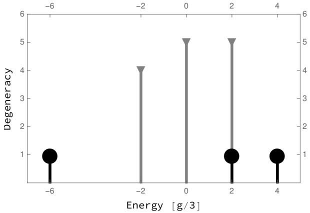
As usual: numerically challenging, except at very small N

But interesting target: symmetric, $N = 3$

$\sim 10^5$ states, compared to $\sim 10^8$ in KT $\mathrm{O}(6)$ model.

Explicit diagonalization 1

$N = 1$, symmetric $\rightarrow$ **16** Fock states, **3** singlets.

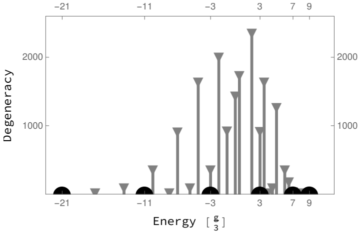


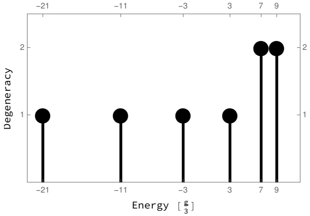
Energy $\tilde{H}$ $[g/3]$	$\mathrm{U}(1)$ charge $\mathcal{Q}$
-6	2
2	-2
4	0

3 singlets

Explicit diagonalization 2

$3 = 3$, antisymmetric traceless $\rightarrow$ **16384** Fock states, **8** singlets.





Energy $\tilde{H}$ $[g/3]$	U(1) charge $\mathcal{Q}$
-21	7
-11	5
-3	3
3	1
7 7	-7 -1
9 9	-5 -3

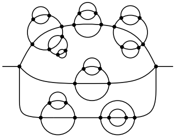
8 singlets

Summary and outlook

Tensor T_{abc}

$\frac{\lambda}{N^{3/2}} T_{aeb} T_{bfc} T_{ced} T_{dfa}$

Melon diagrams



Tractable

- **Third universal class** of large N methods.
- Melons lie in a sweet spot: **both tractable and rich**!
- **Robust** methods:

  colored $\rightarrow$ **irreducible tensor models**
  $\mathrm{U}(N)$, $\mathrm{O}(N)$ $\rightarrow$ $\mathrm{Sp}(N)$

Future:

- enumeration of $\mathrm{Sp}(N)$ invariants: algebraic methods ?
- spectrum of tensor QM: symmetric $\mathrm{Sp}(3)$ model ?

- further extensions of the melonic limit ?
- landscape of **large N tensor QFTs** in higher d ?
- **holography**: higher spins ?