Towards Singlet Spectrum in Holographic Tensor Models

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Structure of the talk

- Introduction/ Motivation
- SYK model and its features
- SYK-like Tensor models
- Numerical results
- Random matrix classification
- Singlet spectrum
  a. Diagonalizing the Hamiltonian
  b. Finding the gauge invariant states
- Conclusions

Holography

- The most complete description of Quantum gravity that we have currently is via holography.
- According to holography, gravity in $(d+1)$ dimensions and (non-gravitational) field theory on the d-dimensional boundary are dual descriptions of the same theory
- Typically, in the holographic field theories that we know, we have a parameter N(loosely can be thought of as a dimensionless Planck's constant).
- In the (classical) limit $N \to \infty$, things are usually under control perturbatively in $1/N$.

Finite-N

- But, as of yet, there are no holographic field theories with sufficiently rich dynamics that are solvable at finite-N.
- The knowledge of finite-N effects is not only necessary to understand quantum gravity more completely but also to resolve problems like information paradox
- Also, information paradox is related to the dynamics of blackholes and to study the dynamics of black holes we need a strongly interacting theory in the boundary

SYK model

- SYK model emerged as one such model that has the potential to shed some light on these issues
- It was proposed by Kitaev based on an old work by Sachdev and Ye and is a model of N Majorana fermions in (0+1) dimensions
- The Hamiltonian of SYK model is[1]

$$H = i^{q/2} \sum_{a<b<\ldots<q} J_{ab\ldots q}\, \psi_a \psi_b \ldots \psi_q$$

where $J_{a\ldots q}$ are picked from a random Gaussian distribution with mean $\langle J_{abcd} \rangle = 0$ and variance $\langle J_{abcd}^2 \rangle = \frac{J^2 (q-1)!}{N^{q-1}}$

[1] [Polchinkski,Rosenhaus: 1601.06768],[Maldacena,Stanford: 1604.07818]

Why is SYK interesting?

- Solvable in large N limit
- Emergent conformal symmetry in infra-red
- Maximally chaotic i.e., SYK model saturates the chaos bound (This is reminiscent of black holes)

$\rightarrow$ The second feature implies that we can construct a holographic dual and the third point indicates that the holographic dual is a black hole.

Hurdle

- While calculating the correlators, we take an average over the entire ensemble of J: disorder averaging
- Because of this disorder averaging, SYK model is not unitary
- But, the disorder averaging is necessary for the solvability of SYK and hence cannot be avoided

Gurau-Witten Model

- To address this issue, Witten proposed a model[2] based on the work of Gurau and collaborators on colored (bosonic) tensor models :**Gurau-Witten Model**
- Gurau-Witten model is dependent on two independent integers– d & n
- Basic building blocks of the model are the **fermionic** tensors of the form $\psi_A^{i_1 i_2 \dots i_d}$
- The index A corresponds to color and runs from 0 to d whereas the indices $(i_1, i_2, \dots i_d)$ take the values from 1 to n
- Number of independent fermionic fields are $N = (d+1)n^d$

[2] [Witten:1610.09758]

GW Hamiltonian for $d = 3$

- For the case of $d = 3$, the symmetry group of GW model is $O(n)^6$ i.e.,

$$G \sim O(n)_{01} \times O(n)_{02} \times O(n)_{03} \times O(n)_{12} \times O(n)_{13} \times O(n)_{23}$$

- The Hamiltonian for $d = 3$ is given as

$$H_{GW} = \frac{J}{n^{3/2}} \psi_0^{abc} \psi_1^{ade} \psi_2^{fbe} \psi_3^{fdc}$$

Gurau-Witten vs SYK

- The large N perturbation theory (in the leading order) of Gurau-Witten model has a similar behaviour as that of SYK model.
  - In particular, in both cases, the “melonic” diagrams dominate in the large N limit
- Like SYK model, Gurau-Witten model is also maximally chaotic
- Further, Gurau-Witten model is a true quantum mechanical model in the sense that the correlators are computed in the usual field-theoretic way (without any averaging)
- Hence, this model is a better candidate to study quantum black holes

Uncolored model

- Klebanov and Tarnopolsky suggested that one could construct a tensor model[3] with a smaller symmetry group that still enjoys the salient features of Gurau-Witten model
- This model is an "uncolored" tensor model in the sense that we do not distinguish among the fermions and build the entire model using a single fermionic tensor $\psi^{i_1 i_2 \cdots i_d}$
- We will still distinguish the tensorial indices though and as in the Gurau-Witten model, each i_m can take values from 1 to n
- The interaction term is similar to that of Gurau-Witten model except that the color labels $(A, B, \ldots)$ are not present

[3] [Klebanov, Tarnopolsky:1611.08915; Carrozza, Tanasa:1512.06718]

Uncolored Hamiltonian for $d = 3$

- For the case of $d = 3$, the symmetry group is $O(n)^3$ i.e.,

$$G \sim O(n) \times O(n) \times O(n)$$

- ψ^{ijk} transform under the vector representation of each of the $O(n)$'s i.e.,

$$\psi^{ijk} \to M_1^{i'i} M_2^{j'j} M_3^{k'k} \psi^{ijk}$$

  where $M_i \in O(n)_i$
- The anti-commutation relations are: (Clifford Algebra)

$$\{\psi^{abc}, \psi^{def}\} = \delta^{ad}\delta^{be}\delta^{cf}$$

- The Hamiltonian for $d = 3$ uncolored model is:

$$H = \frac{J}{n^{3/2}} \psi^{abc}\psi^{ade}\psi^{fbe}\psi^{fdc}$$

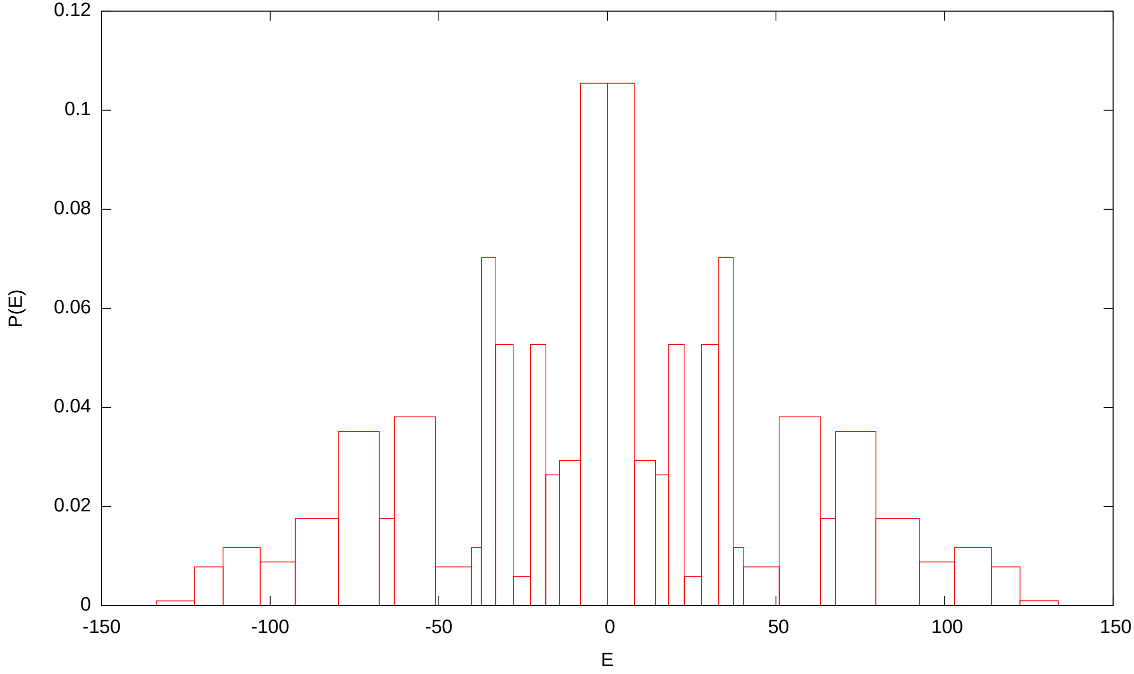
Numerical Diagonalization

- What is the (eigenvalue) spectrum of these SYK-like tensor model Hamiltonians?
- Before answering this question analytically, we use numerical techniques to find the spectrum for $n = 3$
- We assign the gamma matrices of $SO(27)$ to the fermions as follows:

$$\psi^{ijk} = \gamma_p$$
$$p = 9(i-1) + 3(j-1) + k$$

- We then diagonalize the $n = 3$ uncolored Hamiltonian numerically

Density of States for $n = 3$ uncolored model



Properties of the spectrum

- The spectrum is symmetric about $E = 0$: Spectral Mirror symmetry
- The spectrum has an overall 8-fold degeneracy which we explained by identifying various discrete symmetries
- Further, we have also showed that $n = 3$ uncolored model has chaos

Quantum Chaos and Random Matrices

- The eigenvalue statistics of quantum chaotic systems is same as that of random matrix ensembles
- The particular random matrix ensemble depends on the symmetries of the specific Hamiltonian
- For instance, it has been shown that the SYK Hamiltonian belongs[4] to one of the standard Wigner-Dyson ensembles depending on N
- We have presented a similar classification in the case of tensor models and the ensembles belong to Andreev-Altland-Zirnbauer classification because of the presence of spectral mirror symmetry

[4] [Cotler et al:1611.04650]

$n \pmod 8$	0	1	2	3	4	5	6	7
S^2	1	1	-1	-1	1	-1	-1	1
$\mathcal{T}^2$	1	1	1	-1	1	-1	1	1
Class	BDI	BDI	CI	DIII	BDI	DIII	CI	BDI

Table 1: Symmetry classes of uncolored Hamiltonians for varying n and for $d = 3$

- In the above table, S is the spectral mirror symmetry operator and $\mathcal{T}$ is the time-reversal operator.
- A similar random matrix classification for the GW model(for arbitrary d and arbitrary n) is presented in our paper **(CK, KVPK, SS: 1703.08155)**

Agenda for the rest of the talk

In the rest of the talk, we propose a way to find the singlet spectrum of these SYK-like tensor models as follows:

- First, we will propose a strategy to (analytically) diagonalize the Hamiltonian i.e., we identify the eigenvalues and eigenvectors of the Hamiltonian
- Then we identify the (linear combinations of) energy eigenstates that are gauge invariant under the corresponding symmetry group

Diagonalizing the Hamiltonian

- We explicitly demonstrate the method for even n case of uncolored models and it can be extended to other cases with some modifications
- For even n, we define the creation and annihilation operators as follows:

$$\psi^{ijk^{\pm}} = \frac{1}{\sqrt{2}}\left(\psi^{ijk} \pm i\psi^{ij(k+1)}\right)$$

where $k^{\pm}$ takes values from 1 to $\frac{n}{2}$ and are related to k as follows:

$$k = 2k^{\pm} + 1$$

- In terms of these creation and annihilation operators, the Hamiltonian is given by:

$$H = \frac{n^4}{4} + \sum 2\left(\psi^{ijk^+}\psi^{ilm^+}\psi^{njm^-}\psi^{nlk^-} - \psi^{ijk^+}\psi^{njm^+}\psi^{ilm^-}\psi^{nlk^-}\right)$$

Constructing the basis

- We now define the “Clifford” vacuum as the state that is annihilated by all the annihilation operators:

$$\psi^{ijk^-} |\,\rangle = 0$$

- We can now construct the entire Clifford representation by acting on the ground state with the creation operators.
- A general state at level r can be constructed as follows:

$$\psi^{i_1 j_1 k_1^+} \psi^{i_2 j_2 k_2^+} \ldots \psi^{i_r j_r k_r^+} |\,\rangle$$

- Total number of levels : $\frac{n^3}{2}$
  Total number of states: $2^{n^3/2}$(dimensionality of the Hilbert space)

Level-wise Eigenstates

- The level operator can be defined as follows:

$$L = \sum \psi^{ijk^+} \psi^{ijk^-}$$

- The Hamiltonian commutes with the level operator. So, the eigenstates of the Hamiltonian can be found level by level

Levels- 0 & 1

- Now, we proceed to find eigenstates.
- From the Hamiltonian, we can see that the Clifford vacuum and all the level-1 states are eigenstates of the Hamiltonian with energy $\frac{n^3}{4}$.
- At higher levels, we can find eigenstates using the Young tableaux.

Higher levels

- To start with, we remind ourselves that ψ^{ijk^+} transform under fundamental representation of $O(n) \times O(n) \times SU(n/2)$
- This means that the ψ^+'s can be represented using Young tableaux as follows:

$$\psi^{ijk^+} : \left(\boxed{i} \, , \boxed{j} \, , \boxed{k^+} \right)$$

- So, any state at level r in Clifford representation is obtained by taking tensor product of r fundamental representations of $O(n) \times O(n) \times SU(n/2)$.
- That is, in the Young tableaux language, a level r state can be written as:

$$\left(\boxed{i_1} \otimes \dots \otimes \boxed{i_r} \, , \boxed{j_1} \otimes \dots \otimes \boxed{j_r} \, , \boxed{k_1^+} \otimes \dots \otimes \boxed{k_r^+} \right)$$

H on an arbitrary state

- The action of Hamiltonian on a general state at level r is given by:

$$H'\left(\boxed{i_1}\otimes\dots\otimes\boxed{i_r}\,,\ \boxed{j_1}\otimes\dots\otimes\boxed{j_r}\,,\ \boxed{k_1^+}\otimes\dots\otimes\boxed{k_r^+}\right)$$

$$=4n\sum_{p<q}(-1)^{p+q-1}\left[\left(\bullet_{i_p i_q}\,,\ \boxed{j_q}\otimes\boxed{j_p}\,,\ \boxed{k_p^+}\otimes\boxed{k_q^+}\right)\right.$$

$$\left.-\left(\boxed{i_q}\otimes\boxed{i_p}\,,\ \bullet_{j_p j_q}\,,\ \boxed{k_p^+}\otimes\boxed{k_q^+}\right)\right]$$

$$\otimes\left(\underbrace{\boxed{i_1}\otimes\dots\otimes\boxed{i_r}}_{\text{no } i_p \ \& \ i_q},\ \underbrace{\boxed{j_1}\otimes\dots\otimes\boxed{j_r}}_{\text{no } j_p \ \& \ j_q},\ \underbrace{\boxed{k_1^+}\otimes\dots\otimes\boxed{k_r^+}}_{\text{no } k_p^+ \ \& \ k_q^+}\right)$$

Eigenstates and Eigenvalues

- Note that the Clifford vacuum is invariant under $O(n) \times O(n) \times SU(n/2)$ by definition
- Since the Hamiltonian is a singlet under $O(n) \times O(n) \times SU(n/2)$, the eigenstates can be obtained by comparing the irreducible representations of $O(n) \times O(n) \times SU(n/2)$ on both sides
- The corresponding eigenvalues can be obtained from the knowledge of Clebsh-Gordon coefficients

$r = 2$ case

- As an example of our method, we explicitly find eigenstates and eigenvalues at level-2
- The action of Hamiltonian on a level-2 state is given by:

$$H'\left(\psi^{i_1 j_1 k_1^+}\psi^{i_2 j_2 k_2^+}|\,\rangle\right)$$

$$=4\left[\sum_i \psi^{i j_2 k_1^+}\psi^{i j_1 k_2^+}\delta^{i_1 i_2} - \sum_j \psi^{i_2 j k_1^+}\psi^{i_1 j k_2^+}\delta^{j_1 j_2}\right]|\,\rangle$$

- The LHS in terms of Young tableaux is given by:

$$\left(\begin{array}{|c|c|}\hline i_1 & i_2 \\ \hline \end{array} + \begin{array}{|c|}\hline i_1 \\ \hline i_2 \\ \hline \end{array} + \bullet_{i_1 i_2}\ ,\ \begin{array}{|c|c|}\hline j_1 & j_2 \\ \hline \end{array} + \begin{array}{|c|}\hline j_1 \\ \hline j_2 \\ \hline \end{array} + \bullet_{j_1 j_2}\ ,\ \begin{array}{|c|c|}\hline k_1^+ & k_2^+ \\ \hline \end{array} + \begin{array}{|c|}\hline k_1^+ \\ \hline k_2^+ \\ \hline \end{array}\right)$$

- As far as the RHS is concerned, only some of the above diagrams show up and are given as:

$$\left(\bullet_{i_1 i_2}\ ,\ \begin{array}{|c|c|}\hline j_2 & j_1 \\ \hline \end{array} + \begin{array}{|c|}\hline j_2 \\ \hline j_1 \\ \hline \end{array} + \bullet_{j_1 j_2}\ ,\ \begin{array}{|c|c|}\hline k_1^+ & k_2^+ \\ \hline \end{array} + \begin{array}{|c|}\hline k_1^+ \\ \hline k_2^+ \\ \hline \end{array}\right)$$

$$-\left(\begin{array}{|c|c|}\hline i_2 & i_1 \\ \hline \end{array} + \begin{array}{|c|}\hline i_2 \\ \hline i_1 \\ \hline \end{array} + \bullet_{i_1 i_2}\ ,\ \bullet_{j_1 j_2}\ ,\ \begin{array}{|c|c|}\hline k_1^+ & k_2^+ \\ \hline \end{array} + \begin{array}{|c|}\hline k_1^+ \\ \hline k_2^+ \\ \hline \end{array}\right)$$

- Upon comparing the LHS and RHS, we can conclude that the following states have an eigenvalue of "$+4n$":

$$\left(\begin{array}{|c|}\hline i_1 \\ \hline i_2 \\ \hline \end{array} \,,\ \bullet_{j_1 j_2} \,,\ \begin{array}{|c|c|}\hline k_1^+ & k_2^+ \\ \hline \end{array} \right) ; \left(\bullet_{i_1 i_2} \,,\ \begin{array}{|c|c|}\hline j_1 & j_2 \\ \hline \end{array} \,,\ \begin{array}{|c|}\hline k_1^+ \\ \hline k_2^+ \\ \hline \end{array} \right)$$

- The following have an eigenvalue of "$-4n$":

$$\left(\bullet_{i_1 i_2} \,,\ \begin{array}{|c|}\hline j_1 \\ \hline j_2 \\ \hline \end{array} \,,\ \begin{array}{|c|c|}\hline k_1^+ & k_2^+ \\ \hline \end{array} \right) ; \left(\begin{array}{|c|c|}\hline i_1 & i_2 \\ \hline \end{array} \,,\ \bullet_{j_1 j_2} \,,\ \begin{array}{|c|}\hline k_1^+ \\ \hline k_2^+ \\ \hline \end{array} \right) \tag{1}$$

- The rest of the states have a zero eigenvalue.

Finding Singlets

- Which linear combinations of eigenstates are gauge invariant under $O(n)^3$?
- Singlet states have a zero charge under each of the three $O(n)$'s i.e.,

$$Q_a|\text{singlet state}\rangle = 0$$

  where $Q_{1,2,3}$ are the Noether charges corresponding to three $O(n)$'s.
- The Noether charges are given by:

$$Q_1^{i_1 i_2} = i\ \psi^{i_1 jk}\psi^{i_2 jk}$$
$$Q_2^{j_1 j_2} = i\ \psi^{i j_1 k}\psi^{i j_2 k}$$
$$Q_3^{k_1 k_2} = i\ \psi^{i j k_1}\psi^{i j k_2}$$

- Putting $k_2 = k_1 + 1$ and summing over all odd k_1, we find that all the singlets are at $\frac{n^3}{4}$th level
- Since the total number of levels are $\frac{n^3}{2}$, we found that all the singlets are at mid-level: **Mid-level Condition**

$n = 2$

- For the case of $n = 2$, the Noether charges can be written as:

$$Q_1^{12} = i\left(\psi^{111^+}\psi^{211^-} + \psi^{111^-}\psi^{211^+} + \psi^{121^+}\psi^{221^-} + \psi^{121^-}\psi^{221^+}\right)$$
$$Q_2^{12} = i\left(\psi^{111^+}\psi^{121^-} + \psi^{111^-}\psi^{121^+} + \psi^{211^+}\psi^{221^-} + \psi^{211^-}\psi^{221^+}\right)$$
$$Q_3^{12} = 2 - \psi^{111^+}\psi^{111^-} - \psi^{121^+}\psi^{121^-} - \psi^{211^+}\psi^{211^-} - \psi^{221^+}\psi^{221^-}$$

- The mid-level conditions implies that all the singlets are at level-2
- Imposing $Q_i|\text{singlet}\rangle = 0$, we get the following singlet states:

$$\left(\psi^{111^+}\psi^{211^+} + \psi^{121^+}\psi^{221^+}\right)|\,\rangle$$
$$\left(\psi^{111^+}\psi^{121^+} + \psi^{211^+}\psi^{221^+}\right)|\,\rangle$$

- There are only two states in the singlet spectrum in this case
- These findings are consistent with that of [Loganayagam et al:1705.01930].

Singlets in GW model

- Singlet spectrum can also be constructed for the GW model in a similar way. (A harder problem)
- For the case of $n = 2$ we have identified the singlet spectrum and unlike the case of $n = 2$ uncolored model, the singlet spectrum of $n = 2$ GW model is non-trivial
- There are 140 states in the singlet spectrum and all the eigenvalues we obtained here analytically have counterparts in the numerical diagonalization[5] as well.
- Importantly, we found that the the energy eigenvalue of the lowest energy state of the singlet spectrum is $-2\sqrt{14}$ that exactly matches with the lowest eigenvalue obtained via numerical diagonalization
- That is, the ground state is a part of the spectrum.

[5] Krishnan et al:1612.06330

Eigenvalue	$\pm 2\sqrt{14}$	$\pm 4\sqrt{3}$	$\pm 2\sqrt{6}$	± 4	$\pm 2\sqrt{2}$	0
Degeneracy	1	3	4	6	31	50

Table 2: Eigenvalues & corresponding degeneracy of the singlet eigenstates

- The degeneracies of all the eigenvalues except $\pm 2\sqrt{2}$ and 0 can be explained by the residual (discrete) symmetries of the Hamiltonian
- We could explain the degeneracies of $\pm 2\sqrt{2}$ and 0 partially. To explain the degeneracies completely, we need some extra symmetries which have been not understood as yet.
- Further, we can show that the gauged spectrum of this model has rudimentary signs of chaos− This is a good news for holographic purposes.

Future Directions

- Our method has potential to be generalized to tensor models with arbitrary N. Some work along this direction is done with Chethan and Avinash.
- The first step is to find the singlets by using $Q_i|\text{singlet}\rangle = 0$.
- For arbitrary N, we could solve all but one equation which involves dealing with Clebsch Gordon coefficients of permutation groups.
- For instance, to find the singlets of the simplest non-trivial uncolored model($N = 64$), we need to have knowledge of CG coefficients of S_{16}- which is a hard problem.

A simpler problem?

- There is a simpler problem to try before attacking the most general problem
- Whenever there is a factor of $O(2)$ i.e., say a tensor model with symmetry group $O(n_1) \times O(n_2) \times \ldots O(2)$, then we can list all the singlet states in principle using our method.
- But to identify which are the independent ones among them seems to be the bottleneck[6].
- These are the two directions that we would like to report on in the future.

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[6] However, look Mello Koch et al:1707.01455, Ramgoolam et al:1708.03524

THANK YOU!!!