Holographic Tensors mini-symposium, OIST

Thermofield Dynamics of SYK Model

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imagine the impossible

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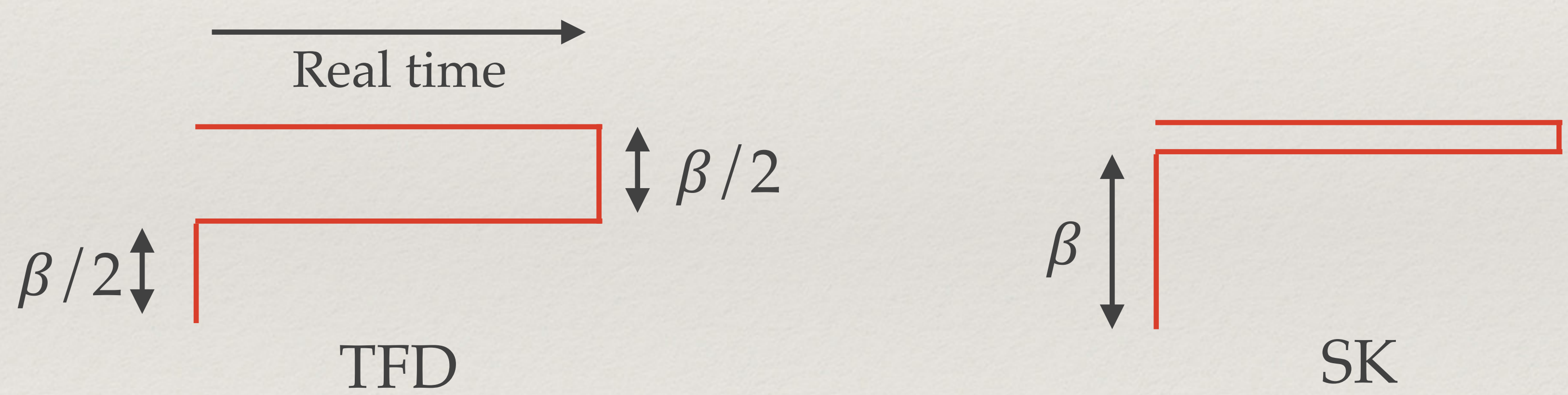
Thermofield Dynamics (TFD)

✴ Real-time Formulation at Finite Temperature

✓ Thermofield Dynamics (TFD): [Takahashi, Umezawa, 1975]

✓ cf) Schwinger-Keldysh Formulation



✓ Restrict to thermal state. cf) general state in SK formulation

✓ Related to Eternal Black hole [Israel, 1976], … , [Maldecena, 2003], …

Basic of TFD

- A specific purification of mixed states (thermal state)
  - ✓ Doubling the Degree of Freedom
- Example: harmonic oscillator
  - ✓ Introduce "fictitious" d.o.f. to purify thermal state

$$a, a^\dagger \longrightarrow a, a^\dagger \qquad \widetilde{a}, \widetilde{a}^\dagger$$

  - ✓ TFD state $|0(\beta)\rangle$: a pure state of which partial trace give thermal state

$$\mathrm{tr}_{\,\tilde{a}} |0(\beta)\rangle\langle 0(\beta)| = \rho_{\text{thermal}}$$

  - ✓ Bogoliubov transformed vacuum

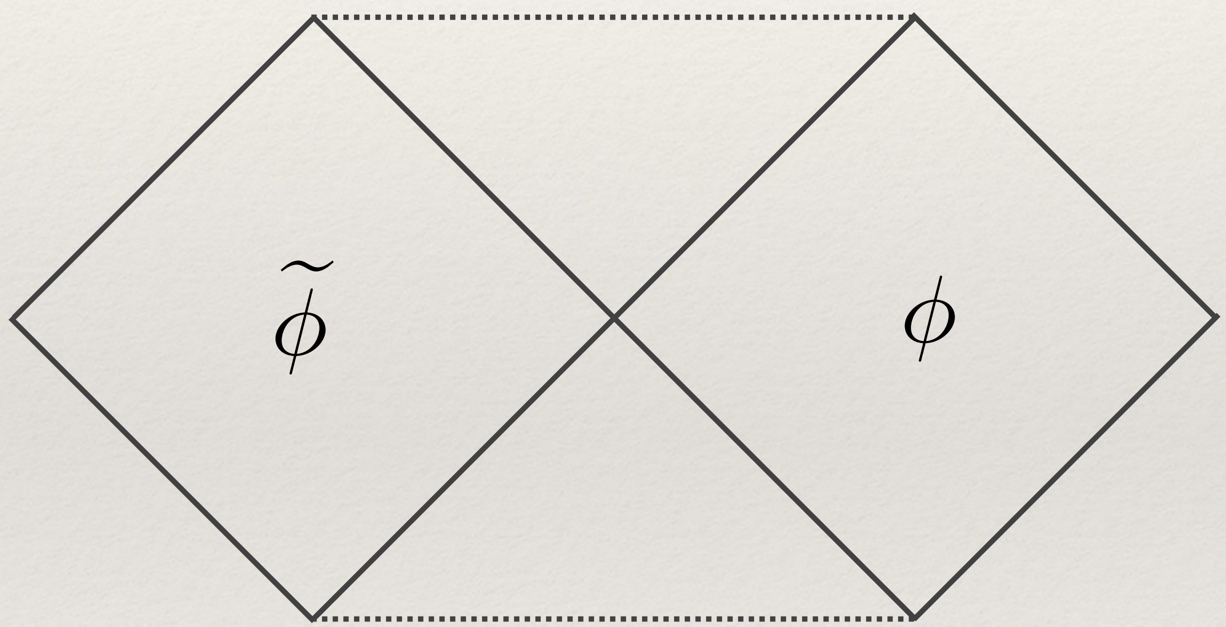
$$|0(\beta)\rangle = e^{\theta(\beta)(a^\dagger \widetilde{a}^\dagger - a\widetilde{a})}|0\rangle = e^{-iG}|0\rangle \qquad \beta\omega = -2\log\tanh\theta$$

  - ✓ Expectation value w.r.t. TFD state gives thermal average:

$$\langle 0(\beta)|\mathcal{O}|0(\beta)\rangle = \frac{\mathrm{Tr}\,\mathcal{O}\rho}{\mathrm{Tr}\,\rho}$$

TFD and Black Hole

- ✴ Thermal vacuum of black hole corresponds to TFD state. [Israel, 1976]
    - ✓ "Fictitious" d.o.f. corresponds to the d.o.f. in the other side of the maximally extended black hole



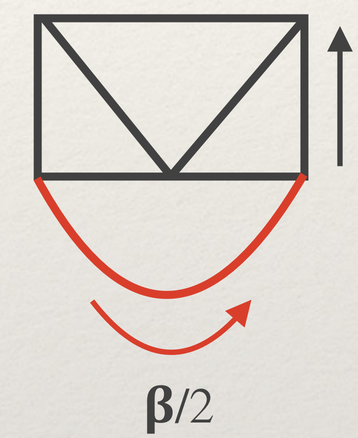
$$|HH\rangle = \frac{1}{\sqrt{Z}} \sum_E e^{-\frac{\beta E}{2}} |E\rangle_\phi |E\rangle_{\tilde{\phi}}$$

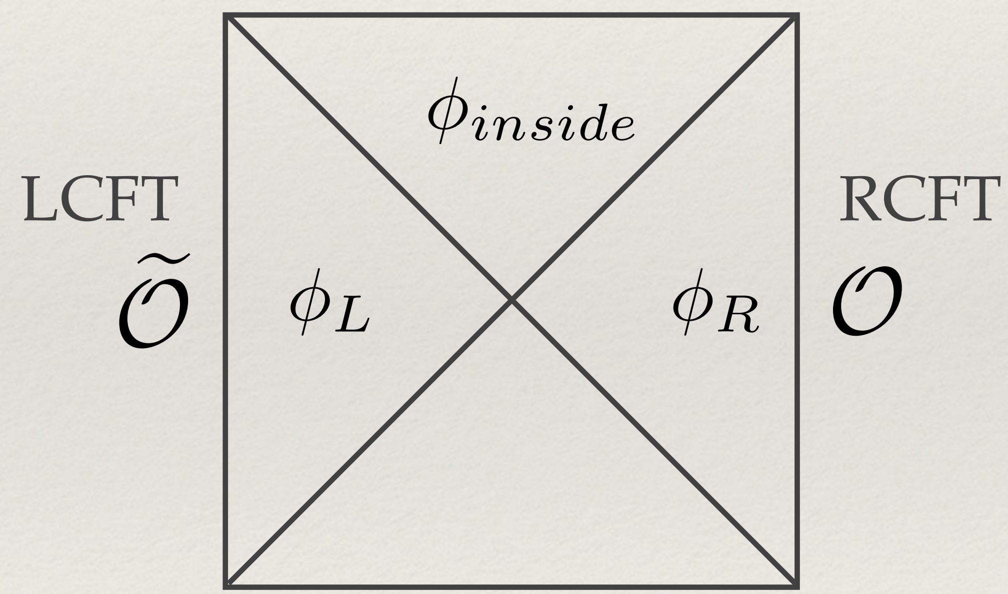
TFD and Black Hole

- ✴ AdS/CFT and TFD [Maldacena, 2003]

  - ✓ Construct TFD state by Euclidean path integral by **β**/2 (Hartle-Hawking state) and evolve the state along the real time.

  - ✓ TFD state: Maximally entangled state of R(ight)CFT and L(eft)CFT





$$\phi_R \sim \mathcal{O}$$

$$\phi_L \sim \tilde{\mathcal{O}}$$

- ✴ Construction of bulk field inside of horizon [Papadodimas, Raju 2012]

  - ✓ Linear combination of $\mathcal{O}$ and $\tilde{\mathcal{O}}$

$$\phi_{inside} \sim \mathcal{O} + \tilde{\mathcal{O}}$$

"Simple" Example of AdS/CFT

- Simple and concrete example of non-SUSY AdS/CFT correspondence:
  - ✓ Three-dimensional U(N)/O(N) Vector model dual to Higher Spin Gravity in AdS_4 [Klebanov, Polyakov, 2002] [Sezgin, Sundell, 2002]
  - ✓ CFT: Singlet Sector of (free/critical) O(N)/U(N) vector model in Large N
  - ✓ Gravity: Infinite tower of higher spin fields in AdS_4 [Fradkin, Vasiliev, 1986 ~]

$$\mathcal{O}_s \longleftrightarrow \mathcal{H}_s \qquad (s = 0, 1, 2, \cdots)$$

conserved spin-s operator — higher spin field

  - ✓ Great playground to understand holography: correlation functions, free energy, bulk construction

Collective Field Theory

* O(N) Singlet Sector can be described by bi-local field

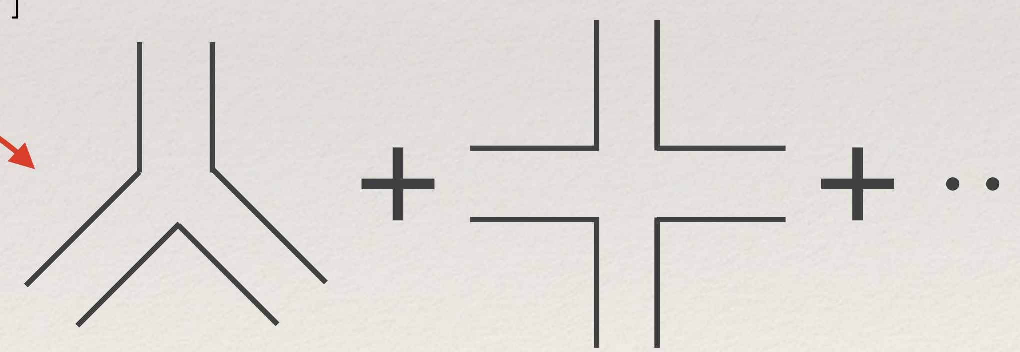
$$\Psi(x_1, x_2) = \sum_{i=1}^{N} \varphi^i(x_1)\varphi^i(x_2)$$

  - ✓ analogous to radial coordinate for rotationally invariant system
  - ✓ Non-linear transformation from vector field to bi-local field

* Collective field theory of O(N) vector model [Sakita, Jevicki, 1979], …

  - ✓ Collective Action: $Z_{\text{CFT}} = \int \mathcal{D}\Psi \; \mu[\Psi] e^{-S_{\text{col}}[\Psi]}$

$$S_{\text{col}} = S_{\text{free scalar}}[\Psi] - \frac{1}{2} N \text{Tr} \log \Psi$$



  - ✓ Measure generates vertices of bi-local fields:
  - ✓ conformal partial wave function [d.M.Koch, Jevicki, Suzuki, JY, 1810.02332]

TFD of O(N) Vector Model

- Natural to consider TFD of O(N) vector model and its holographic dual
    - ✓ 4D HS Black hole? [Didenko, Vasiliev, 2009], …
- Doubling D.o.F. $\phi^i(t,\vec{x}) \longrightarrow \phi^i(t,\vec{x}) \quad \widetilde{\phi}^i(t,\vec{x})$
- Doubling O(N) Symmetry and singlet sector

$$\boxed{O(N)\times O(N)} \quad \phi^i \quad \widetilde{\phi}^i \longrightarrow U^{ij}\phi^j \quad V^{ij}\widetilde{\phi}^j \qquad U,V\in O(N)$$

$$\boxed{\text{diagonal } O(N)} \quad \phi^i \quad \widetilde{\phi}^i \longrightarrow U^{ij}\phi^j \quad U^{ij}\widetilde{\phi}^j \qquad U\in O(N)$$

- O(N) Invariants: $\boxed{\boxed{\phi^i\phi^i \quad \widetilde{\phi}^i\widetilde{\phi}^i} \qquad \phi^i\widetilde{\phi}^i}$
- Bi-local field (for diagonal O(N)): $\Psi_{ab}(t;\vec{x},\vec{y}) \equiv \begin{pmatrix} \phi^i(\vec{x})\phi^i(\vec{y}) & i\phi^i(\vec{x})\widetilde{\phi}^i(\vec{y}) \\ i\widetilde{\phi}^i(\vec{x})\phi^i(\vec{y}) & -\widetilde{\phi}^i(\vec{x})\widetilde{\phi}^i(\vec{y}) \end{pmatrix}$

Collective Description of TFD

- Compact notation:

$$X = (a, \vec{x}) \in \{1,2\} \times \mathbb{R}^2$$

$$\Psi_{ab}(t; \vec{x}, \vec{y}) \equiv \begin{pmatrix} \phi^i(\vec{x})\phi^i(\vec{y}) & i\phi^i(\vec{x})\widetilde{\phi}^i(\vec{y}) \\ i\widetilde{\phi}^i(\vec{x})\phi^i(\vec{y}) & -\widetilde{\phi}^i(\vec{x})\widetilde{\phi}^i(\vec{y}) \end{pmatrix} \longrightarrow \Psi(t; X, Y) = \phi^i(t, X)\phi^i(t, Y)$$

- Collective Hamiltonian [Jevicki, JY, 2015]

$$H_{\mathrm{TFD}} = H - \widetilde{H} = \frac{2}{N}\mathrm{Tr}\left[\Pi \star \Psi \star \Pi\right] + \frac{N}{8}\mathrm{Tr}\left[\Psi^{-1}\right] + \frac{N}{2}\mathrm{Tr}\left[-\nabla^2 \star \Psi\right]$$

- Large N Saddle Point Equation

  ✓ can be solved up to arbitrary function F(p)

$$\Psi_{cl}(\vec{x}, \vec{y}) \sim \begin{pmatrix} \cosh F(\vec{p}) & i\sinh F(\vec{p}) \\ i\sinh F(\vec{p}) & -\cosh F(\vec{p}) \end{pmatrix} e^{i(\vec{x}-\vec{y})\cdot\vec{p}}$$

  ✓ F(p) can be fixed by KMS relation (or, by TFD state): $\tanh F = e^{-\frac{\beta|\vec{p}|}{2}}$

Bi-local Map in TFD

* Bi-local field can be expanded in terms of spin-s operator

$$: \phi^i(x)\phi^i(y) :\sim \sum_s \mathcal{O}_s$$

* By normal ordering of bi-local field w.r.t. vacuum, one can express spin-s operator in terms of bi-local oscillator $\alpha \sim a^i a^i$

$$\mathcal{O}_s \sim \alpha + \alpha^\dagger$$

* In TFD, in addition to (doubled) conserved spin-s Operator O_{11} and O_{22}, we also have off-diagonal spin-s operator O_{12}

$$\mathcal{O}_s^{ab}(x;\alpha) = \sum_{n=0}^{s/2} \frac{(-4)^n}{(2n)!} \sum_{k=0}^{s-2n} \frac{(-1)^k}{k!(s-2n-k)!} : (\alpha\cdot\partial)^{n+k}\phi^i(a,x)\;(\alpha\cdot\partial)^{s-n-k}\phi^i(b,x) :$$

* Normal ordering w.r.t. TFD state $|0(\beta)\rangle$, spin-s operator can be expand as

$$\mathcal{O}^{ab} \sim \alpha_\theta + \tilde{\alpha}_\theta^\dagger + \gamma_\theta + \gamma_\theta^\dagger$$

$$\alpha_\theta \sim a_\theta^i a_\theta^i \qquad \widetilde{\alpha}_\theta \sim \widetilde{a}_\theta^i \widetilde{a}_\theta^i$$

$$\boxed{\gamma_\theta \sim a_\theta^i \widetilde{a}_\theta^i}$$

Extra mode

$$a^i \xrightarrow{\text{Bogoliubov transf}} a_\theta^i$$

Bi-local Map in TFD

$$\mathcal{O}^{ab} \sim \alpha_\theta + \tilde{\alpha}_\theta^\dagger + \gamma_\theta + \gamma_\theta^\dagger$$

Time-like Mode

$\alpha_\theta \sim a_\theta^i a_\theta^i$ $\qquad$ $\widetilde{\alpha}_\theta \sim \widetilde{a}_\theta^i \widetilde{a}_\theta^i$

$$p^0 = \pm(p_1^0 + p_2^0)$$
$$\vec{p} = \pm(\vec{p}_1 + \vec{p}_2)$$

$p_1^0 = |\vec{p}_1|$

$p_2^0 = |\vec{p}_2|$

$$(p^0)^2 - \vec{p}^2 = 2(|\vec{p}_1||\vec{p}_2| - \vec{p}_1 \cdot \vec{p}_2)$$
$$\geqq 0$$

Space-like Mode

$\gamma_\theta \sim a_\theta^i \widetilde{a}_\theta^i$

$$p^0 = \pm(p_1^0 - p_2^0)$$
$$\vec{p} = \pm(\vec{p}_1 - \vec{p}_2)$$

$$(p^0)^2 - \vec{p}^2 = -2(|\vec{p}_1||\vec{p}_2| - \vec{p}_1 \cdot \vec{p}_2)$$
$$\leqq 0$$

Evanescent Mode in Black Hole

$$\gamma_\theta \sim a^i_\theta \widetilde{a}^i_\theta$$

$$p^0 = \pm(p^0_1 - p^0_2)$$
$$\vec{p} = \pm(\vec{p}_1 - \vec{p}_2)$$
$$(p^0)^2 - \vec{p}^2$$
$$= -2(|\vec{p}_1||\vec{p}_2| - \vec{p}_1 \cdot \vec{p}_2)$$
$$\leqq 0$$

- Space-like Mode of Spin-s Current

$$\mathcal{O}^{ab} \sim \alpha_\theta + \tilde{\alpha}^\dagger_\theta + \gamma_\theta + \gamma^\dagger_\theta$$

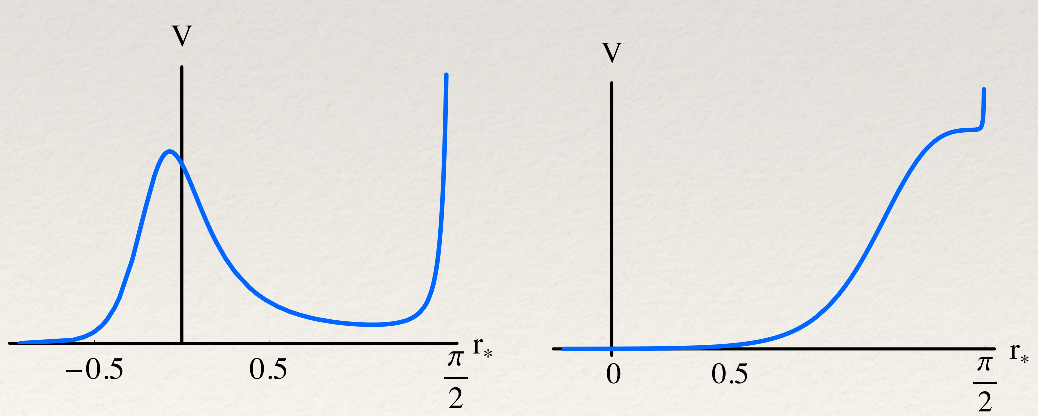
Time-like Space-like

- Exponentially Decay in z direction of Bulk $(p^z)^2 \simeq (p^0)^2 - \vec{p}^2$

$$\mathcal{H}_s \sim \alpha_\theta + \tilde{\alpha}^\dagger_\theta + \gamma_\theta + \gamma^\dagger_\theta$$

Propagating Decaying

- It might be related to Evanescent Mode in Black Hole



Effective potential of AdS-Schwarzschild black hole

[Rey, Rosenhaus, 2014]

SYK Model

- Quantum Mechanics of O(N) Majorana fermions with random coupling [Sachdev, Ye, 1992], [Kitaev, 2015], [Polchinski, Rosenhaus, Jevicki, Suzuki, JY, Maldacena, Stanford, Gross, …]

$$S=\int d\tau\left[\frac{1}{2}\sum_{i=1}^{N}\chi^i\partial_\tau\chi^i-\sum_{i,j,k,l=1}^{N}J_{ijkl}\chi^i\chi^j\chi^k\chi^l\right]$$

- Maximally chaotic
  - ✓ Saturation of Chaos Bound (Lyapunov exponent) [Maldacena, Shenker, Stanford, 2015] [Kitaev, 2015]
  - ✓ Random Matrix Behavior [Shenker, Hanada et al 2016], …, [Nosaka, Rosa, JY 2018]
  - ✓ Holographic dual to black hole
- Collective Action of bi-local field $\Psi(\tau_1,\tau_2)=\frac{1}{N}\sum_{i=1}^{N}\chi^i(\tau_1)\chi^i(\tau_2)$ [Jevicki, Suzuki, JY, 2016]

$$S_{col}=\frac{N}{2}\mathrm{Tr}\left[-D\star\Psi+\log\Psi\right]-\frac{NJ^2}{8}\int d\tau_1 d\tau_2\ \left[\Psi(\tau_1,\tau_2)\right]^4$$

-

TFD of SYK Model

* Action for Lorentzian SYK model

$$\mathcal{L} = i\chi_i \partial_t \chi_i + i^{\frac{q}{2}} J_{i_1 \cdots i_q} \chi_{i_1} \cdots \chi_{i_q}$$

* Doubling of D.o.F. : $\chi \longrightarrow \chi \quad \widetilde{\chi}$

$$\widetilde{\mathcal{L}} = i\tilde{\chi}_i \partial_t \tilde{\chi}_i + i^{\frac{q}{2}} J_{i_1 \cdots i_q} \tilde{\chi}_{i_1} \cdots \tilde{\chi}_{i_q}$$

  - ✓ Issue in sign of the copied Lagrangian (fermionic case) [Ojima, 1981]

* Action of TFD: $\mathcal{S}_{TFD} = \int dt \, (\mathcal{L} - \widetilde{\mathcal{L}})$

* O(N) Singlet Sector: Diagonal O(N) symmetry (vs O(N)×O(N) symmetry)

* For diagonal O(N) singlet sector, we can define invariant bi-local field as

$$\Psi_{ab}(t_1, t_2) \equiv \frac{1}{N} \begin{pmatrix} \chi^i(t_1)\chi^i(t_2) & \chi^i(t_1)\widetilde{\chi}^i(t_2) \\ \widetilde{\chi}^i(t_1)\chi^i_1(t_2) & -\widetilde{\chi}^i(t_1)\widetilde{\chi}^i(t_2) \end{pmatrix}$$

Collective Description of TFD

- Collective Action for TFD of SYK model

$$iS_{col} = \frac{N}{2}\mathbf{Tr}\,[\mathcal{D}\circledast\Psi] - \frac{N}{2}\mathbf{Tr}\,\log\Psi - \frac{NJ^2}{2q}\sum_{a,b=1}^{2}\int_{-\infty}^{\infty} dt_1 dt_2\; \gamma_q^{ab}[\Psi^{ab}(t_1,t_2)]^q$$

$$\gamma_q^{ab} \equiv \begin{cases} (-1)^{a+b} & \text{for } \; q\equiv 0 \pmod 4 \\ 1 & \text{for } \; q\equiv 2 \pmod 4 \end{cases}$$

- Large N Saddle Point Equation

$$(\mathcal{D}\circledast\Psi)^{ab}(t_1,t_2) - \delta^{ab}\delta(t_1-t_2) - J^2\sum_{c=1}^{2}\int_{-\infty}^{\infty} dt_3\; \gamma_q^{ab}[\Psi^{ac}(t_1,t_3)]^{q-1}\Psi^{cb}(t_3,t_2) = 0$$

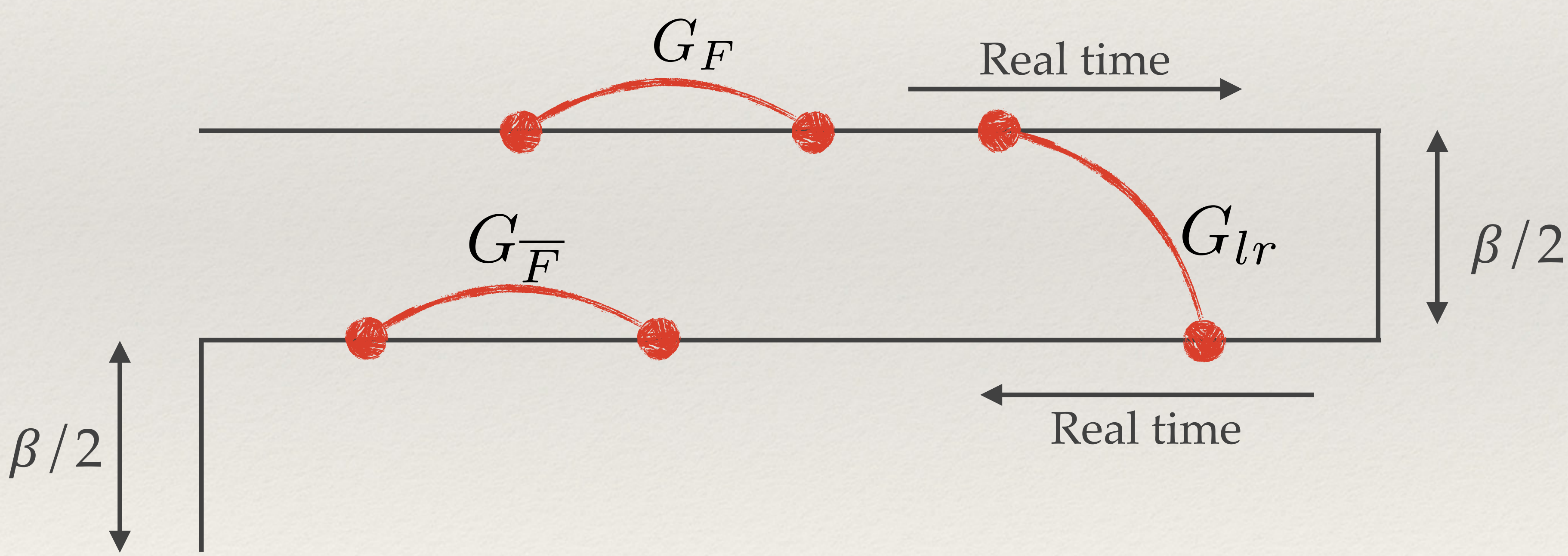
  - ✓ Schwinger-Dyson Equation for two point function
  - ✓ cf) Two coupled Euclidean SYK model [Maldacena, Qi, 2018]
  - ✓ Coupled integral equation: Difficult to solve in general

Real-time Correlation Functions

* Real-time correlation functions can be evaluated by analytic continuation of Euclidean correlation function

$$G_>(t_1,t_2) = e^{-\frac{i\pi}{q}} \frac{\Lambda\left(\frac{\pi}{\beta}\right)^{\frac{2}{q}}}{\left(\sinh\frac{\pi t_{12}}{\beta}\right)^{\frac{2}{q}}} \qquad G_<(t_1,t_2) = -e^{\frac{i\pi}{q}} \frac{\Lambda\left(\frac{\pi}{\beta}\right)^{\frac{2}{q}}}{\left(\sinh\frac{\pi t_{12}}{\beta}\right)^{\frac{2}{q}}} \qquad G_F(t_1,t_2) = G_<(t_1,t_2) + G_R(t_1,t_2) \quad \text{etc.}$$

  ✓ Retarded, advanced, Feynman, anti-Feynman, Wightman correlation functions



* Solution of SD equation: $\Psi^{ab}_{cl}(t_1,t_2) = \begin{pmatrix} G_F(t_1,t_2) & G_{lr,1}(t_1,t_2) \\ -G_{lr,2}(t_1,t_2) & -G_{\bar{F}}(t_1,t_2) \end{pmatrix}$

Large q Limit

* Large q Limit simplify SD equation for two point function
  - ✓ Rescaling the coupling: $\mathcal{J}^2 = \dfrac{qJ^2}{2^{q-1}}$
  - ✓ 1/q expansion of Large N classical solution:

$$\Psi^{ab}(t_1,t_2) = \frac{1}{2}\begin{pmatrix} \mathrm{sgn}\,(t_{12})\left[1+\frac{1}{q}\phi^{11}(t_{12})+\cdots\right] & -\left[1+\frac{1}{q}\phi^{12}(t_{12})+\cdots\right] \\ -\left[1+\frac{1}{q}\phi^{21}(t_{12})+\cdots\right] & \mathrm{sgn}\,(t_{12})\left[1+\frac{1}{q}\phi^{22}(t_{12})+\cdots\right] \end{pmatrix}$$

* SD equation becomes "Liouville" equations

$$\partial_t^2\begin{pmatrix} \mathrm{sgn}\,(t)\phi^{11}(t) & -\phi^{12}(t) \\ -\phi^{21}(t) & \mathrm{sgn}\,(t)\phi^{22}(t) \end{pmatrix} = -2\mathcal{J}^2\begin{pmatrix} \mathrm{sgn}\,(t)e^{\frac{1}{2}[\phi^{11}(t)+\phi^{11}(-t)]} & -e^{\frac{1}{2}[\phi^{12}(t)+\phi^{21}(-t)]} \\ -e^{\frac{1}{2}[\phi^{21}(t)+\phi^{12}(-t)]} & \mathrm{sgn}\,(t)e^{\frac{1}{2}[\phi^{22}(t)+\phi^{22}(-t)]} \end{pmatrix}$$

  - ✓ Solution: $e^{\phi^{11}(t)} = \dfrac{\cos^2\frac{\pi v}{2}}{\cosh^2\left[\pi v\left(\frac{|t|}{\beta}+\frac{i}{2}\right)\right]}$ $\qquad \dfrac{\pi v}{\cos\frac{\pi v}{2}} = \beta\mathcal{J}$

Generalizations

- TFD of Complex SYK model: $\bar{\chi}_{1,i}\ ,\ \chi_1^i$ and $\bar{\chi}_{2,i}\ ,\ \chi_2^i$
  - ✓ Bi-local field: $\Psi_{ab}(t_1,t_2) \equiv \frac{1}{N}\begin{pmatrix} \bar{\chi}_{1,i}(t_1)\chi_1^i(t_2) & \bar{\chi}_{1,i}(t_1)\chi_2^i(t_2) \\ \bar{\chi}_{2,i}(t_1)\chi_1^i(t_2) & \bar{\chi}_{2,i}(t_1)\chi_2^i(t_2) \end{pmatrix}$
  - ✓ Large q limit: $\frac{qJ^2}{2^{q-1}}[1-\gamma^2]^{\frac{q}{2}-1} = \mathcal{J}^2$ $\quad e^{\phi^{11}(t)} = \frac{\cos^2\frac{\pi v}{2}}{\cosh^2\left[\pi v\left(\frac{|t|}{\beta}+\frac{i}{2}\right)\right]}$ $\quad \frac{\pi v}{\cos\frac{\pi v}{2}} = \beta\mathcal{J}$

    similar to [Sachdev, Jensen, Davison et al, 2016]
- Multiple copies of SYK models

Discussion

- Real time formulation of SYK model at Finite temperature
- Fluctuation around the classical solution: $\Psi^{ab}(t_1, t_2) = \Psi^{ab}_{cl}(t_1, t_2) + \frac{1}{\sqrt{N}}\eta^{ab}(t_1, t_2)$
    - ✓ can perform 1/N expansion of collective action
    - ✓ Quadratic action gives two point function of bi-locals (four point functions of fermions)
- Meaning of extra mode
- Explicit interaction between two SYK model
    - ✓ Traversable wormhole: [Maldacena, Qi, 2018]

Thanks

Questions?