Tensor models, algebras and topological holography

Sanjaye Ramgoolam

Queen Mary, University of London

Mini-symposium "Holographic Tensors"
OIST, Japan, 2018

"Counting Tensor Model Observables and Branched Covers of the 2-Sphere," Joseph Ben Geloun, Sanjaye Ramgoolam, https://arxiv.org/abs/1307.6490, AIHPD, 2014
"Tensor Models, Kronecker coefficients and Permutation Centralizer Algebras," Joseph Ben Geloun, Sanjaye Ramgoolam, https://arxiv.org/abs/1708.03524, JHEP 1711 (2017) 092

P. Mattioli and S. Ramgoolam, “Permutation Centralizer Algebras and Multi-Matrix Invariants,” Phys. Rev. D **93** (2016) no.6, 065040 [arXiv:1601.06086 [hep-th]].
P. Diaz and S. J. Rey, “Invariant Operators, Orthogonal Bases and Correlators in General Tensor Models,” Nucl. Phys. B **932** (2018) 254, [arXiv:1801.10506 [hep-th]].
A. Mironov and A. Morozov, “Correlators in tensor models from character calculus,” Phys. Lett. B **774** (2017) 210 [arXiv:1706.03667 [hep-th]].
H. Itoyama, A. Mironov and A. Morozov, “From Kronecker to tableau pseudo-characters in tensor models,” arXiv:1808.07783 [hep-th].

plus earlier papers on tensor models, authored by many of you here, which motivated our investigations (and are cited in the papers with Joseph).

Introduction : Permutation algebras

Tensor models have an underlying algebraic structure related to permutation groups.

The group S_n is the group of all permutations of n distinct objects, e.g $\{1, 2, \cdots, n\}$. There is an algebra $\mathbb{C}(S_n)$, consisting of linear combinations with complex coefficients, of group elements.

$$\sum_{\sigma \in S_n} a_\sigma \, \sigma \sum_{\tau \in S_n} b_\tau \, \tau = \sum_{\sigma, \tau} a_\sigma b_\tau (\sigma \cdot \tau)$$

The algebras for tensor models are related to these group algebras, which we will describe concretely for complex bosonic tensor models.

An important property of these algebras is that they admit a Fourier transformation, given by representation theory.

Take an irrep V_R labelled by a Young diagram with n boxes. We have, with a choice of orthonormal basis, matrices

$$D^R_{ij}(\sigma)$$

for every group element. The i, j indices range over $1 \leq i, j \leq d_R$ where d_R is the dimension of the irrep.

$$Q^R_{ij} = \frac{d_R}{n!} \sum_{\sigma} D^R_{ij}(\sigma)\sigma^{-1} \in \mathbb{C}(S_n)$$

and give a complete basis.

$$n! = \sum_R d_R^2$$

Semi-simple algebra, WA decomposition

Using orthogonality propertries of the matrix elements, can show that

$$Q^R_{ij} Q^S_{kl} = \delta^{R,S} \delta_{jk} Q_{il}$$

For fixed $R = S$, this is like the multiplication of elementary matrices

$$E_{ij} E_{kl} = \delta_{jk} E_{il}$$

The Q^R_{ij} form a basis for $\mathbb{C}(S_n)$ which shows that it is a direct sum labelled by irreps (Young diagrams). For each R, we have $d_R \times d_R$ elements in $\mathbb{C}(S_n)$, which is thus a direct sum of matrix algebras.

This is an example of the Wedderburn-Artin theorem at work. Any semi-simple associative algebra has a matrix decomposition.

Algebra : vector space with an associative product.

semi-simple : The vector space also has a non-degenerate bilinear pairing.

Here

$$< \sigma_1, \sigma_2 >= \delta(\sigma_1 \sigma_2^{-1})$$

δ is defined as

$$\begin{aligned} \delta(\sigma) \quad &= 1 \quad \text{for} \quad \sigma = \text{ identity group element} \\ &= 0 \quad \text{for} \quad \sigma \neq \text{ identity} \end{aligned}$$

See refs to math books and online math notes in the papers with Joseph for general proof and discussion of the WA theorem

Lattice gauge theory with finite gauge group, and a topological action, give topological gauge theory. Can define this on cell decomposition of 2D surfaces, or more general 2-complexes. Sum over group variables for each edge.

Plaquette action :

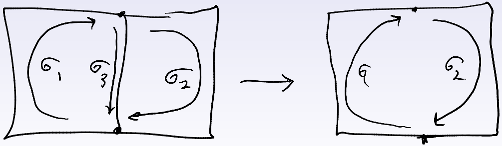
$$\begin{aligned} Z_P(\sigma_P) &= \delta(\sigma_P) \\ \delta(\sigma) &= 1 \ \text{ if } \sigma = 1 \\ &= 0 \quad \text{otherwise} \end{aligned}$$

Partition function :

$$Z = \frac{1}{n!^V} \sum_{\{\sigma_E\}} \prod_P Z_P(\sigma_P)$$

Introduction : Topological lattice gauge theory - invariance

04 April 2013
13:29



$$\sum_{\sigma_3} \delta(\sigma_1 \sigma_3)\, \delta(\sigma_3^{-1} \sigma_2) \longrightarrow \delta(\sigma_1 \sigma_2)$$

integrating out an edge $\longrightarrow$ Plaquette weight of new Plaquette

This partition function calculates a sum of homomorphisms from π_1 of the cell complex into the gauge group (here S_n) , e.g. on a torus

$$\frac{1}{n!} \sum_{\sigma_1, \sigma_2 \in S_n} \delta(\sigma_1 \sigma_2 \sigma_1^{-1} \sigma_2^{-1})$$

Counting pairs σ_1, σ_2 which commute. If define equivalence classes by simultaneous conjugation,

$$\sigma_1, \sigma_2) \sim (\gamma \sigma_1 \gamma^{-1}, \gamma \sigma_2 \gamma^{-1})$$

can write this as

$$\sum_{\text{equiv classes}} \frac{1}{\text{Aut}}$$

Also counts unbranched covers of $T^2 \to T^2$ of degree n, weighted by inverse automorphisms of the cover.

Introduction : Topological lattice gauge theory - covers

For sphere with d boundaries, we can fix the conjugacy classes of the permutations at the boundaries, and obtain topological amplitudes

$$\mathcal{Z}(T_1, T_2, \cdots, T_d) = \frac{1}{n!} \sum_{\sigma_1 \in T_1} \sum_{\sigma_2 \in T_2} \cdots \sum_{\sigma_d \in T_d} \delta(\sigma_1 \sigma_2 \cdots \sigma_d)$$

These amplitudes (and their higher genus generalizations provide examples of Frobenius algebras - algebracic defintion of topological field theory by Atiyah; subsequently discussed in physics literature - refs in papers with Joseph).

This has a geometrical interpretation in terms of counting branched covers of the 2-sphere with up to d branch points. Example one boundary, $n = 2, d = 2$

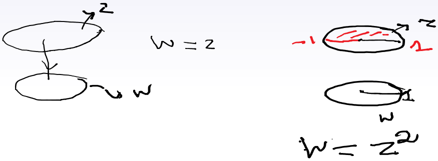
$$\mathcal{Z}(T_1, T_2) = \sum_{\sigma_1 \in T_1} \sum_{\sigma_2 \in T_2} \delta(\sigma_1 \sigma_2)$$

$T_1 = T_2$ conjugacy class of $(1)(2)$. Or $T_1 = T_2$ conjugacy class of $(1, 2)$.

Branched covers : Quick review

A holomorphic map from a Riemann surface to another is a branched cover. With fixed branch locus, the counting of branched covers

01 November 2018 15:27



As go round w = 0 ;
inverse image 1 → 2
2 → 1

The above gives the local description of a branch point - a point on the target space. At degree 2, we can have (1)(2) or (12) as permutations of the inverse images.

If we have $w = z^2$ globally on P^1, then we have (12) locally around $w = 0$ and (12) around $w = \infty$.

Formally and more generally, the holomorphic maps from P^1 to P^1, with degree n and fixed branch locus, are given by $Hom(\pi_1(P^1 \setminus \text{ Branch locus })) \to S_n$.

see, for example, Wikipedia article on "Riemann existence theorem" .

The sphere with d branch points removed has a fundamental group which is generated by d generators $\alpha_1, \alpha_2, \cdots, \alpha_d$, and one relation

$$\alpha_1 \alpha_2 \cdots \alpha_d = 1$$

The string theory of 2dYM - large N expansion of 2D $U(N)$ gauge theory - was developed by using group theory and branched cover theory by Gross and Taylor 1994.

This was the second example of gauge-string duality from 1990's.

So although no extra dimension, it is gauge-string duality, and thus holographic. In the tensor model cases, there are emergent dimensions from the counting and correlators in zero dimensions, so perhaps more deservedly holographic.

In the first example, namely old matrix models, multiple descriptions of the same physics.

E.g. matrix dynamics in one dimension with an inverted harmonic oscillator potential,

$c = 1$ matter coupled to Liouville theory,

string sigma model in a 1D dilaton background;

topological field theory on the worldsheet;

Collective field theory of Das-Jevicki (a sort of string field theory derived from matrices);

Topological theory with conifold background (Ghoshal and Vafa);

See Review article, e.g. by Ginsparg and Moore

OUTLINE

Part 1 : Tensor models and Algebras

- The bosonic complex tensor model.
- Counting and algebras: general tensor invariants.
- Counting and algebras: color symmetrised subspace
- Color-symmetrisation and new integer sequences
- Possible tensor model approach to some algebra/rep theory/combinatorics problems.

OUTLINE

Part 2 : Algebras and Holographic geometry

- Counting and branched covers of sphere.
- Correlators and covers of more general 2-complexes.
- Physical and topological holography: lessons from old matrix models?

Part I : The tensor variables

We will be primarily interested in complex tensor models, with complex tensor $\Phi_{i_1,i_2,\cdots,i_d}$ transforming as $\bar{\Phi}^{i_1,i_2,\cdots,i_d}$.

The complex tensor transforms as

$$V_N \otimes V_N \otimes \cdots \otimes V_N = V_N^{\otimes d}$$

of $U(N) \times U(N) \cdots \times U(N) = U(N)^d$. The complex conjugate tensor transforms as

$$\bar{V}_N \otimes \bar{V}_N \otimes \cdots \otimes \bar{V}_N = \bar{V}_N^{\otimes d}$$

of $U(N) \times U(N) \cdots \times U(N) = U(N)^d$.

Part I : The invariant theory problem

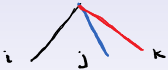
Consider polynomial functions of these tensor variables, invariant under $U(N)^{\times d}$. Invariants are constructed by contracting the V_N indices of Φ with the $\bar{V}_N$ indices of $\bar{\Phi}$. So we need an equal number of each - let this number be n.

Problem 1: Find the dimension of the space of invariants as a function of n.

Problem 2: Refined counting problem: Find the dimensions of subspaces which transform under specific irreducible representations (irreps) of S_d. e.g. color-symmetrised invariants, which transform under the trivial rep of S_d.

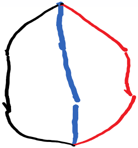
01 November 2018 08:21

$\Phi_{ijk} \longrightarrow$

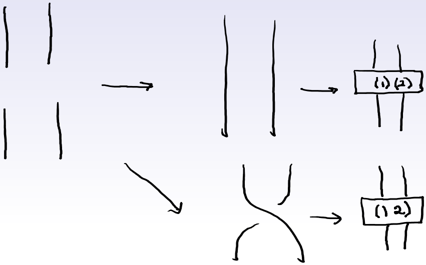


$\bar{\Phi}^{i'j'k'} \longrightarrow$

$\Phi_{ijk}\bar{\Phi}^{ijk} \longrightarrow$



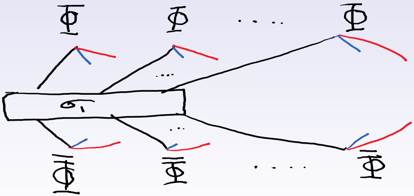
01 November 2018 10:14



$\sigma \in S_n \to \mathrm{End}(V_N^{\otimes n})$

Here $n = 2$

01 November 2018 10:25



$(\sigma_1, \sigma_2, \sigma_3)$

The bosonic symmetry means invariance under $\gamma \in S_n$ and $\mu \in S_n$. These imply equivalences of the triples.

$$(\sigma_1, \sigma_2, \sigma_3) \sim (\gamma\sigma_1\mu, \gamma\sigma_2\mu, \cdots, \gamma\sigma_3\mu)$$

For rank d tensors, the counting of invariants is given by

$$(\sigma_1, \sigma_2, \cdots, \sigma_d) \sim (\gamma\sigma_1\mu, \gamma\sigma_2\mu, \cdots, \gamma\sigma_d\mu)$$

So studying tensor model invariants amounts to studying the double cosets

$$Diag(S_n) \setminus (S_n \times S_n \times S_n) / Diag(S_n)$$

Associated with the set of equivalences classes, there is a sub-algebra of $\mathbb{C}(S_n) \otimes \mathbb{C}(S_n) \otimes \mathbb{C}(S_n)$. This is the subspace which is invariant under left action by S_n and invariant under right action of S_n.

This subspace is a closed algebra $\mathcal{K}(n)$, which is associative and semi-simple. Associativity and the non-degenerate bilinear pairing is inherited from the parent algebra $\mathbb{C}(S_n) \otimes \mathbb{C}(S_n) \otimes \mathbb{C}(S_n)$.

By the Wedderburn-Artin theorem, there is a matrix basis. So the number of invariants is

$$\sum_{R,S,T \vdash n} (C(R,S,T))^2$$

$C(R,S,T)$ is the Clebsch-Gordan multiplicity for $R \otimes S$. How many times does T appear, when we decompose $R \otimes S$ into irreps of the diagonal S_n. Also equal to multiplicity of the one-dimensional rep in $R \otimes S \otimes T$.

Also called the Kronecker coefficient.

R, S, T are Young diagrams with n boxes. The finite N counting is given by restricting $l(R) \leq N$.

Color-symmetrised counting

Color-symmetrised counting.

$$
\begin{aligned}
&(\sigma_1, \sigma_2, \sigma_3) + (\sigma_2, \sigma_1, \sigma_3) + \cdots \\
&= \sum_{\alpha \in S_3} (\sigma_{\alpha(1)}, \sigma_{\alpha(2)}, \sigma_{\alpha(3)})
\end{aligned}
$$

Could also take a Young diagram with 3 boxes Y, and consider projecting to Y

$$
\frac{d(Y)}{3!} \sum_{\alpha} \chi_Y(\alpha) \ \ (\sigma_{\alpha(1)}, \sigma_{\alpha(2)}, \sigma_{\alpha(3)})
$$

Projecting to this subspace commutes with the projectors to the invariants under left and right action of S_n. So we can do the left/right projection, along with this S_3 projection.

The color-symmetric subspace is a closed sub-algebra. It has a WA decomposition.

$$\dim(\mathcal{K}_{Y_0}(n)) = \sum_{R \neq S \neq T} (C(R,S,T))^2 + \sum_{R \neq S} (\text{Mult}(\text{Sym}^2(R), S))^2 + (\text{Mult}(\Lambda^2(R), S))^2 + \sum_{R} \sum_{\Lambda} (\text{Mult}(R^{\otimes 3}, [n] \otimes \Lambda))^2$$

Hence a sum of squares; and we can write a Matrix basis, using appropriate “Clebsch-Gordan coefficients” for the symmetric groups.

$$\dim(\mathcal{K}_{Y_0}(n)) = \frac{1}{6} S^{(3)}_{[1^3]}(n) + \frac{1}{2} S^{(3)}_{[2,1]}(n) + \frac{1}{3} S^{(3)}_{[3]}(n)$$

$$S^{(3)}_{[1^3]} = \mathrm{tr}_{\mathcal{K}(\mathrm{n})}((1)(2)(3))$$
$$S^{(3)}_{[2,1]} = \mathrm{tr}_{\mathcal{K}(\mathrm{n})}((12))$$
$$S^{(3)}_{[3]} = \mathrm{tr}_{\mathcal{K}(\mathrm{n})}((123))$$

Also

$$S_{[2,1]} = \sum_{R,S} C(R,R,S)$$
$$= \frac{1}{n!} \sum_{S} \sum_{\tau} \chi^S(\tau)\mathrm{Sym}(\tau) = \sum_{p} \sum_{S} \chi^S(\tau_p)$$

This is the sum of all entries in the character table of S_n.

The sum is a positive integer that can be constructed from tenor models.

How about the refined R-dependent quantity. (dropping the sum over R).

$$\sum_S C(R, R, S) = \sum_p \chi_R(\tau_p)$$

Also integer, known from representation theory (because sum of Clebsch-multiplicity). An open problem of Stanley : combinatoric construction of this sum of characters.

Tensor model invariants (in the large N limit) give a construction of the number

$$\sum_{R,S,T} (C(R,S,T))^2$$

Color-symmetrised tensors give a construction of

$$\sum_{R,S} C(R,R,S)$$

DO tensor invariants, appropriately refined, give a construction of

$$\sum_{p} \chi_R(\tau_p)$$

Or more ambitiously for $C(R,S,T)$ in general. Old problem with partial recent success (maths literature motivated by P v/s NP problem) for certain infinite classes of Young diagrams.

Part II : S_n TFT2 and covering space holography

Counting tensor invariants is the counting of equivalence classes of permutation triples $(\sigma_1, \sigma_2, \sigma_3)$ where two triples are in the same equivalence class if

$$(\sigma_1, \sigma_2, \sigma_3) \sim (\gamma\sigma_1\mu, \gamma\sigma_2\mu, \gamma\sigma_3\mu)$$

for some $\gamma, \mu \in S_n$.

Can think of this as follows. There is a set of $(n!)^3$ triples, which is organised into orbits - a sort of gauge equivalence where the $S_n \times S_n$ acts to produce the some observable.

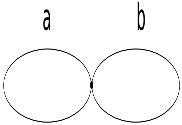
Burnside Lemma says : Can count orbits by counting fixed points i.e. number of solutions to

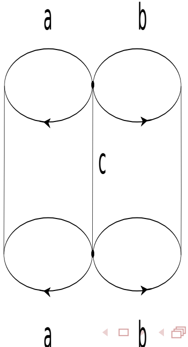
$$\gamma\sigma_1\mu = \sigma_1$$
$$\gamma\sigma_2\mu = \sigma_2$$
$$\gamma\sigma_3\mu = \sigma_3$$

In other words, number of tensor invariants at degree n is

$$\frac{1}{(n!)^2}\sum_{\sigma_1\in S_n}\sum_{\sigma_2\in S_n}\sum_{\sigma_3\in S_n}\delta(\gamma\sigma_1\mu\sigma_1^{-1})\delta(\gamma\sigma_2\mu\sigma_2^{-1})\delta(\gamma\sigma_3\mu\sigma_3^{-1})$$

Joseph showed some 2-complexes such that S_n TFT2 on those complexes has partition function equal to the above. The 2-complex has 4-faces joining at a line, so can be 2-skeleton of a cell decomposition of a 3-manifold, not a 2-manifold.





After some simplifications of the delta function sums and re-writing, we found (1307 paper) that this is counting all branched covers of the sphere, with three branch points.

In matrix model problems, one tends to find counting of branched covers, each one counted with an inverse automorphism factor . Here we find counting each branched cover with weight one.

some stringy holography with 2D target and also something higher dimensional with 3D target - due to a relation between delta functions which count branched covers with and without $1/Aut$.

INTRIGUING !! and should be understood better.

Color symmetrised counting and branched covers

Branched covers of the two sphere of degree n, with 3 branch points are counted using triples $\tau_1, \tau_2, \tau_3 \in S_n$ such that

$$\tau_1 \tau_2 \tau_3 = 1$$

There is an action of S_3 on these - called spherical braid group action - :

$$(\tau_1, \tau_2, \tau_3) \to (\tau_2, \tau_2^{-1} \tau_1 \tau_2, \tau_3)$$
$$(\tau_1, \tau_2, \tau_3) \to (\tau_1, \tau_3, \tau_3^{-1} \tau_2 \tau_3)$$

Counting color-symmetrised orbits is the same as counting braid orbits on the branched covers.

Checked for $d = 3$. Everything before this slide has generalizations for any d. The connection to spherical braud group higher d – plausible but calculations remain to be done.
interesting to generalize this ...

Branched covers and correlators

Compute one-point function of the observables $\mathcal{O}_{\sigma_1,\cdots,\sigma_d}$ in the Gaussian model. Can express simply in terms of delta functions.

Express as S_n TFT on some 2-complex. We find

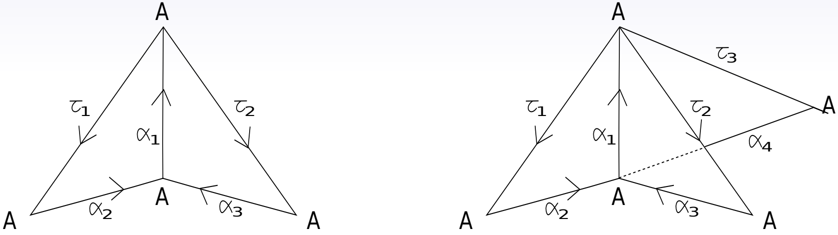


Figure: 2-cell for S_n TFT2 interpretation of correlator at $d = 3$ (left) and $d = 4$ (right). The 2-cells in the 2-complexes are $(\alpha_1 \tau_l \alpha_l)$, $l = 2, 3, 4$.

Open questions

- 2-complexes suggestive of 3D manifolds come up in the topological (perm TFT) interpretation of counting and correlators. There is intriguing interplay between 2D and 3D in these topological interpretations.
- Old matrix models : a combination of perspectives, topological and physical, e.g. Witten - topological phase of 2D gravity.
- Perhaps some of these intriguing hints from the topological view of S_n TFT are signals of things that can be explored in other views of the holographic dual - AdS, black holes, Vassiliev's proposal, ...

$\mathcal{K}(n)$ is related to tensor model counting, and the size of the blocks is equal to Kronecker coefficients $C(R, S, T)$.

Above, all Young diagrams all have n boxes.

In the space of all Young diagrams, there is another roduct given by the Littlewood-Richardson coefficient $g(R, S, T)$. Here, if R has m boxes, and S has n boxes, then T has $m + n$ boxes.

This is the Clebsch multiplicity for $U(N)$.

There is an algebra $\mathcal{A}(m,n)$, defined in terms of permutation equivalence classes,

$$\sigma \in S_{m+n}$$
$$\sigma \sim \gamma\sigma\gamma^{-1} \text{ for } \gamma \in S_m \times S_n$$

This algebra controls counting for 2-matrix invariants. Matrix variables X, Y which are $N \times N$, and

$$X \to UXU^\dagger$$
$$Y \to UYU^\dagger$$

This algebra has many applications in AdS/CFT, in connection with the construction of local operators in CFT related to branes (called giant gravitons) whose properties are very sensitive to finite N effects in AdS.

Both for $\mathcal{K}(n)$ and for $\mathcal{A}(m,n)$, finite N effects are controlled by $l(R) \leq N$. This simple implementation of finite N comes from Schur-Weyl duality.

These algebras $\mathcal{A}(m,n)$ - along with ideas of Fourier transformation, Schur-Weyl duality - were used to prove some Young diagram identities which are needed in some quantum information theory problems.

$$d_r n(n+1) = \sum_{R \vdash (n+1)} d_R \, g(r, \square, R) \, (c_{\square}(R, r))^2$$

Sanjaye Ramgoolam, Michal Sedlak,
"Quantum Information Processing and Composite Quantum Fields,"

arXiv:1809.05156 [hep-th]

The quantum information problem has to to with something related to approximate cloning - more precisely perfect probabilistic cloning - of a unitary operator.

$\mathcal{A}(m,n)$ has a lot of information about counting and correlators in 2-matrix system, of relevance to AdS/CFT. It also knows, through this identity, about approximate cloning in quantum information.

Is this a mathematical accident ? or is there an interpretation of the approximate cloning problem encoded in the correlators of