From Higher-Spin Gauge Theory to Strings and Tensor Models

M.A.Vasiliev

Lebedev Institute, Moscow

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Plan

Properties of HS theories

HS symmetries

Framed oscillator algebras

Nonlinear HS theories

Coxeter groups and Cherednik algebras

From Coxeter HS theories to strings and matrix models

Conclusions

Quantum Gravity Challenge

QG effects should matter at ultra-high energies of Planck scale

$$m_P^2 = \frac{hc}{G} \qquad m_P \sim 10^{19} GeV$$

Given little hope to test QGR experimentally a unique chance is to conjecture that the regime of ultra high (transPlanckian) energies exhibits some high symmetries that are spontaneously broken at low energies

Idea: to understand what kind of higher symmetries can be introduced in relativistic theory and to see consequences

HS gauge theory: theory of higher symmetries beyond usual space-time and inner (super)symmetries, still being consistent with unitary QFT

Fronsdal Fields

Fronsdal fields 1978

All $m = 0$ HS fields are gauge fields

$\varphi_{n_1 \dots n_s}$ is a rank s symmetric tensor obeying $\varphi^k{}_k{}^m{}_{m n_5 \dots n_s} = 0$

Gauge transformation:

$$\delta \varphi_{n_1 \dots n_s} = \partial_{(n_1} \varepsilon_{n_2 \dots n_s)} \,, \qquad \varepsilon^m{}_{m n_3 \dots n_{s-1}} = 0$$

Fronsdal action $S(\varphi)$ implies field equations: $G_{n_1 \dots n_s}(\varphi) = 0$ with Einstein-like tensor

$$G_{n_1 \dots n_s}(\varphi) := \Box \varphi_{n_1 \dots n_s}(x) - s \partial_{(n_1} \partial^m \varphi_{n_2 \dots n_s m)}(x) + \frac{s(s-1)}{2} \partial_{(n_1} \partial_{n_2} \varphi^m{}_{n_3 \dots n_s m)}(x)$$

In 60-70th it was argued (Weinberg, Coleman-Mandula) that HS symmetries cannot be realized in a nontrivial local field theory in Minkowski space

Green light: AdS background with $\Lambda \neq 0$ Fradkin, MV, 1987

In agreement with no-go statements the limit $\Lambda \to 0$ is singular

AdS/CFT Correspondence

That HS gauge theory is formulated in AdS_4 fits naturally AdS/CFT

Klebanov and Polyakov conjecture (2002):

AdS_4 HS theory is dual to $3d$ vectorial conformal σ-models of φ^i, ψ^i_α.

Checked Giombi and Yin (2009) and extended to Chern-Simons BCFT

Aharony, Gur-Ari, Yacoby and Giombi, Minwalla, Prakash, Trivedi, Wadia, Yin (2011)

$$S^B = \frac{k}{4\pi} S_{CS} + \frac{1}{2}\int d^3x D_n\phi_i D^n\phi^i\,, \qquad S^F = \frac{k}{4\pi} S_{CS} + \int d^3x \bar{\psi}_i\gamma^n D_n\psi^i\,, \qquad i = 1, \dots N$$

$3d$ bosonization as a consequence of duality

$$\varphi = \frac{\pi}{2}\lambda_B\,, \qquad \varphi = \frac{\pi}{2}(1-\lambda_F) \qquad (\eta = \exp i\varphi) \qquad \lambda := \frac{N}{k}$$

Unexpected possibility of lab tests of Quantum Gravity

AdS_3/CFT_2 HS correspondence Gaberdiel and Gopakumar (2010)

Tensor models: $\phi^i \to \phi^{i_1\dots i_p}$ with $(\phi_i\phi^i)^2 \to \phi^{ijk}\phi_i{}^{ln}\phi_{jl}{}^{m}\phi_{knm}$

CFT dual?

HS Gauge Theory Versus String Theory

HS theories: $\Lambda \neq 0$, $m = 0$, **symmetric fields** $s = 0, 1, 2, \dots \infty$

First Regge trajectory

String Theory: $\Lambda = 0$, $m \neq 0$ **except for a few zero modes**

Infinite set of Regge trajectories

What is a HS symmetry of a string-like extension of HS theory?

MV 2012, Gaberdiel and Gopakumar 2014-2018

String Theory as spontaneously broken HS theory?! $(s > 2, m > 0)$

AdS/CFT **Boundary interpretation**

HS: $\phi^i(x)$ **Vector fields**

ST: $A^i{}_j(x)$ **Matrices**

Tensor models ?!: $\phi^{ijk\dots}(x)$ **Tensors**

Global CFT_3 HS Symmetry

Conformal HS symmetry in d dimensions = HS symmetry in AdS_{d+1}

Maximal symmetry of a d-dimensional free conformal field(s)

What are symmetries of KG equation? Shaynkman, MV 2001 $3d$; Eastwood 2002 $\forall d$

$3d$ coordinates: $x^{\alpha\beta} = x^{\beta\alpha} = \tau_n^{\alpha\beta} x^n \,, \qquad \alpha, \beta = 1, 2\,,\ \tau_n^{\alpha\beta} = \left(\delta^{\alpha\beta}, {\sigma_1}^{\alpha\beta}, {\sigma_3}^{\alpha\beta}\right)$

$$\det |x^{\alpha\beta}| := \epsilon_{\alpha\beta}\epsilon_{\gamma\delta} x^{\alpha\gamma} x^{\beta\delta} = x^n x_n \,, \qquad \epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}\,, \qquad \epsilon_{12} = 1$$

$3d$ Lorentz algebra: $o(2,1) \sim sp(2,R) \sim sl_2(R)$. α, β are spinor indices

Unfolded Klein-Gordon equation $\Box C(x) = 0$ for a scalar $C(x)$

$$\partial_{\alpha\beta} C(x) = C_{\alpha\beta}(x)\,, \qquad \partial_{\alpha\beta} C_{\gamma\delta}(x) = C_{\alpha\beta;\gamma\delta}(x)\,, \qquad \partial_{\alpha\beta} = \frac{\partial}{\partial x^{\alpha\beta}}$$

$$\Box C(x) = 0 \longrightarrow \epsilon^{\alpha\gamma}\epsilon^{\beta\delta}\partial_{\alpha\beta}\partial_{\gamma\delta} C(x) = 0$$

implies $C_{\alpha\beta;\gamma\delta}(x)$ is totally symmetric: $C_{\alpha\beta;\gamma\delta}(x) = C_{\alpha\beta\gamma\delta}(x)$. Continuation:

$$dx^{\alpha_1\alpha_2}\partial_{\alpha_1\alpha_2} C_{\beta_1\ldots\beta_n}(x) = dx^{\alpha_1\alpha_2} C_{\alpha_1\alpha_2\beta_1\ldots\beta_n}(x)$$

$C_{\beta_1\ldots\beta_n}$ parameterize all on-shell nontrivial higher derivatives of C.

Unfolded Form of $3d$ Massless Equations

Packing all symmetric multispinors into a generating function of commuting spinor variables y^α

$$C(y|x) = \sum_{n=0}^{\infty} C^{\alpha_1 \dots \alpha_{2n}}(x) y_{\alpha_1} \dots y_{\alpha_{2n}}$$

unfolded $3d$ massless equations take the form

$$dx^{\alpha\beta} \left(\frac{\partial}{\partial x^{\alpha\beta}} + \frac{\partial^2}{\partial y^\alpha \partial y^\beta} \right) C(y|x) = 0$$

$3d$ conformal HS algebra is the algebra of various differential operators $\epsilon(y, \frac{\partial}{\partial y})$ obeying $\epsilon(-y, -\frac{\partial}{\partial y}) = \epsilon(y, \frac{\partial}{\partial y})$

$$\delta C(y|x) = \epsilon(y, \frac{\partial}{\partial y}|x) C(y|x)$$

$$\epsilon(y, \frac{\partial}{\partial y}|x) = \exp\left[-x^{\alpha\beta} \frac{\partial^2}{\partial y^\alpha \partial y^\beta} \right] \epsilon_{gl}(y, \frac{\partial}{\partial y}) \exp\left[x^{\alpha\beta} \frac{\partial^2}{\partial y^\alpha \partial y^\beta} \right]$$

$\epsilon_{gl}(y, \frac{\partial}{\partial y})$: $3d$ HS symmetry algebra is Weyl algebra with spinor generators

Properties of HS Algebras

Global symmetry of symmetric vacuum of HS theory Fradkin, MV 1986

Let T_s be a homogeneous polynomial of degree $2(s-1)$ in $y^\alpha, \frac{\partial}{\partial y^\alpha}$

$$[T_{s_1}, T_{s_2}] = T_{s_1+s_2-2} + T_{s_1+s_2-4} + \ldots + T_{|s_1-s_2|+2}$$

Once spin $s>2$ appears, the HS algebra contains an infinite tower of higher spins: $[T_s, T_s]$ gives rise to T_{2s-2} as well as T_2 of $o(3,2) \sim sp(4)$.

Usual symmetries: spin-$s \leq 2$ $u(1) \oplus o(3,2)$: maximal finite-dimensional subalgebra of $hu(1,0|4)$. $u(1)$ is associated with the unit element.

Three types of HS algebras: $hu(n,m|4)$, $ho(n,m|4)$, $husp(2n,2m|4)$

Since HS symmetries do not commute with space-time symmetries

$$[T^n, T^{HS}] = T^{HS}, \qquad [T^{nm}, T^{HS}] = T^{HS} \quad \Longrightarrow \quad \delta_{HS}\varphi_{nm} \sim \varphi_{HS}$$

Riemann geometry is not appropriate for HS theory

How (non)local is HS gauge theory?

HS Algebra and Modules

Free field analysis: realization of the HS algebra as Weyl algebra

$$[y_\alpha\,, y_\beta]_* = 2i\varepsilon_{\alpha\beta}\,, \qquad [\bar{y}_{\dot\alpha}\,, \bar{y}_{\dot\beta}]_* = 2i\varepsilon_{\dot\alpha\dot\beta}$$

Fradkin, MV 1987

AdS_4 **algebra** $sp(4) \sim o(3,2)$

Naive way to extend the spectrum of fields

$$y_\alpha \to y_\alpha^n$$

does not lead to physically acceptable HS theories

Let hs_1 be a HS algebra with the single set of oscillators

The Fock hs_1-module F_1 describes free boundary conformal fields

$$D|0\rangle = h_1|0\rangle$$

The lowest weight representations of the naively extended algebras hs_p built from p copies of oscillators have too high weights

Framed Oscillator Algebras

The problem is resolved in the framed oscillator algebras

replacing usual oscillator algebra

$$[y^n_\alpha , y^m_\beta]_* = 2i\delta^{nm}\epsilon_{\alpha\beta} I \,,$$

where I is the unit element by

$$[y^n_\alpha , y^m_\beta]_* = 2i\delta^{nm}\epsilon_{\alpha\beta} I_n$$

"Units" I_n are assigned to each specie of the oscillators forming a set of commutative central idempotents

$$I_i I_j = I_j I_i \,, \qquad I_i I_i = I_i$$

This allows us to consider Fock modules F_i obeying

$$I_j F_i = \delta_{ij} F_i$$

equivalent to those of the single-oscillator case:

massless HS fields are in the spectrum

Unfolded dynamics

First-order form of differential equations

$$\dot{q}^i(t) = \varphi^i(q(t)) \qquad \text{initial values:} \quad q^i(t_0)$$

Unfolded dynamics: multidimensional covariant generalization

$$\frac{\partial}{\partial t} \to \mathsf{d}\,, \qquad q^i(t) \to W^\Omega(x) = dx^{n_1} \wedge \ldots \wedge dx^{n_p} W_{n_1 \ldots n_p}(x)$$

$$\mathbf{dW^\Omega(x) = G^\Omega(W(x))\,, \qquad d = dx^n \partial_n \qquad\qquad MV \quad 1988}$$

$G^\Omega(W)$: **function of "supercoordinates"** W^Φ

$$G^\Omega(W) = \sum_{n=1}^{\infty} f^\Omega{}_{\Phi_1 \ldots \Phi_n} W^{\Phi_1} \wedge \ldots \wedge W^{\Phi_n}$$

$d > 1$: **Nontrivial compatibility conditions**

$$G^\Phi(W) \wedge \frac{\partial G^\Omega(W)}{\partial W^\Phi} \equiv 0$$

Holographic duality between theories in different dimensions:

universal unfolded system admits different space-time interpretations.

Massless Fields in AdS_4

Infinite set of spins $s = 0, 1/2, 1, 3/2, 2\dots$ Fermions need doubling of fields

Doubled Weyl algebra connection: $\omega^{ii}(y, \bar{y} \mid x)\,, \qquad i = 0, 1$

Twisted adjoint module: $C^{i1-i}(y, \bar{y} \mid x)$,

$$\bar{\omega}^{ii}(y, \bar{y} \mid x) = \omega^{ii}(\bar{y}, y \mid x)\,, \qquad \bar{C}^{i\,1-i}(y, \bar{y} \mid x) = C^{1-i\,i}(\bar{y}, y \mid x)$$

$$A(y, \bar{y} \mid x) = i \sum_{n,m=0}^{\infty} \frac{1}{n!m!} y_{\alpha_1} \dots y_{\alpha_n} \bar{y}_{\dot{\beta}_1} \dots \bar{y}_{\dot{\beta}_m} A^{\alpha_1 \dots \alpha_n}{}_{,}{}^{\dot{\beta}_1 \dots \dot{\beta}_m}(x)$$

Nonlinear HS equations demand doubling of spinors and Klein operator

$$\omega(Y|x) \longrightarrow W(Z; Y; K|x)\,, \qquad C(Y|x) \longrightarrow B(Z; Y; K|x)$$

Some of the nonlinear HS equations determine the dependence on Z_A in terms of "initial data" $\omega(Y; K|x)$ and $C(Y; K|x)$ $S(Z; Y; K|x) = dZ^A S_A(Z; Y; K|x)$ is a connection along Z^A

Klein operators $K = (k, \bar{k})$: k generates chirality automorphisms

$$kf(A) = f(\tilde{A})k\,, \qquad A = (a_\alpha\,, \bar{a}_{\dot{\alpha}}): \qquad \tilde{A} = (-a_\alpha\,, \bar{a}_{\dot{\alpha}})$$

Nonlinear HS Equations

HS star product

$$(f * g)(Z,Y) = \int dSdT \exp iS_A T^A f(Z+S, Y+S) g(Z-T, Y+T)$$

$[Y_A, Y_B]_* = -[Z_A, Z_B]_* = 2iC_{AB}\,,$ $Z-Y\,:\,Z+Y$ **normal ordering**

Inner Klein operators:

$\kappa = \exp iz_\alpha y^\alpha\,,$ $\bar\kappa = \exp i\bar z_{\dot\alpha}\bar y^{\dot\alpha}\,,$ $\kappa * f = \tilde f * \kappa\,,$ $\kappa * \kappa = 1$

$$\begin{cases} \mathrm{d}W + W * W = 0 \\ \mathrm{d}B + W * B - B * W = 0 \\ \mathrm{d}S + W * S + S * W = 0 \\ \mathbf{S * B - B * S = 0} \\ \mathbf{S * S = i(dZ^A dZ_A + \eta dz^\alpha dz_\alpha B * k * \kappa + \bar\eta d\bar z^{\dot\alpha} d\bar z_{\dot\alpha} B * k * \bar\kappa)} \end{cases}$$

1992

Dynamical content is located in the x-independent twistor sector

The non-zero curvature has the form of Z_2-Cherednik algebra

This form comes out from the consistency of nonlinear HS equations

Coxeter Groups and Cherednik Algebras

A rank-p Coxeter group $\mathcal{C}$ is generated by reflections with respect to root vectors $\{v_a\}$ in a p-dimensional Euclidean vector space V. An elementary reflection associated with v_a

$$R_{v_a} x^i = x^i - 2v_a^i \frac{(v_a, x)}{(v_a, v_a)}, \qquad R_{v_a}^2 = I$$

Cherednik deformation of the semidirect product of the Weyl (oscillator) algebra with the group algebra of $\mathcal{C}$ is

$$[q_\alpha^n, q_\beta^m] = -i\epsilon_{\alpha\beta}\left(2\delta^{nm} + \sum_{v\in\mathcal{R}} \nu(v)\frac{v^n v^m}{(v,v)} k_v\right), \qquad k_v q_\alpha^n = R_v{}^n{}_m q_\alpha^m k_v$$

It respects Jacobi identities $(\alpha = 1, 2,\ n = 1; \ldots, p)$

Coupling constants $\nu(v)$ are invariants of $\mathcal{C}$ being constant on the conjugacy classes of root vectors under the action of $\mathcal{C}$.

General framed Cherednik algebra

$$[q_\alpha^n, q_\beta^m] = -i\epsilon_{\alpha\beta}\left(2\delta^{nm} I_n + \sum_{v\in\mathcal{R}} \nu(v)\frac{v^n v^m}{(v,v)} \hat{k}_v\right), \qquad \hat{k}_v := k_v \prod I_{i_1(v)} \ldots I_{i_k(v)}$$

B_p–Coxeter System

Important case of the Coxeter root system is B_p with the roots

$$R_1 = \{\pm e^n \qquad 1 \le n \le p\}\,, \qquad R_2 = \{\pm e^n \pm e^m \quad 1 \le n < m \le p\}\,.$$

Apart from permutations B_p contains reflections of basis axes $v^n_\pm = e^n$.

R_1 and R_2 form two conjugacy classes of B_p.

The Coxeter group of $3d$ HS theory is $A_1 \sim B_1$.

B_2 underlies the string-like HS models.

B_p is conjectured to underly tensor-like HS models.

Lorentz symmetry of HS theories follows from the fundamental property of Cherednik algebras that for any Coxeter root system the generators

$$t_{\alpha\beta} := \frac{i}{4}\sum_{n=1}^{p}\{q^n_\alpha\,, q^n_\beta\}$$

obey the $sp(2)$ commutation relations properly rotating all indices α

$$[t_{\alpha\beta}\,, q^n_\gamma] = \epsilon_{\beta\gamma} q^n_\alpha + \epsilon_{\alpha\gamma} q^n_\beta$$

Coxeter HS Equations

Unfolded equations for 1804.06520 $\mathcal{C}$-HS theories remain the same except

$$iS * S = dZ^{An} dZ_{An} + \sum_i \sum_{v \in \mathcal{R}_i} F_{i*}(B) \frac{dZ_n^{\alpha} v^n dZ_{\alpha\, m} v^m}{(v,v)} * \kappa_v$$

κ_v are generators of $\mathcal{C}$ acting trivially on all elements except for $dZ_{\alpha n}$

$$\kappa_v * dZ_\alpha^n = R_v{}^n{}_m dZ_\alpha^m * \kappa_v$$

$F_{i*}(B)$ is any star-product function of the zero-form B on the conjugacy classes R_i of $\mathcal{C}$. In the important case of the Coxeter group B_p

$$S * S = dZ_{An} dZ^{An} + \sum_{v \in \mathcal{R}_1} F_{1*}(B) \frac{dZ_n^{\alpha} v^n dZ_{\alpha\, m} v^m}{(v,v)} * \kappa_v + \sum_{v \in \mathcal{R}_2} F_{2*}(B) \frac{dZ_n^{\alpha} v^n dZ_{\alpha\, m} v^m}{(v,v)} * \kappa_v$$

with arbitrary $F_{1*}(B)$ and $F_{2*}(B)$ responsible for the

HS and stringy/tensorial features, respectively

$F_{2*}(B) \neq 0$ for $p \geq 2$.

The framed construction leads to a proper massless spectrum.

Jacobi for Cherednik imply consistency of field equations

Extensions

W, S and B can be valued in any associative algebra A: A_∞ structure.

Multi-particle extensions are associated with the semi-simple Coxeter groups. The simplest option with $\mathcal{C} = B_p^{\mathcal{N}}$ is the product of $\mathcal{N}$ of B_p systems

$$B_p^{\mathcal{N}} := \underbrace{B_p \times B_p \times \ldots}_{\mathcal{N}} .$$

The limit $\mathcal{N} \to \infty$ along with the graded symmetrization of the product factors expressing the spin-statistics is related to the (graded symmetric) multi-particle algebra $M(h(\mathcal{C}))$ of the HS algebra $h(\mathcal{C})$

$M(h(\mathcal{C})) = U(h(\mathcal{C}))$: Hopf algebra.

Multi-particle algebra $M(h(\mathcal{C}))$ is conjectured to underly the full fledged string and tensor bulk models

Klein Operators and Single-Trace operators

Enlargement of the field spectra of the rank-$p>1$ Coxeter HS models:

$C(Y^n_\alpha; k_v)$ **depend on p copies of oscillators Y^n_α and Klein operators k_v**

Qualitative agreement with enlargement of the boundary operators in tensorial boundary models.

Klein operators of Coxeter reflections permute master field arguments

At $p=2$ the star product of two master fields $C(Y_1,Y_2|x)k_{12}$ gives

$$(C(Y_1,Y_2|x)k_{12}) * (C(Y_1,Y_2|x)k_{12}) = C(Y_1,Y_2|x) * C(Y_2,Y_1|x)\,.$$

$p=2$ **system: strings of fields with repeatedly permuted arguments**

$$C^n_{string} := \underbrace{C(Y_1,Y_2|x) * C(Y_2,Y_1|x) * C(Y_1,Y_2|x)\ldots}_{n}\,.$$

are analogous of the single-trace operators in AdS/CFT.

$C(Y_1,Y_2|x)$ **and** $C(Y_1,Y_2|x) * C(Y_2,Y_1|x)$: **single-trace-like**

$C(Y_1,Y_2|x) * C(Y_1,Y_2|x)$: **double-trace-like.**

For $p>2$ fields carry p arguments permuted by S_p generated by k_{ij}

Conclusion

HS gauge theories contain gravity along with infinite towers of other fields with various spins including ordinary matter fields.

Main principle: formal consistency & massless fields in the spectrum

Coxeter HS theories extend minimal HS theories to String-like B_2 models and tensor-like B_p models of any rank p

$\mathcal{N}=4$ SYM is argued to be a natural dual of the B_2–HS model

The spectrum of the B_2 HS model is analogous to that of String Theory with the infinite set of Regge trajectories.

Single-trace-like strings of operators and their tensor generalizations.

Multi-particle states of a lower-dimensional model = elementary states in a higher-dimensional (particularly, $10d$ model)

riginal $3d$ and $4d$ spinorial HS theories: branes in the $10d$ theory

Main problem on the Agenda:

spontaneous breaking of HS symmetries in the Coxeter HS models

Vacuum and Fluctuations

h: a Lie algebra. $\omega = \omega^\alpha T_\alpha$: a h-valued 1-form h.

$$G(\omega) = -\omega \wedge \omega \equiv -\frac{1}{2}\omega^\alpha \wedge \omega^\beta [T_\alpha, T_\beta]$$

The unfolded equation with $W = \omega$ has the zero-curvature form

$$d\omega + \omega \wedge \omega = 0\,.$$

Background geometry in a coordinate independent way.

Minkowski or AdS_d space-time: h is Poincare or AdS algebra Let W^α contain p-forms $\mathcal{C}^i$ (e.g. 0-forms) and G^i be linear in ω and $\mathcal{C}$

$$G^i = -\omega^\alpha (T_\alpha)^i{}_j \wedge \mathcal{C}^j\,.$$

The compatibility condition: $\mathcal{C}^i$ form some h-module.

The unfolded equation

$$D_\omega \mathcal{C} = 0$$

$D_\omega \equiv d + \omega$: covariant derivative in the h-module

Unfolding and Holographic Duality

Unfolded formulation unifies various dual versions of the same system.

Duality in the same space-time:

ambiguity in what is chosen to be dynamical or auxiliary fields.

Holographic duality between theories in different dimensions:

universal unfolded system admits different space-time interpretations.

Extension of space-time without changing dynamics by letting the differential d and differential forms W to live in a larger space

$$\mathsf{d} = dX^n \frac{\partial}{\partial X^n} \to \tilde{\mathsf{d}} = dX^n \frac{\partial}{\partial X^n} + d\hat{X}^{\hat{n}} \frac{\partial}{\partial \hat{X}^{\hat{n}}}\,, \qquad dX^n W_n \to dX^n W_n + d\hat{X}^{\hat{n}} \hat{W}_{\hat{n}}\,,$$

$\hat{X}^{\hat{n}}$ are additional coordinates

$$\tilde{\mathsf{d}} W^{\Omega}(X, \hat{X}) = G^{\Omega}(W(X, \hat{X}))$$

Framed Star Product

x-dependent fields W, S and B depend on p sets of variables Y^n_A, Z^n_A $(A = 1, \dots M)$, I_n, anticommuting differentials dZ^A_n $(n = 1, \dots p)$ and Klein-like operators $\widehat{k}_v$ associated with all roots of $\mathcal{C}$. Coxeter HS field equations are formulated in terms of the star product

$$(f{*}g)(Z;Y;I) = \frac{1}{(2\pi)^{pM}} \int d^{pM}S\, d^{pM}T \exp\left[iS^A_n T^B_m \delta^{nm} C_{AB}\right] f(Z_i{+}I_iS_i;Y_i{+}I_iS_i;I)g$$

$$I_n * Y^n_A = Y^n_A * I_n = Y^n_A\,, \qquad I_n * Z^n_A = Z^n_A * I_n = Z^n_A\,, \qquad I_n * I_n = I_n$$

Implying

$$[Y^n_A, Y^m_B]_* = -[Z^n_A, Z^m_B]_* = 2iC_{AB}\delta^{nm} I_n\,, \qquad [Y^n_A, Z^m_B]_* = 0\,.$$

This star product admits inner Coxeter-Klein operators

$$\exp i\frac{v^n v^m Z_{\alpha n} Y^\alpha{}_m}{(v,v)}$$

Unitarity

Covariant derivative for a rank-two field $C(Y_1, k_1; Y_2, k_2) := C_{0,1}(Y_1, Y_2) k_2$

$$D_0(C_{0,1}(Y_1,Y_2)) = \left(D^{\mathsf{L}} - h^{\alpha\dot\beta}\left(y_{1\alpha}\frac{\partial}{\partial \bar y_1^{\dot\beta}} + \frac{\partial}{\partial y_1^{\alpha}}\bar y_{1\dot\beta} - i y_{2\alpha}\bar y_{2\dot\beta} + i\frac{\partial^2}{\partial y_2^{\alpha}\partial \bar y_2^{\dot\beta}}\right)\right) C_{0,1}(Y_1,Y_2)$$

$C_{0,1}(Y_1, Y_2)$ is valued in the tensor product of the Y_1-adjoint module and Y_2-twisted adjoint module.

Twisted adjoint module and its tensor products correspond to unitary multi-particle-like states.

Zero-form fields containing an adjoint module a factor do not form a unitary particle-like representation except for the Y_1-independent $C_{1,2}(Y_1, Y_2)$ which decribes I_1 unitary massless states in the Y_2 sector.

Non-singlet states in the adjoint module factors can be truncated away: non-singlet elements of the adjoint factor are never generated.

Idempotent Extension

Let A be an associative algebra with the star product and a set of idempotents

$$\pi_i * \pi_i = \pi_i \,, \qquad \pi_i \in A \,.$$

$$a_i{}^j \in A_i{}^j : \quad a_i{}^j = \pi_i * a * \pi_j \,, \quad a \in A \,.$$

The matrix-like composition law

$$(a * b)_i{}^j = \sum_k a_i{}^k * b_k{}^j$$

A is the algebra of functions of dx, dZ, Z, Y, k_v, x

π_i: Z-independent Fock idempotents of the star-product algebra.

The set of idempotents π_i has to be $\mathcal{C}$-invariant

The idempotent-extended $\mathcal{C}$-HS equations have the same form with the replacement of $A \to A_{\{\pi\}}$.

Vector-Like Models

Fock idempotent in the $4d$ HS theory

$$\pi_i^{star} = 4I_i \exp y_{i\alpha}\bar{y}_i^{\alpha}$$

$$(y_{i\alpha} - i\bar{y}_{i\alpha}) * \pi_i^{star} = 0\,, \qquad \pi_i^{star} * (y_{i\alpha} + i\bar{y}_{i\alpha}) = 0\,.$$

For HS fields carrying matrix indices

$$\pi_i = \pi_i^{star}\pi_i^{color}\,, \qquad \pi_i^{color} = \delta_1^u\delta_v^1\,.$$

${A_0}^i$**-module describes** $3d$ **conformal fields=** $4d$ **singletons:**

Idempotent realization of Klebanov-Polyakov AdS_4/CFT_3

vector model HS holography checked by Giombi and Yin in 2009

Idempotent extensions of the Coxeter HS systems describe

lower-dimensional brane-like objects.

$N=4$ SUSY

$4d$ conformal massless fields are valued in the Fock module π 2002

$$a_\alpha * \pi = 0\,,\qquad \bar{b}^{\dot\beta} * \pi = 0\,,\qquad \phi_i * \pi = 0\,,\qquad \pi * \bar{a}^{\dot\alpha} = 0\,,\ldots$$

$$[a_\alpha, b^\beta]_* = \delta_\alpha^\beta\,,\qquad [\bar{a}_{\dot\gamma}, \bar{b}^{\dot\beta}]_* = \delta_{\dot\gamma}^{\dot\beta}\,,\qquad \{\phi_i, \bar\phi^j\}_* = \delta_i{}^j\,,$$

$i,j = 1,\ldots N$. Bilinears: $su(2,2;N)$. Clifford oscillators: color $Mat_{2^{2N}}$.

B_2-HS theory contains $y_{i\alpha}$, $\bar{y}_{i\dot\alpha}$. Vacuum π is defined by ϕ_i, $\bar\phi^j$ and

$$a_\alpha = y_{1\alpha} + i y_{2\alpha}\,,\quad b_\alpha = \frac{1}{4i}(y_{1\alpha} - i y_{2\alpha})\,,\quad \bar{a}_{\dot\alpha} = \bar{y}_{1\dot\alpha} - i\bar{y}_{2\dot\alpha}\,,\quad \bar{b}_{\dot\alpha} = \frac{1}{4i}(\bar{y}_{1\dot\alpha} + i\bar{y}_{2\dot\alpha})$$

$4d$ massless conformal fields are valued in the Fock modules.

Reflection $Y_1^A \leftrightarrow Y_2^A$ maps π to the opposite idempotent $\tilde\pi$

$$b_\alpha * \tilde\pi = 0\,,\qquad \bar{a}^{\dot\beta} * \tilde\pi = 0\,,\qquad \bar\phi^i * \tilde\pi = 0\,,\qquad \tilde\pi * \bar{b}^{\dot\alpha} = 0\,,\ldots$$

Both π and $\tilde\pi$ have to be present. Elements $\pi * a * \tilde\pi$ are ill defined: at $N=0$, $\pi * \tilde\pi = \infty$. Bosons and fermions contribute with opposite signs. The compensation occurs at $N=4$ when $\#_B = \#_F$. $N=4$ SYM is the only $N=4$ massless conformal system with spins $s \le 1$.

Extension to Higher Forms and Invariant Functionals

$W + S \to \mathcal{W}$: **forms of all odd degrees**

$B \to \mathcal{B}$: **forms of all even degrees**

$\mathcal{L}$ **are Lagrangian-type closed forms generating invariants**

$$\mathcal{W} * \mathcal{W} = -i\Big(dZ^{An} dZ_{An} + F_*(\mathcal{B}, \gamma_i)\Big) + \mathcal{L}(x) I\,, \tag{1}$$

$$[\mathcal{W}\,, \mathcal{B}]_* = 0\,, \tag{2}$$

where

$$\gamma_i = \sum_{v \in \mathcal{R}_i} \frac{dZ_n^{\alpha} v^n dZ_{\alpha\, m} v^m}{(v, v)} * \widehat{\kappa}_v\,. \tag{3}$$

are central.

Resulting theory describes also higher differential forms

Algebraically, this constructions leads to an interesting generalization of Cherednik algebras

Some Reviews

MV, hep-th/0401177; 9910096; 9611024

X. Bekaert, S. Cnockaert, C. Iazeolla and MV, hep-th/0503128

D. Sorokin, arXiv:hep-th/0405069

A. Fotopoulos and M. Tsulaia, 0805.1346

X. Bekaert, N.Boulanger and P.Sundell, 1007.0435

M. R. Gaberdiel and R. Gopakumar, 1207.6697

S. Giombi and X. Yin, 1208.4036