Counting U(N) and O(N) invariants and graph algebras of tensor models

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Based on

JHEP **1711**, 092 (2017) and
Ann. Inst. HP D Comb. Phys. Interact. **1**, 77-138 (2014) with S. Ramgoolam (QMUL)

"Counting O(N) invariants," work in Progress, with R. C. Avohou (Jerusalem U & ICMPA, Benin), N Dub (LIPN)

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Holographic Tensors,
OIST, Okinawa

Outline

1. Introduction: Permutation groups and Tensor models

2. Complex tensors and $U(N)$ invariants

3. Extending the results: Real tensors and $O(N)$ invariants

Outline

Observables and Correlators in TM

• Key objects for tensor models: interactions/observables $\equiv$ invariants of classical Lie groups, $U(N), O(N)$.

• Invitation: new formulation of tensor model observables and correlators in terms of the symmetric group theoretic language and its representation theory.

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Switching to the Symmetric group and its representation theory

It allows us

$\rightarrow$ to embed our results in a different set theoretic formulation
$\rightarrow$ to gain confidence when implementing computations by software resources which could have been otherwise very difficult to handle.

$\rightarrow$ to shed a different light on our results
$\rightarrow$ to discover new and genuine effects
$\rightarrow$ to bridge theories and therefore discover new correspondences between countings, which from the outset look rather different (new bijections)

$\rightarrow$ to ask new questions
$\rightarrow$ to push towards new software experiments to guide our intuition

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Switching to the symmetric group and its representation theory

• In practice, what symmetric group formulation has brought to the understanding of TM

⇒ Mathematics/Computer Science side:

→ Enumeration the observables/invariants (via computer softwares), generating functions are given (complex, and real)

→ Connection to topological field theory (TFT) with a geometrical interpretation of the counting (complex)

→ Understanding tensor invariants have a particularly rich algebra with interesting properties (matrix decomposition, bases, graduation)

→ Simplification and discovery new integer sequences [OEIS]

→ Discover computable sectors in TM

→ Establish a link with Computational Complexity Theory (via representation theoretic formulation of the counting): Counting of tensor invariants is related to the problem of positivity of the Kronecker coefficient problem (NP vs P).

→ Matrices & linguistic [S Ramgoolam et al].

⇒ Physics side:

→ For Matrix Models, much more [S Ramgoolam & R. de Mello Koch, Pablo Diaz, ...], from instance, understanding the half-BPS sector of $N = 4$ SYM.

→ Quantum information processing [Ramgoolam & Sedlak].

→ Highlight intriguing correspondences between countings in TM, Matrix Model (and therefore in String theory).

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Goals

• Review some main results in complex TM using symmetric group formulation and its representation theory [work with Sanjaye, JHEP 2017; AIHP D 2014]

• Show preliminary results about the extension of these results to real TM, Carrozza-Tanasa-Klebanov-Tarnolposky. [wip with Avohou and Dub].

Outline

Building blocks: Complex tensors and unitary invariants

• A covariant complex tensor $T_{p_1,\dots,p_d}$ with transformation rule

$$T_{p_1,\dots,p_d} = \sum_{q_k} U^{(1)}_{p_1 q_1} \dots U^{(d)}_{p_d q_d} T_{q_1,\dots,q_d}\,, \qquad U^{(a)} \in U(N_a) \tag{1}$$

• Tensor contractions = unitary invariants

$$S^{\rm int}_{\mathbf{b}}(T,\bar T) = \mathrm{Tr}_{\mathbf{b}}(\bar T \cdot T \dots \bar T \cdot T) \tag{2}$$

• An input: T is viewed as a $(d-1)$-simplex. $S^{\rm int}_{\mathbf{b}}$ "is" a gluing of simplexes and represents a d-polytope geometry

• In the following, all illustrations are made at fixed rank $d = 3$ but generally extends in any d.

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Unitary invariants

$$S_{\mathbf{b}}^{\text{int}}(T, \bar{T}) = \mathrm{Tr}_{\mathbf{b}}(\bar{T} \cdot T \dots \bar{T} \cdot T)$$

- Coding unitary invariants: **b** bi-partite colored graphs

- Rank $D = 1$,Vectors: $||\phi|| = \sum_a |\phi_a|^2$, 1 invariant.
- Rank $D = 2$, Matrices: $\mathrm{Tr}[(M^\dagger M)^n]$, $\forall n \geq 1$, cyclic graphs (show on the board), $n \in \mathbb{N}$.

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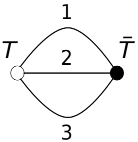

$\mathrm{Tr}_2(\bar{T}\, T)$

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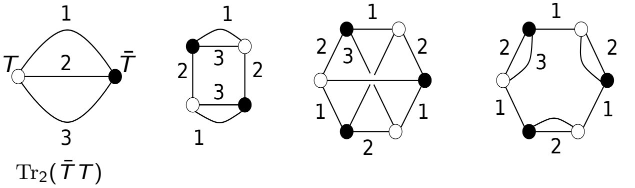

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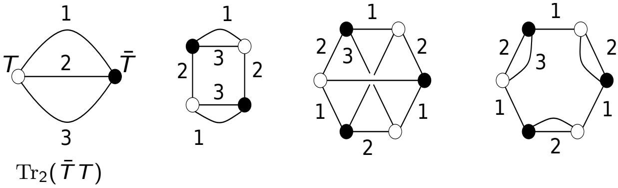

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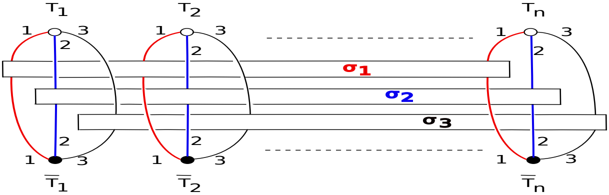
Counting observables

[BG, Ramgoolam, AIHP D '14] Illustration in rank 3:



- Counting permutation triples $(\sigma_1, \sigma_2, \sigma_3) \in (S_n \times S_n \times S_n)$ up to the equivalence

$$(\sigma_1, \sigma_2, \sigma_3) \sim (\gamma_1\sigma_1\gamma_2,\ \gamma_1\,\sigma_2\gamma_2,\ \gamma_1\sigma_3\gamma_2), \qquad \gamma_i \in S_n\,. \tag{3}$$

- Elément of the double quotient $\mathrm{Diag}(S_n)\backslash(S_n \times S_n \times S_n)/\mathrm{Diag}(S_n)\,.$

Counting orbits: Apply Burnside's lemma

$$|H_1 \backslash G / H_2| = \frac{1}{|H_1||H_2|} \sum_{h_1 \in H_1} \sum_{h_2 \in H_2} \sum_{g \in G} \delta(h_1 g h_2 g^{-1}) \tag{4}$$

- Number of invariants

$$Z_3(n) = \frac{1}{(n!)^2} \sum_{\sigma_{1,2,3} \in S_n} \sum_{\gamma_1, \gamma_2 \in S_n} \delta(\gamma_1 \sigma_1 \gamma_2^{-1} \sigma_1^{-1}) \delta(\gamma_1 \sigma_2 \gamma_2^{-1} \sigma_2^{-1}) \delta(\gamma_1 \sigma_3 \gamma_2^{-1} \sigma_3^{-1}) \tag{5}$$

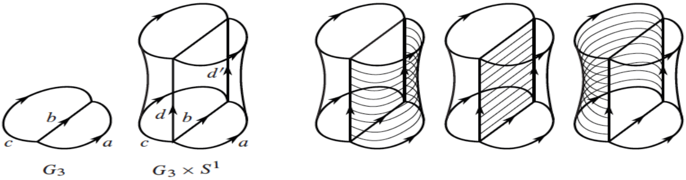
$\rightarrow$ Programming in Gap, and Mathematica [OEIS: A110143 (isomorphism of graph coverings)] Illustration at rank $d = 3$

$$1; 4; 11; 43; 161; 901; 5579; 43206; 378360; 3742738, \ldots \tag{6}$$

Topological Field Theory

→ TFT_2 on toirodal lattice

$$Z_3(n) = \frac{1}{(n!)^2} \sum_{\sigma_{1,2,3} \in S_n} \sum_{\gamma_1, \gamma_2 \in S_n} \delta(\gamma_1 \sigma_1 \gamma_2^{-1} \sigma_1^{-1}) \delta(\gamma_1 \sigma_2 \gamma_2^{-1} \sigma_2^{-1}) \delta(\gamma_1 \sigma_3 \gamma_2^{-1} \sigma_3^{-1}) \qquad (7)$$



After some manipulations (gauge fixing one σ_i and introduce another variable), one arrives at

$$Z_3(n) = \frac{1}{n!} \sum_{\tau_0, \tau_1, \tau_2 \in S_n} \sum_{\gamma \in S_n} \delta(\gamma \tau_1 \gamma^{-1} \tau_1^{-1}) \delta(\gamma \tau_2 \gamma^{-1} \tau_2^{-1}) \delta(\gamma \tau_0 \gamma^{-1} \tau_0^{-1}) \delta(\tau_0 \tau_1 \tau_2) \qquad (8)$$

3 generators with a single relation, that is the fundamental group of S^2 with 3 punctures.

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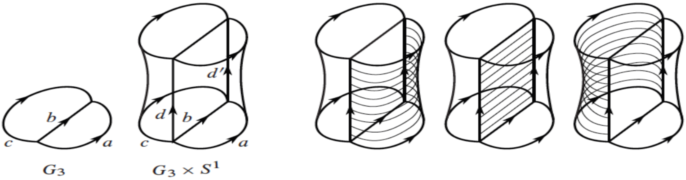

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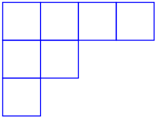
- Irreps of symmetric group S_n are labelled by Young diagrams or $R \vdash n$ partition of n.

$$n = 7, \quad R = (1,2,4) = \tag{9}$$



- Expansion of the delta

$$\delta(\sigma) = \sum_R \frac{d(R)}{n!} \chi^R(\sigma)\,, \qquad d(R) = \frac{n!}{h(R)}\,, \quad h(R) = \prod_{i,j} \mathrm{hook} - \mathrm{Length}_{i,j}$$

$$\sum_{\sigma \in S_n} \chi^R(\sigma)\chi^S(\sigma) = n!\, \delta_{RS}\,; \qquad \sum_{\gamma \in S_n} \delta(\gamma\sigma\gamma^{-1}\tau) = \sum_{R \vdash n} \chi^R(\sigma)\chi^R(\tau) \tag{10}$$

Revisiting the counting

- A small calculation

$$Z_3(n) = \frac{1}{(n!)^2} \sum_{\sigma_i \in S_n} \sum_{\gamma_1,\gamma_2 \in S_n} \delta(\gamma_1\sigma_1\gamma_2^{-1}\sigma_1^{-1})\delta(\gamma_1\sigma_2\gamma_2^{-1}\sigma_2^{-1})\delta(\gamma_1\sigma_3\gamma_2^{-1}\sigma_3^{-1})$$
$$= \frac{1}{(n!)^2} \sum_{\gamma_i \in S_n} \sum_{R_i \vdash n} \chi^{R_1}(\gamma_1)\chi^{R_1}(\gamma_2)\chi^{R_2}(\gamma_1)\chi^{R_2}(\gamma_2)\chi^{R_3}(\gamma_1)\chi^{R_3}(\gamma_2)$$
$$= \sum_{R_1,R_2,R_3 \vdash n} (C(R_1,R_2,R_3))^2 \quad (11)$$

where the symbol

$$C(R_1,R_2,R_3) = \frac{1}{n!} \sum_{\sigma \in S_n} \chi^{R_1}(\sigma)\chi^{R_2}(\sigma)\chi^{R_3}(\sigma) \quad (12)$$

is the Kronecker coefficient.

→ The dimension of an algebra $\mathcal{K}(n)$.

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&= \frac{1}{(n!)^2} \sum_{\gamma_i \in S_n} \sum_{R_i \vdash n} \chi^{R_1}(\gamma_1) \chi^{R_1}(\gamma_2) \chi^{R_2}(\gamma_1) \chi^{R_2}(\gamma_2) \chi^{R_3}(\gamma_1) \chi^{R_3}(\gamma_2) \\
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Counts
$\rightarrow$ multiplicity of the irreps R_3 in the tensor product of the irreps $R_1 \otimes R_2$
$\rightarrow$ multiplicity of the one-dimensional (trivial) representation in the tensor product $R_1 \otimes R_2 \otimes R_3$.

• Link with Computational Complexity theory:
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$\mathcal{K}(n)$, the double coset graph algebra

- Group algebra $\mathbb{C}(S_n)$, i.e. an element of which writes $a = \sum_{\sigma \in S_n} \lambda_\sigma \sigma, \;\; \lambda_\sigma \in \mathbb{C}$
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$$(\sigma_1, \sigma_2, \sigma_3) \sim (\gamma_1 \sigma_1 \gamma_2, \gamma_1 \sigma_2 \gamma_3, \gamma_1 \sigma_3 \gamma_2) \tag{13}$$

- Define $\mathcal{K}(n) \subset \mathbb{C}(S_n)^{\otimes 3}$ is the vector space over $\mathbb{C}$

$$\mathcal{K}(n) = \mathrm{Span}_{\mathbb{C}} \Big\{ \sum_{\gamma_1, \gamma_2 \in S_n} \gamma_1 \sigma_1 \gamma_2 \otimes \gamma_1 \sigma_2 \gamma_2 \otimes \gamma_1 \sigma_3 \gamma_2, \; \sigma_1, \sigma_2, \sigma_3 \in S_n \Big\} \tag{14}$$

→ A fact $\dim_{\mathbb{C}} \mathcal{K}(n) = Z_3(n)$.
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$$\delta_3(\otimes_{i=1}^3 \sigma_i ; \otimes_{i=1}^3 \sigma'_i) = \prod_{i=1}^{3} \delta(\sigma_i \sigma_i'^{-1}) \tag{15}$$

→ Explain the sum of squares: the Wedderburn-Artin theorem

$$\sum_{R_1, R_2, R_3 \vdash n} (C(R_1, R_2, R_3))^2 \tag{16}$$

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Decomposition of $\mathcal{K}(n)$ in direct subspaces...

Fourier basis of $\mathcal{K}(n)$

• Consider $D^R_{ij}(\sigma) = \langle R, j|\sigma|R, i\rangle$ the matrix representation of σ in the irrep $R \vdash n$ (dimension $d(R)$)

$$\text{Orthogonality}: \quad \sum_{\sigma\in S_n} D^R_{ij}(\sigma) D^S_{kl}(\sigma) = \frac{n!}{d(R)}\, \delta^{RS}\, \delta_{ik}\delta_{jl}\;;$$
$$\text{Reality}: \quad D^R_{ij}(\sigma^{-1}) = D^R_{ji}(\sigma)\,; \tag{17}$$
$$\text{Clebsch}-\text{Gordan}: \quad \sum_{\sigma\in S_n} D^{R_1}_{i_1 j_1}(\sigma) D^{R_2}_{i_2 j_2}(\sigma) D^{R_3}_{i_3 j_3}(\sigma) = \frac{n!}{d(R_3)} \sum_{\tau} C^{R_1,R_2;R_3,\tau}_{i_1,i_2;i_3} C^{R_1,R_2;R_3,\tau}_{j_1,j_2;j_3}$$
$$\tau \in [\![1, \mathrm{C}(R_1, R_2, R_3)]\!]$$

Fourier basis of $\mathcal{K}(n)$ algebra

- Introduce the Fourier basis of $\mathbb{C}(S_n)$

$$Q^R_{ij} = \frac{\kappa_R}{n!} \sum_{\sigma \in S_n} D^R_{ij}(\sigma)\sigma \tag{18}$$

$$\sum_{i_l, j_l, k} C^{R_1,R_2;R_3,\tau}_{i_1,i_2;i_3} C^{R_1,R_2;R_3,\tau'}_{j_1,j_2;j_3} \sum_{\sigma_1,\sigma_2} \rho_L(\sigma_1)\rho_R(\sigma_2) Q^{R_1}_{i_1 j_1} \otimes Q^{R_2}_{i_2 j_2} \otimes Q^{R_3}_{i_3 j_3} = Q^{R_1,R_2,R_3}_{\tau,\tau'} \tag{19}$$

- The set $\{Q^{R_1,R_2,R_3}_{\tau,\tau'}\}$ forms an orthogonal basis of $\mathcal{K}(n)$.
- At fixed $[R_1, R_2, R_3]$, $Q^{R_1,R_2,R_3}_{\tau,\tau'}$ is matrix with $C(R_1, R_2, R_3)^2$ entries.

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The basis $Q^{R_1,R_2,R_3}_{\tau,\tau'}$

- Invariant $(\gamma_1 \otimes \gamma_2) \cdot Q^{R,S,T}_{\tau_1,\tau_2} \cdot (\gamma_1 \otimes \gamma_2) = Q^{R,S,T}_{\tau_1,\tau_2}$
- Multiply like matrices:

$$Q^{R,S,T}_{\tau_1,\tau_2} Q^{R',S',T'}_{\tau_2',\tau_3} = \delta^{RR'} \delta^{SS'} \delta^{TT'} \delta_{\tau_2 \tau_2'} Q^{R,S,T}_{\tau_1,\tau_3} \tag{20}$$

- Orthogonal under the pairing

$$\boldsymbol{\delta}_2(Q^{R,S,T}_{\tau_1,\tau_1'}; Q^{R',S',T'}_{\tau_2,\tau_2'}) = \kappa_{R,S} d(T) \delta^{RR'} \delta^{SS'} \delta^{TT'} \delta_{\tau_1 \tau_2} \delta_{\tau_1' \tau_2'} \tag{21}$$

$\rightarrow$ This guarantees that there is a non degenerate bilinear mapping from a trace form. Hence, the Wedderburn-Artin theorem applies, the algebra is semi-simple.

The center of $\mathcal{K}(n)$

- Trace over Q-basis

$$P^{R,S,T} = \sum_{\tau} Q^{R,S,T}_{\tau,\tau} \tag{22}$$

- $\{P^{R,S,T}\}$ is a basis of the center $\mathcal{Z}(\mathcal{K}(n))$

$$P^{R,S,T} Q^{R',S',T'}_{\tau,\tau'} = Q^{R',S',T'}_{\tau,\tau'} P^{R,S,T} \tag{23}$$

- $\{P^{R,S,T}\}$ is orthogonal

$$\boldsymbol{\delta}_2(P^{R,S,T}; P^{R',S',T'}) = \kappa_{R,S}\, d(T)\, \mathrm{C}(R,S,T)\delta^{RR'}\delta^{SS'}\delta^{TT'} \tag{24}$$

Proposition

The set $\{P^{R,S,T}\}$ is a basis of $\mathcal{Z}(\mathcal{K}(n))$ and

$$\dim \mathcal{Z}(\mathcal{K}(n)) = \text{ number of non vanishing Kronecker coefficients} \tag{25}$$

- For (R,S,T), $P^{R,S,T}$ is the sum of diagonal elements of $Q^{R,S,T}_{\tau,\tau'}$. Collecting all possible diagonals hence $P^{R,S,T}$ spans the centre $\mathcal{Z}(\mathcal{K}(n))$.
- A triple (R,S,T), such that $\mathrm{C}(R,S,T) \neq 0$ yields a non vanishing $Q^{R,S,T}_{\tau,\tau'}$ and contributes to a single $P^{R,S,T}$. The result on the dimension of $\mathcal{Z}(\mathcal{K}(n))$ follows.

The graph algebra

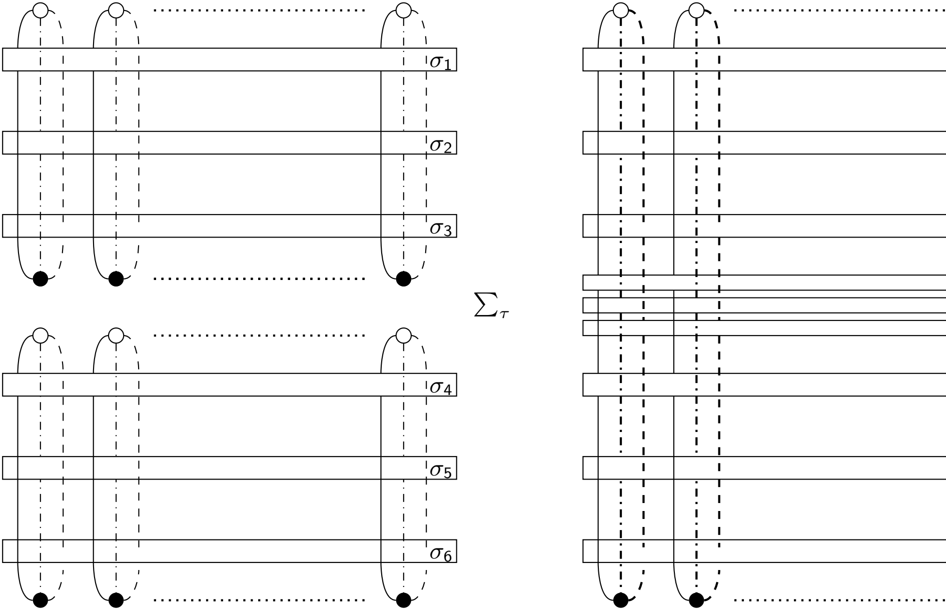
- Convenient normalization

$$A_{\sigma_1,\sigma_2,\sigma_3} = \frac{1}{(n!)^2} \sum_{\gamma_1,\gamma_2 \in S_n} \gamma_1\sigma_1\gamma_2 \otimes \gamma_1\sigma_2\gamma_2 \otimes \gamma_1\sigma_3\gamma_2 \qquad (26)$$

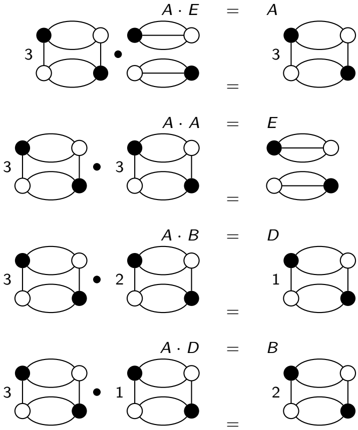
- Multiplication

$$A_{\sigma_1,\sigma_2,\sigma_3} A_{\sigma_4,\sigma_5,\sigma_6} = \frac{1}{n!} \sum_{\tau \in S_n} A_{\sigma_1\tau\sigma_4,\, \sigma_2\tau\sigma_5,\, \sigma_3\tau\sigma_6} \qquad (27)$$

Multiplication of graphs



The algebra $\mathcal{K}(n=2)$



(29)

$$\mathcal{K}(2) = \mathbb{C}(S_2) \otimes \mathbb{C}(S_2) \tag{30}$$

Correlators

At rank $d = 3$, consider the Gaussian model

$$\mathcal{Z} = \int d\Phi d\bar{\Phi}\ e^{-\frac{1}{2}\sum_{i_l} \Phi_{i_1 i_2 i_3} \bar{\Phi}_{i_1 i_2 i_3}} \qquad (31)$$

$$\mathcal{O}_{\sigma_1,\sigma_2,\sigma_3} = \sum_{i_l,j_l,k_l} \Phi_{i_1 j_1 k_1} \Phi_{i_2 j_2 k_2} \dots \Phi_{i_n j_n k_n}\ \bar{\Phi}_{i_{\sigma_1(1)} j_{\sigma_2(1)} k_{\sigma_3(1)}} \bar{\Phi}_{i_{\sigma_1(2)} j_{\sigma_2(2)} k_{\sigma_3(2)}} \dots \bar{\Phi}_{i_{\sigma_1(n)} j_{\sigma_2(n)} k_{\sigma_3(n)}} \quad (32)$$

- The Wick theorem

$$\begin{aligned}
\langle \mathcal{O}_{\sigma_1,\sigma_2,\sigma_3} \rangle &= \frac{1}{\mathcal{Z}} \int d\Phi d\bar{\Phi}\ e^{-\frac{1}{2}\sum_{i,j,k} \Phi_{ijk}\bar{\Phi}_{ijk}} \mathcal{O}_{\sigma_1,\sigma_2,\sigma_3} \\
&= \sum_{i_l,j_l,k_l} \sum_{\mu \in S_n} \delta_{i_1 i_{\mu(\sigma_1(1))}} \delta_{i_2 i_{\mu(\sigma_1(2))}} \dots \delta_{i_n i_{\mu(\sigma_1(n))}} \\
&\times \delta_{j_1 j_{\mu(\sigma_2(1))}} \delta_{j_2 j_{\mu(\sigma_2(2))}} \dots \delta_{j_n j_{\mu(\sigma_2(n))}}\ \delta_{k_1 k_{\mu(\sigma_3(1))}} \delta_{k_2 k_{\mu(\sigma_3(2))}} \dots \delta_{k_n k_{\mu(\sigma_3(n))}} \\
&= \sum_{\mu \in S_n} N^{\mathbf{c}(\mu\sigma_1)+\mathbf{c}(\mu\sigma_2)+\mathbf{c}(\mu\sigma_3)} = N^{\#\text{Faces}} \qquad (33)
\end{aligned}$$

$\mathbf{c}(\alpha)$ is the number of cycles of α.

Computable correlators in representation

$$\langle \mathcal{O}_{\sigma_1,\sigma_2,\sigma_3}\rangle = \sum_{\gamma} N^{\mathbf{c}(\gamma\sigma_1)+\mathbf{c}(\gamma\sigma_2)+\mathbf{c}(\gamma\sigma_3)} \tag{34}$$

$$= \sum_{\gamma} \sum_{\alpha_1,\alpha_2,\alpha_3} N^{\mathbf{c}(\alpha_1)+\mathbf{c}(\alpha_2)+\mathbf{c}(\alpha_3)} \delta(\gamma\sigma_1\alpha_1)\delta(\gamma\sigma_2\alpha_2)\delta(\gamma\sigma_3\alpha_3)$$

$$= \sum_{\gamma} \sum_{R_l} \Big[\prod_{l=1}^{3} \mathrm{Dim}_N(R_l)\Big] \chi^{R_1}(\gamma\sigma_1)\chi^{R_2}(\gamma\sigma_2)\chi^{R_3}(\gamma\sigma_3)$$

$$\mathcal{O}_{S_1,S_2,S_3} = \frac{1}{(n!)^3} \sum_{\sigma_l \in S_n} \chi^{S_1}(\sigma_1)\chi^{S_2}(\sigma_2)\chi^{S_3}(\sigma_3)\langle \mathcal{O}_{\sigma_1,\sigma_2,\sigma_3}\rangle$$

$$= \Big[\prod_{l=1}^{3} \frac{\mathrm{Dim}_N(S_l)}{d(S_l)}\Big] \sum_{\gamma} \chi^{S_1}(\gamma)\chi^{S_2}(\gamma)\chi^{S_3}(\gamma)$$

$$= n! \Big[\prod_{l=1}^{3} \frac{\mathrm{Dim}_N(S_l)}{d(S_l)}\Big] \, \mathrm{C}(S_1, S_2, S_3) \tag{35}$$

$\mathcal{O}_{S_1,S_2,S_3} \propto$ the Kronecker coefficients.

Outline

Non-orientable invariants

(Work in progress, with Nicolas Dub and Avohou R. Cocou)

• Recent interest: SYK-Type models, Carrozza-Tanasa, Klebanov-Tarnolpolsky $O(N)$ models.

• A real tensor $T_{p_1,\dots,p_d}$ with transformation rule

$$T_{p_1,\dots,p_d} = \sum_{q_k} O^{(1)}_{p_1 q_1} \dots O^{(d)}_{p_d q_d} T_{q_1,\dots,q_d} \,, \qquad O^{(a)} \in O(N_a) \tag{36}$$

• Tensor contractions = orthogonal invariants

$$S^{\rm int}_{\mathbf{b}}(T) = {\rm Tr}_{\mathbf{b}}(T \cdot T \dots T \cdot T) \tag{37}$$

• Coding orthogonal invariants $\mathbf{b}$ colored graphs

This one is new!

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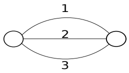
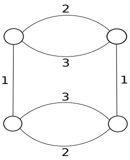
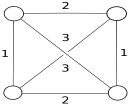


$\mathrm{Tr}_2(T^2)$





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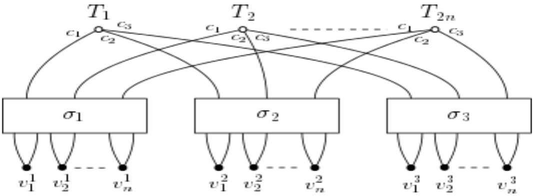
Counting orthogonal invariants

Illustration in rank 3:



- Counting permutation triples $(\sigma_1, \sigma_2, \sigma_3) \in (S_{2n} \times S_{2n} \times S_{2n})$ up to the equivalence

$$(\sigma_1, \sigma_2, \sigma_3) \sim (\, \gamma_1 \sigma_1 \gamma, \; \gamma_2 \, \sigma_2 \gamma, \; \gamma_3 \sigma_3 \gamma \,) \,, \qquad \gamma_i \in S_n[S_2], g \in S_n \,. \quad (38)$$

- Elément of the double quotient $S_n[S_2] \times S_n[S_2] \times S_n[S_2] \backslash (S_n \times S_n \times S_n)/\mathrm{Diag}(S_n)$.
- Number of invariants

$$Z_{o;3}(2n) = \frac{1}{((n!(2!)^n)^3(2n!))} \sum_{\sigma_i, \gamma \in S_{2n}} \sum_{\gamma_i \in S_n[S_2]} \delta(\gamma_1 \sigma_1 \gamma^{-1} \sigma_1^{-1}) \delta(\gamma_2 \sigma_2 \gamma^{-1} \sigma_2^{-1}) \delta(\gamma_3 \sigma_3 \gamma^{-1} \sigma_3^{-1}) \; (39)$$

[R.C. Read, "The enumeration of locally restricted graphs," Journal London Math.Soc. 34 (1959), 417-436]

1; 5; 16; 86; 448; 3580; 34981; 448628; 6854130; 121173330

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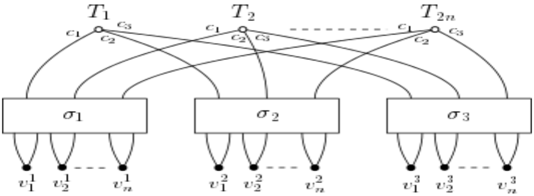

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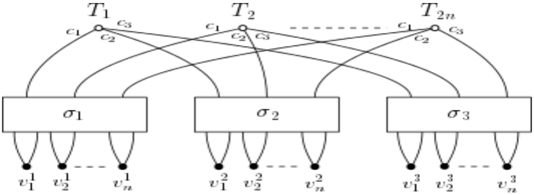

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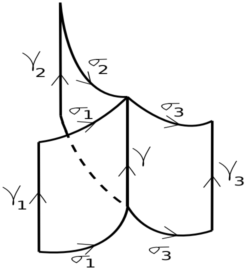
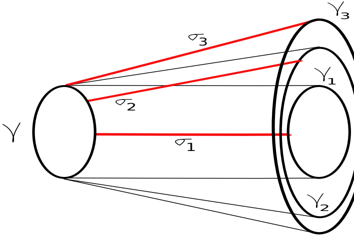
TFT$_2$


γ_2
σ_2
σ_1
σ_3
γ
γ_3
γ_1
σ_1
σ_3


γ_3
σ_3
γ_1
σ_2
γ
σ_1
γ_2

The algebra $\mathcal{K}_o(n)$

- Define $\mathcal{K}_o(n) \subset \mathbb{C}(S_{2n})^{\otimes 3}$ is the vector space over $\mathbb{C}$

$$\mathcal{K}(n) = \mathrm{Span}_{\mathbb{C}}\Big\{ \sum_{\gamma_i \in S_n[S_2]; \gamma \in S_{2n}} \gamma_1\sigma_1\gamma \otimes \gamma_2\sigma_2\gamma \otimes \gamma_3\sigma_3\gamma, \; \sigma_1, \sigma_2, \sigma_3 \in S_{2n} \Big\} \qquad (40)$$

- Dimension:

$$Z_{o;3}(n) = \frac{1}{(n!(2!)^n)^3} \times \sum_{R_i \vdash 2n} C(R_1, R_2, R_3) \Big[\sum_{\gamma_1 \in S_n[S_2]} \chi^{R_1}(\gamma_1) \Big] \Big[\sum_{\gamma_2 \in S_n[S_2]} \chi^{R_2}(\gamma_2) \Big] \Big[\sum_{\gamma_3 \in S_n[S_2]} \chi^{R_3}(\gamma_3) \Big] \qquad (41)$$

$\rightarrow \mathcal{K}_o(n)$ is an associative unital subalgebra of $\mathbb{C}(S_n)^{\otimes 3}$.

Question

Given $R \vdash 2n$, how to expand R in irreps representation of $S_n[S_2]$?

The algebra $\mathcal{K}_o(n)$

- Introducing the notation $\vec{X} = (X_1, X_2, X_3)$, $\vec{D}^{R}_{ij}(\vec{\sigma}) = \prod_{l=1}^{3} D^{R_l}_{i_l j_l}(\sigma_l)$, $\vec{\sigma}^{\otimes} = \sigma_1 \otimes \sigma_2 \otimes \sigma_3$
- Invariant generators:

$$Q^{\vec{R};\tau}_{\vec{i}} = \kappa_{\vec{R};\vec{i}} \sum_{j_l} C^{R_1,R_2;R_3,\tau'}_{j_1,j_2;j_3} \sum_{\gamma_l,\gamma} (P^L_{\gamma_1} \otimes P^L_{\gamma_2} \otimes P^L_{\gamma_2}) P^R_{\gamma} \left[Q^{R_1}_{i_1 j_1} \otimes Q^{R_2}_{i_2 j_2} \otimes Q^{R_3}_{i_3 j_3} \right]$$

$$= \kappa_{\vec{R};\vec{i}} \sum_{\sigma_l \in S_{2n}} \sum_{p_l, q_l} W^{\vec{R}}_{\vec{i};\,\vec{p}}\, C^{R_1,R_2;R_3,\tau'}_{q_1,q_2;q_3}\, \vec{D}^{R}_{pq}(\vec{\sigma})\, \vec{\sigma}^{\otimes}$$

$$W^{\vec{R}}_{\vec{i};\vec{p}} = \Big[\sum_{\gamma_1 \in S_n[S_2]} D^{R_1}_{i_1 p_1}(\gamma_1) \Big] \Big[\sum_{\gamma_2 \in S_n[S_2]} D^{R_2}_{i_2 p_2}(\gamma_2) \Big] \Big[\sum_{\gamma_3 \in S_n[S_2]} D^{R_3}_{i_3 p_3}(\gamma_3) \Big] \qquad (42)$$

- The set $\{Q^{\vec{R};\tau}_{\vec{i}}\}$ forms an invariant generating system of $\mathcal{K}_o(n)$:

$$(\gamma_1 \otimes \gamma_2 \otimes \gamma_3) \cdot Q^{\vec{R};\tau}_{\vec{i}} \cdot \gamma^{\otimes 3} = Q^{\vec{R};\tau}_{\vec{i}} \qquad (43)$$

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$$\delta_3(Q_{\vec{i}}^{\vec{R};\tau}; Q_{\vec{i}'}^{\vec{R}';\tau'}) = \frac{(2n)!^3}{d(\vec{R})} \times \kappa_{\vec{R},\vec{i}}\,\kappa_{\vec{R},\vec{i}'}\,\delta_{\vec{R},\vec{R}'}\,\delta_{\tau,\tau'}\,W^{\vec{R}}_{\vec{i},\vec{i}'} \tag{44}$$

→ Hence depending on $W^{\vec{R}}_{\vec{i};\vec{i}'}$, and the pairing δ_3 might be degenerate on this basis.

→ We still have block decomposition in the $\vec{R}$ and τ sector.

→ Block reduction is tensor-like and not matrix-like.

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Gaussian correlators

- Consider observables

$$\mathcal{O}_{\sigma_1,\sigma_2,\sigma_3} = \sum_{i_l,j_l,k_l} \Phi_{i_{\sigma_1(1)} j_{\sigma_2(1)} k_{\sigma_3(1)}} \Phi_{i_{\sigma_1(2)} j_{\sigma_2(2)} k_{\sigma_3(2)}} \dots \Phi_{i_{\sigma_1(2n)} j_{\sigma_2(2n)} k_{\sigma_3(2n)}} \quad (45)$$

$\sigma_i^2 = id$, pairing.

- The Wick theorem for an observable $\mathcal{O}_{\sigma_1,\sigma_2,\sigma_3}$:

$$\langle \mathcal{O}_{\sigma_1,\sigma_2,\sigma_3} \rangle = \sum_{\mu \in S_n} N^{\mathbf{c}(\mu\sigma_1)+\mathbf{c}(\mu\sigma_2)+\mathbf{c}(\mu\sigma_3)} = N^{\#\mathrm{F}aces} \quad (46)$$

μ is again a pairing, $\mu^2 = id$.

Conclusion

- The counting of unitary invariants gives a sum of squares of Kroneckers.
- Unitary invariants span a semi-simple graph algebra, that is an algebra which decomposes in matrix subalgebras (WA theorem).
- WA orthogonal bases of the graph algebra have been found using representation theory, and the dimension of the algebra well reflects in the base label.
- Correlators can be computed in the same formulation. Fourier transform of the correlators gives function which are exactly computable (via softwares).
- Much more results in [JBG & Ramgoolam, JHEP 17]: for instance, the algebra become graded by irreps representations of S_d, where d is the tensor rank.

- Orthogonal invariants: More numerous and can be enumerated using similar symmetric group techniques.
- In the represention, the counting expresses as Kroneckers convoluted with characters (a bit more/less involved).
- The underlying TFT of the counting: a TFT on toroidal lattice with defects (effects of $S_n[S_2]$ subgroup).
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- They span a graph algebra that might be not semi-simple. We have the simplest a set of invariant generators that we conjecture to be overcomplete.

Conclusion

- The counting of unitary invariants gives a sum of squares of Kroneckers.
- Unitary invariants span a semi-simple graph algebra, that is an algebra which decomposes in matrix subalgebras (WA theorem).
- WA orthogonal bases of the graph algebra have been found using representation theory, and the dimension of the algebra well reflects in the base label.
- Correlators can be computed in the same formulation. Fourier transform of the correlators gives function which are exactly computable (via softwares).
- Much more results in [JBG & Ramgoolam, JHEP 17]: for instance, the algebra become graded by irreps representations of S_d, where d is the tensor rank.

- Orthogonal invariants: More numerous and can be enumerated using similar symmetric group techniques.
- In the represention, the counting expresses as Kroneckers convoluted with characters (a bit more/less involved).
- The underlying TFT of the counting: a TFT on toroidal lattice with defects (effects of $S_n[S_2]$ subgroup).
- They span a graph algebra that might be not semi-simple. We have the simplest a set of invariant generators that we conjecture to be overcomplete.