Brief Review and New Results in Melonic Quantum Mechanics

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Okinawa Institute of Science and Technology
Mini Symposium on Holographic Tensors
OIST, November 2nd, 2018

Plan of the talk:

1. Introduction: Holography, Matrix Models, Large N and Large D

2. (Auxiliary field and Tait graphs)

3. Phase diagrams: critical points, Lyapunov exponents, quantum criticality

4. New IR behaviour: surprising solutions to Schwinger-Dyson equations

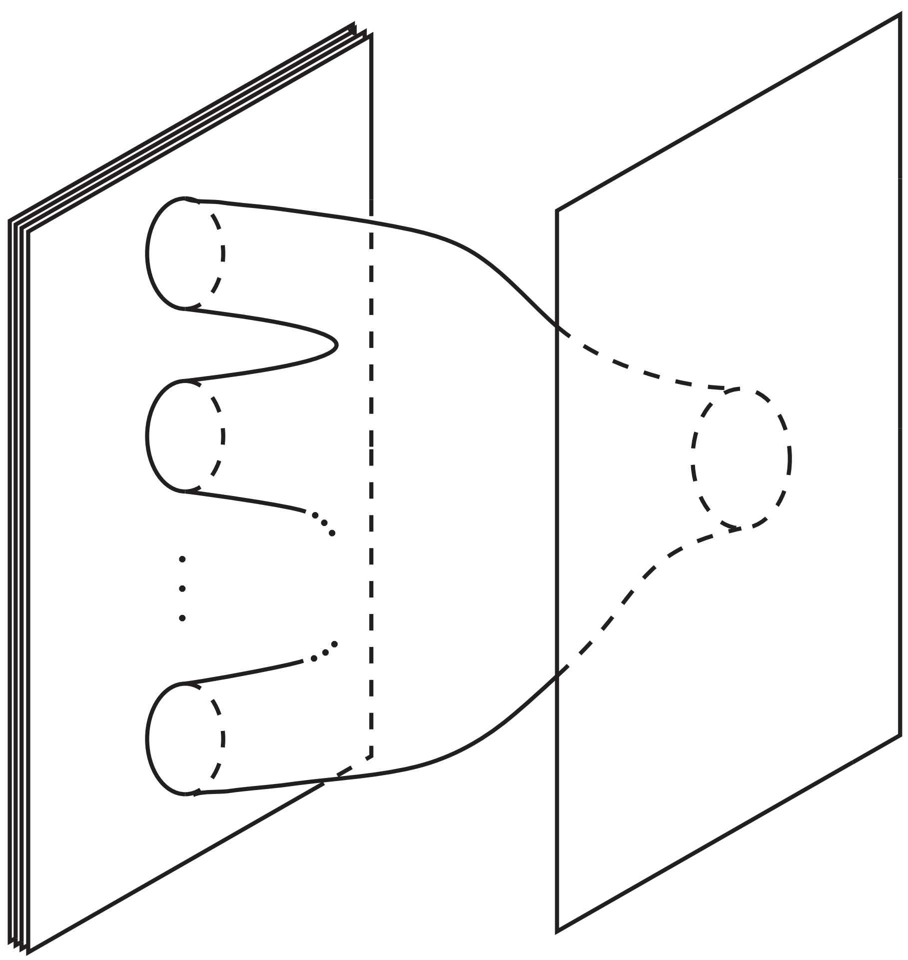
5. Space of solutions of Schwinger-Dyson equations and non-linear system of differential equations

Work partly in collaboration with Paolo Gregori (ULB and Torino, soon in Lisbon), Fidel Schaposnik (IBS, then ULB, now at IHÉS), Guillaume Valette (ULB), Gregory Kozyreff (ULB, for part 5).

References: 1701.01171; 1707.03431 (with FS and T. Azeyanagi); 1709.07366 (with V. Rivasseau and GV); 1710.07263 (with T.A., P.G., G.V., Laeticia Leduc; papers to appear in the coming weeks.

Large N and large D

Matrix model with variables $X^{a}{}_{b\,\mu}$ and a generic symmetry group $\mathrm{U}(N) \times \mathrm{O}(D)$ (which may be enhanced to $\mathrm{U}(N)^2 \times \mathrm{O}(D)$)



$$1 \leq \mu \leq D = d - p - 1$$

Finite N is “impossible”. We look at large N (this is also associated with the classical limit of gravity in holography)

$$F = \sum_{g \geq 0} N^{2-2g} F_g$$

This is still too difficult… Except in “trivial” cases, the sum over planar diagrams cannot be done.

Few exceptions: some simple zero dimensional examples (but even simple-looking zero dimensional matrix models are not possible to solve at large N); the one matrix quantum mechanics in the singlet sector (but this is NOT a black hole, contrary to the generic matrix quantum mechanics).

It’s a very frustrating situation that lasted over 40 years.

$$F = \sum_{g\geq 0} N^{2-2g} F_g$$

Idea (not new): since we also have D, why not look at the large D limit too? We then get an approximation to the approximation: a large D expansion of the sum over planar diagrams (or of the sum over fixed genus graphs).

$$S = N\,\mathrm{tr}\big(X_\mu X_\mu + \lambda\, X_\mu X_\nu X_\mu X_\nu + \kappa\, X_\mu X_\mu X_\nu X_\nu\big)$$

$$\Big\downarrow \quad \lambda \to \lambda/D\,, \quad \kappa \to \kappa/D$$

$$S = ND\,\mathrm{tr}\big(X_\mu X_\mu + \lambda\, X_\mu X_\nu X_\mu X_\nu + \kappa\, X_\mu X_\mu X_\nu X_\nu\big)$$

$$F_g = \sum_{\ell\geq 0} D^{1-\ell} F_{g,\ell}$$

Uninteresting: vector model physics. This new approximation kills too many graphs. In particular, at leading order, the tetrahedric coupling does not contribute. The expected qualitative behaviour of the full sum over planar diagrams is not at all reproduced.

$$F = \sum_{g \geq 0} N^{2-2g} F_g$$

Idea (new): enhance some couplings so that more graphs can contribute.

$$S = ND\,\mathrm{tr}\big(X_\mu X_\mu + \lambda\, X_\mu X_\nu X_\mu X_\nu + \kappa\, X_\mu X_\mu X_\nu X_\nu\big)$$

$$\Big\downarrow \quad \lambda \to \sqrt{D}\lambda$$

$$S = ND\,\mathrm{tr}\big(X_\mu X_\mu + \sqrt{D}\lambda\, X_\mu X_\nu X_\mu X_\nu + \kappa\, X_\mu X_\mu X_\nu X_\nu\big)$$

We get a new large D limit, not vector-like. Remarkably, it is consistent: graphs of fixed genus turn out to have an upper bound on the possible power of D.

$$S = ND\,\mathrm{tr}\big(X_\mu X_\mu + \sqrt{D}\lambda\, X_\mu X_\nu X_\mu X_\nu + \kappa\, X_\mu X_\mu X_\nu X_\nu\big)$$

$$F = \sum_{g\geq 0} N^{2-2g} F_g$$

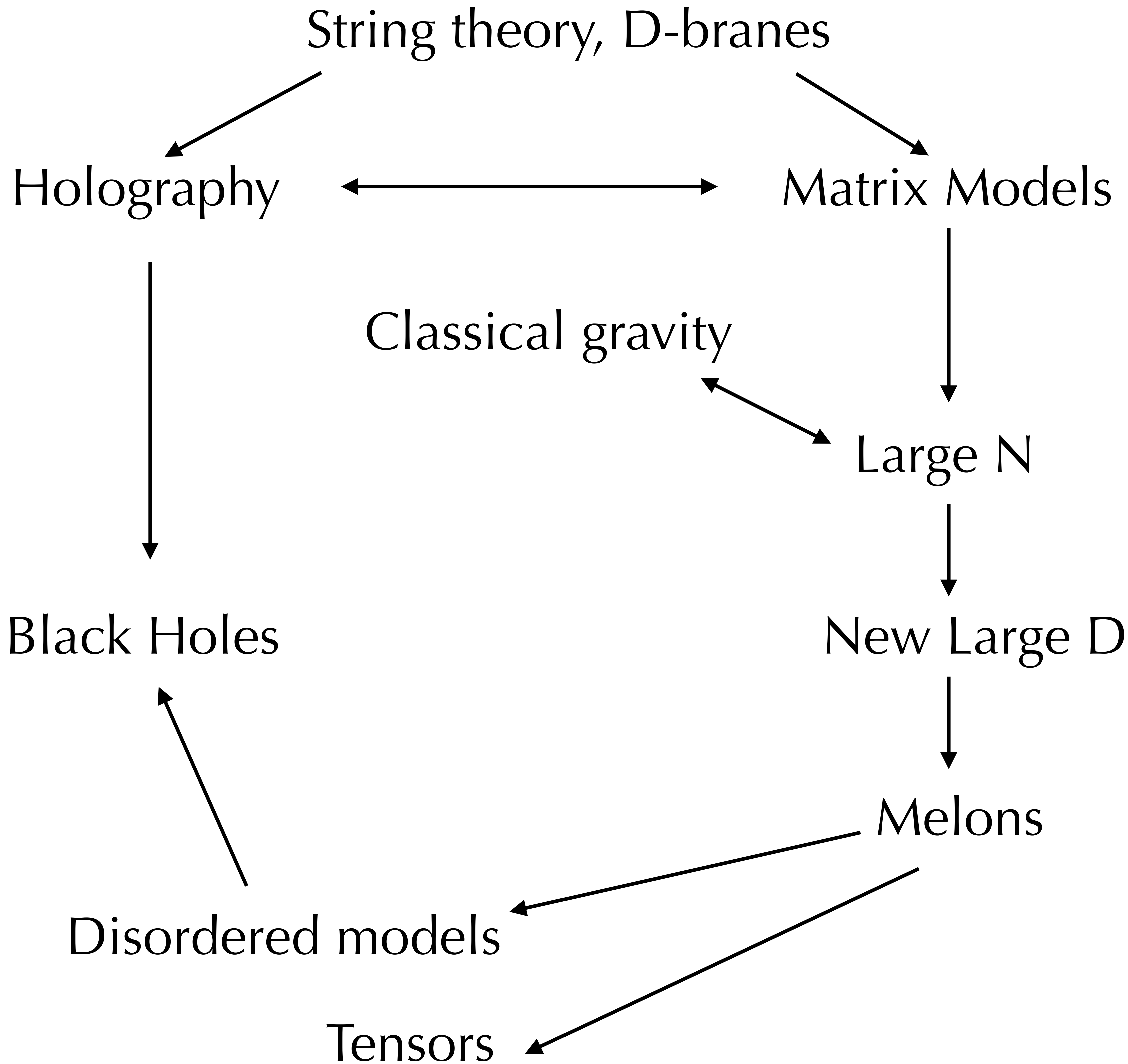
$$F_g = \sum_{\ell\geq 0} D^{1+g-\ell/2} F_{g,\ell}$$

Important property: the new large D limit does not commute with the large N limit, because at fixed N there exist graphs with arbitrarily high powers of D.

Its existence relies on a subtle interplay between the vector and matrix properties.

Strong claim: the leading order $F_{0,0}$ of this approximation correctly captures the qualitative physics associated with the full sum over planar diagrams.

Moreover, the leading order is dominated by melonic graphs.

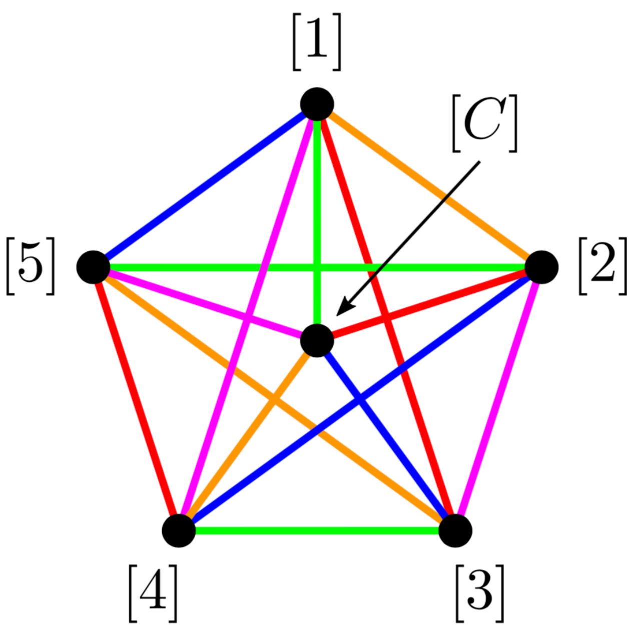

String theory, D-branes
Holography
Matrix Models
Classical gravity
Large N
Black Holes
New Large D
Melons
Disordered models
Tensors

Phase diagrams

$$H_1 = ND\,\mathrm{tr}\Big(m\psi_\mu^\dagger\psi_\mu + \frac{1}{2}\sqrt{D}\lambda\,\psi_\mu\psi_\nu^\dagger\psi_\mu\psi_\nu^\dagger\Big)$$

$$H_2 = ND\,\mathrm{tr}\Big(m\psi_\mu^\dagger\psi_\mu + \frac{1}{2}\sqrt{D}\big(\lambda\,\psi_\mu\psi_\nu\psi_\mu\psi_\nu^\dagger + \mathrm{H.c.}\big)\Big)$$

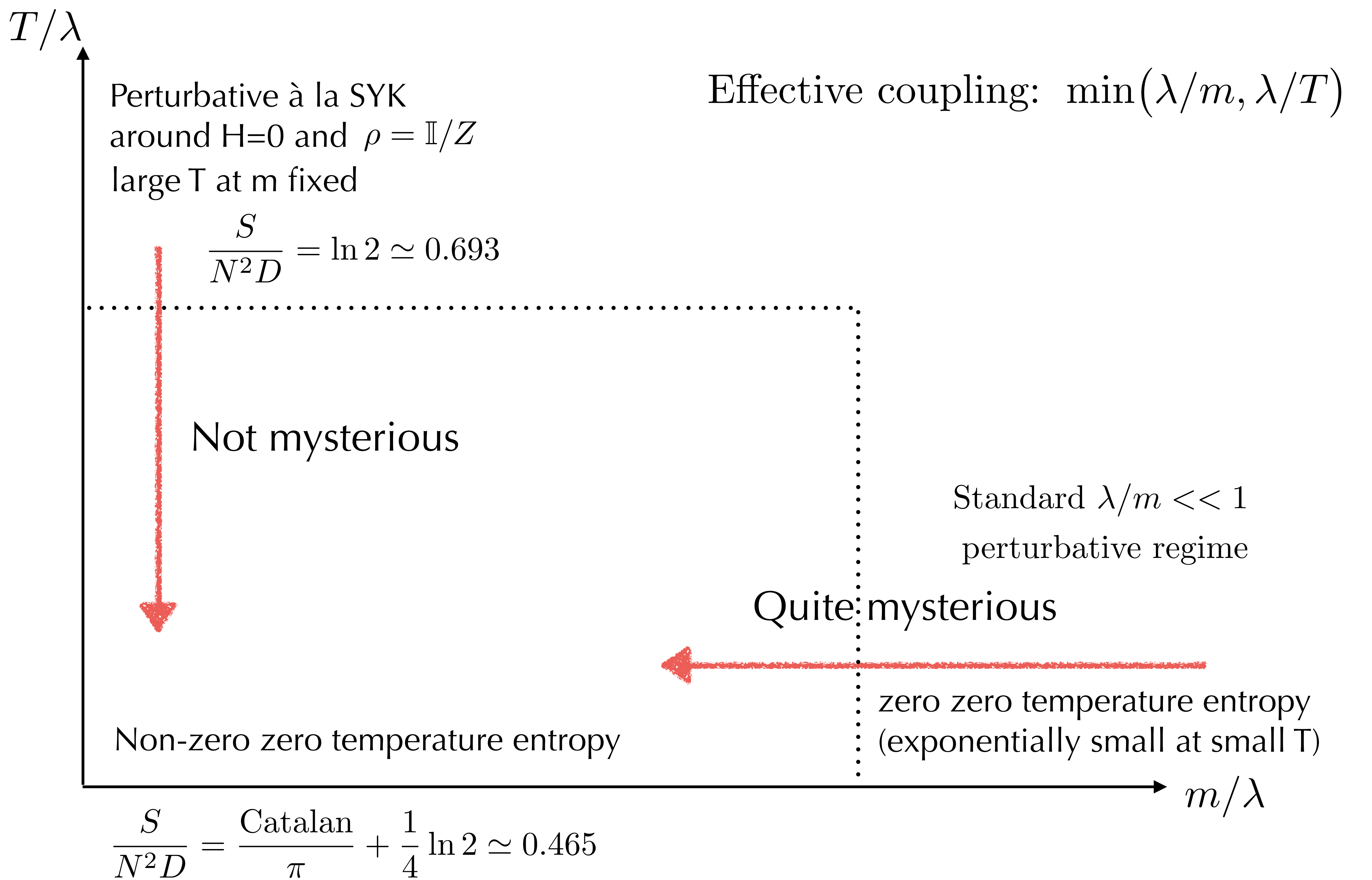
or more general (r,s) models



There is also a disordered version, with a Hamiltonian of the form

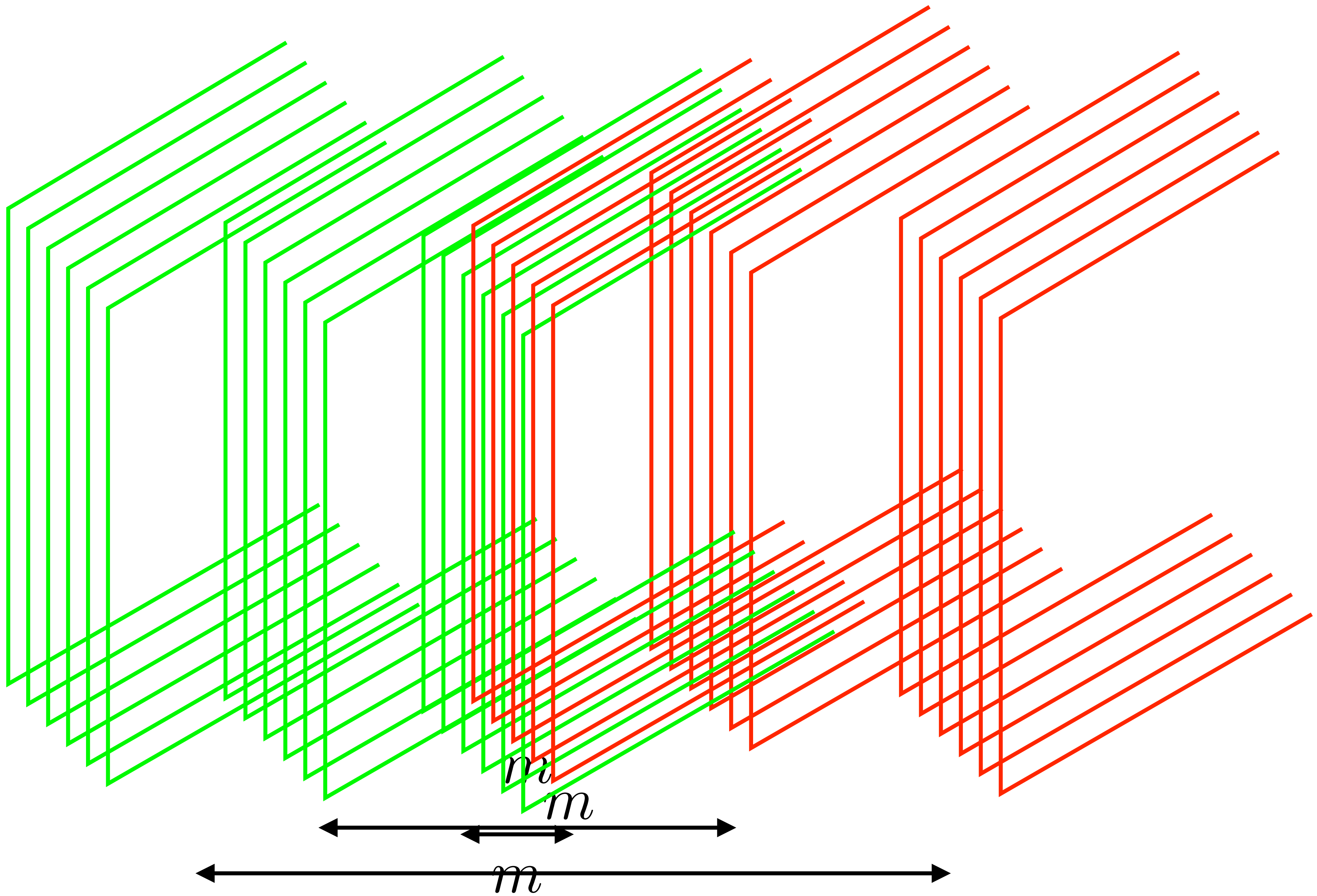
$$H = m\psi_i^\dagger\psi^i + \sqrt{N}g_{i_1\cdots i_r}^{j_1\cdots j_s}\psi_{j_1}^\dagger\cdots\psi_{j_s}^\dagger\psi^{i_1}\cdots\psi^{i_r} + \mathrm{H.\ c.}$$

One motivation:

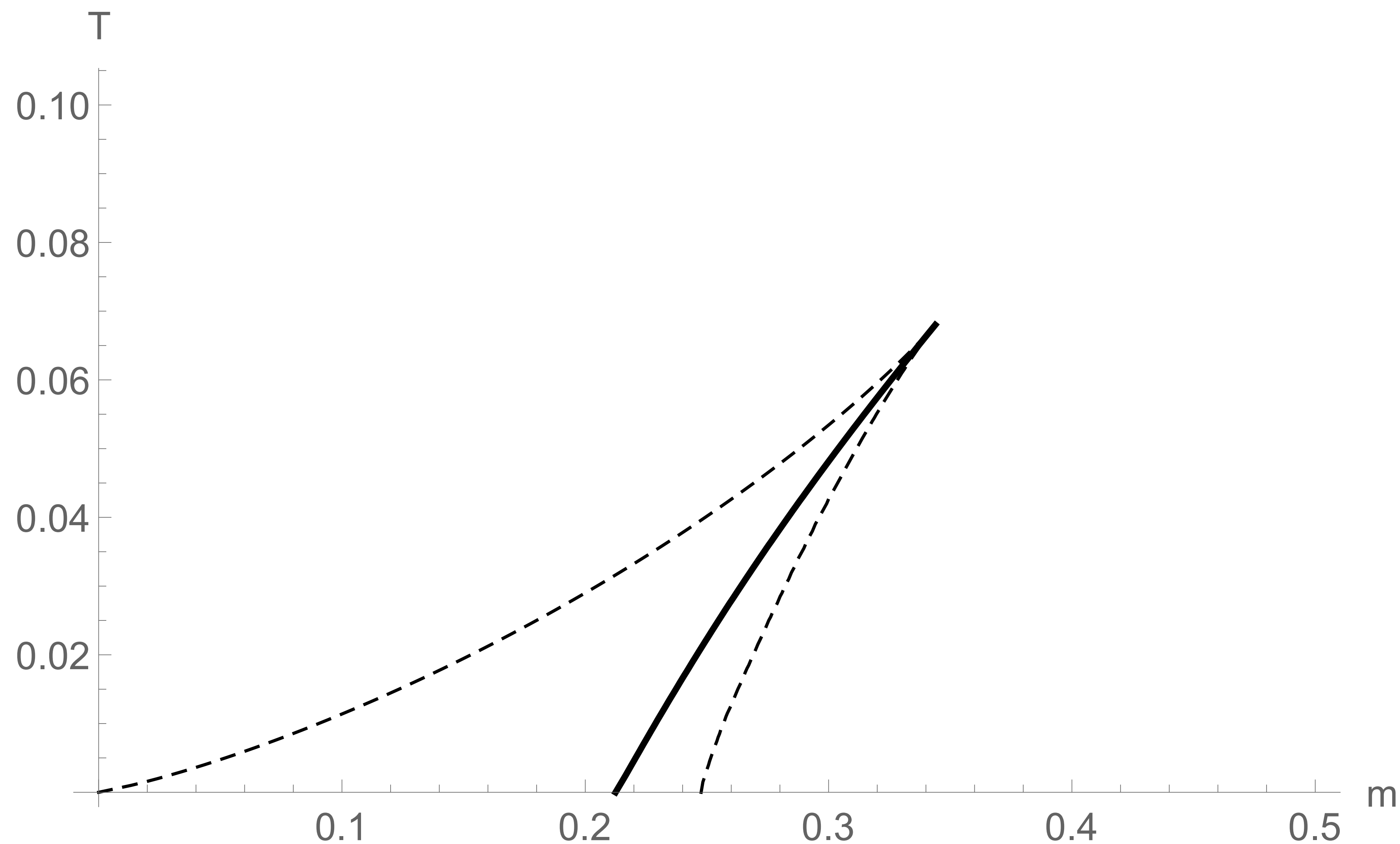


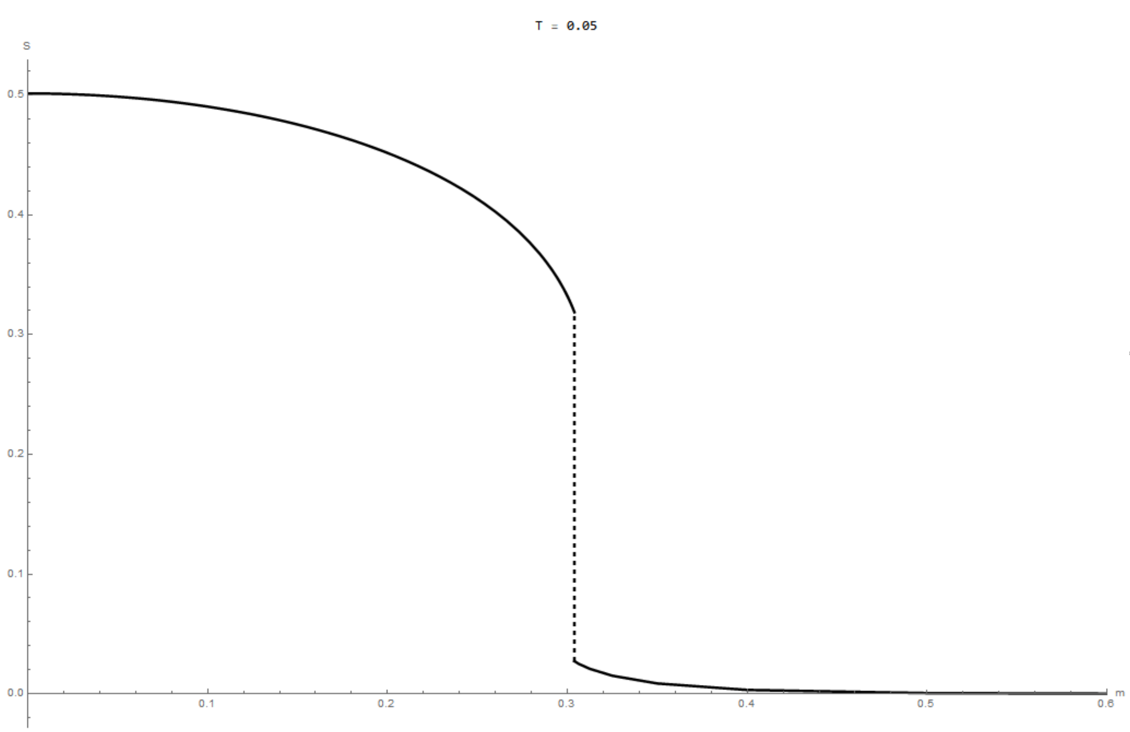
Another motivation: matrix models have a brane-like interpretation in which the mass term represent the distance between the brane.

$$H_1 = ND\,\mathrm{tr}\Big(m\psi_\mu^\dagger\psi_\mu + \frac{1}{2}\sqrt{D}\lambda\,\psi_\mu\psi_\nu^\dagger\psi_\mu\psi_\nu^\dagger\Big)$$



We may expect that, below some critical distance, the stack of branes gravitationally collapse to a black hole.

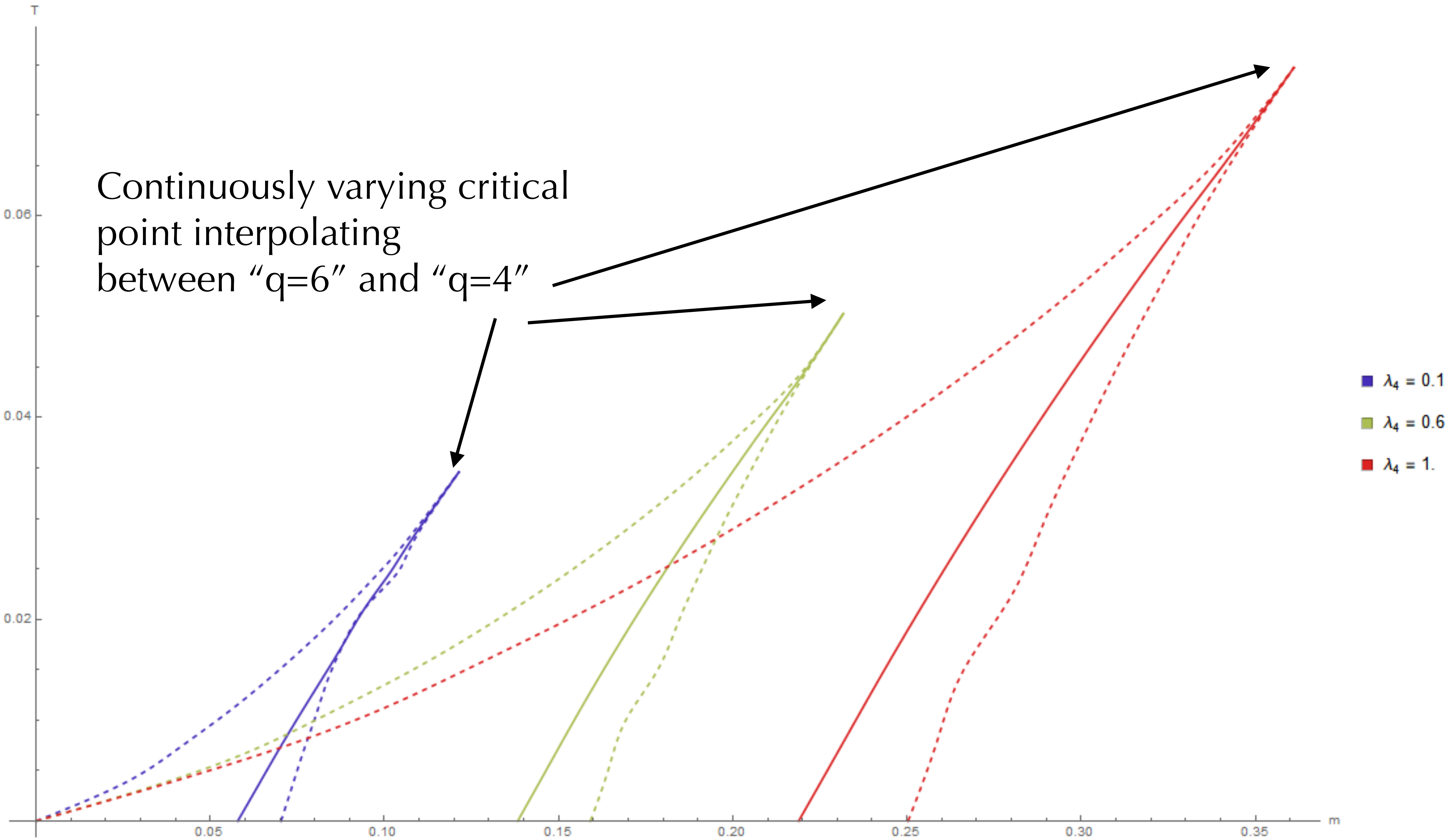




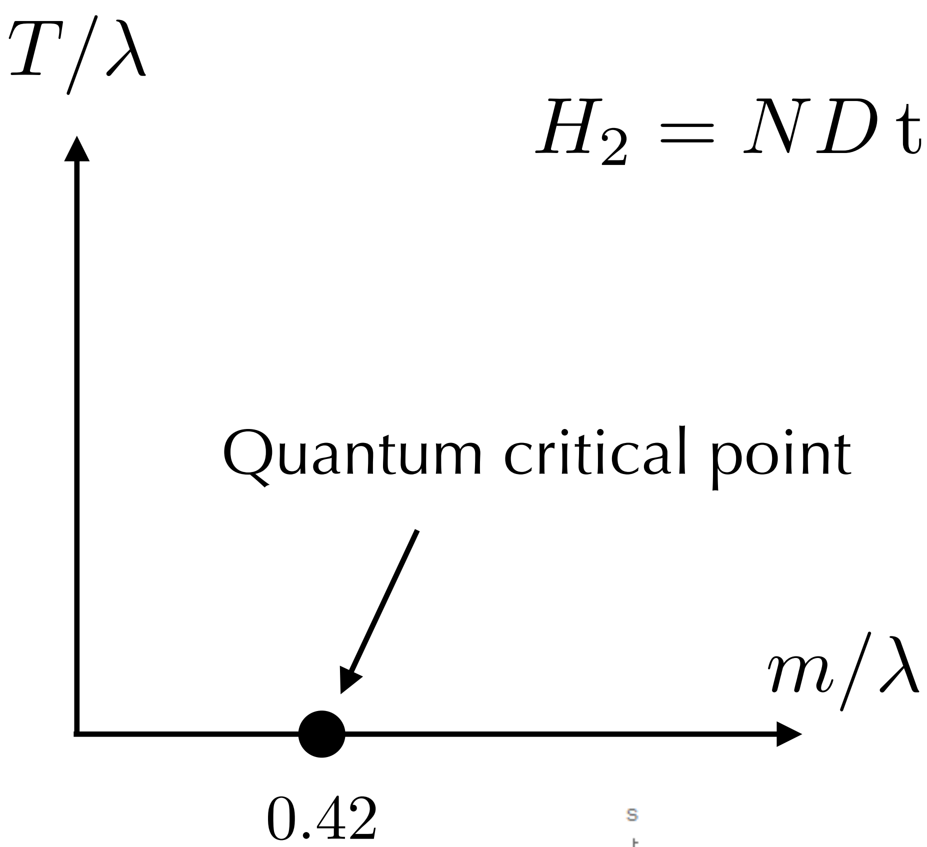
Monodromy in the space of solution of the Schwinger-Dyson equations

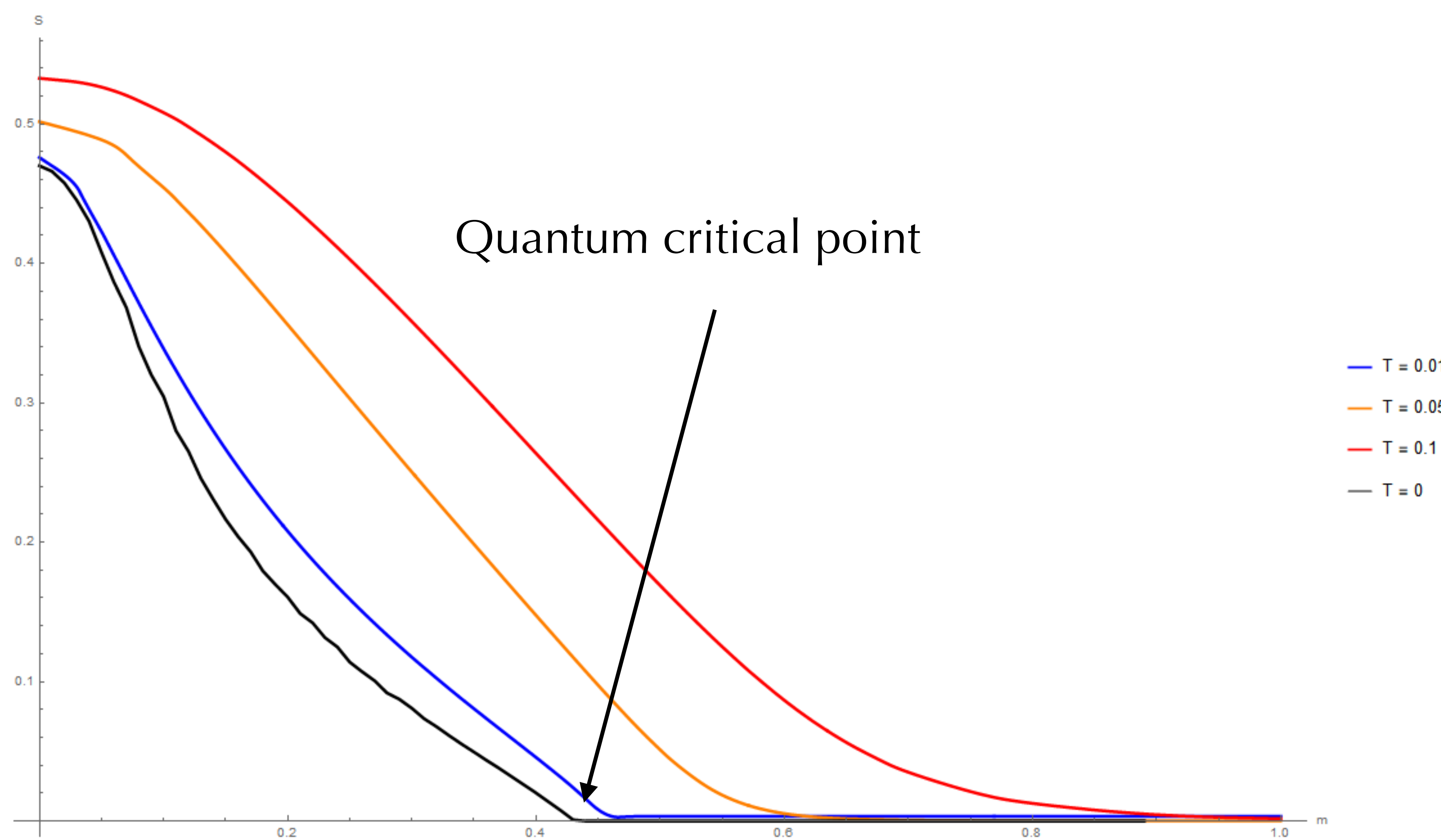
Entropy-mass plot

Model with both a degree four and a degree six interaction term

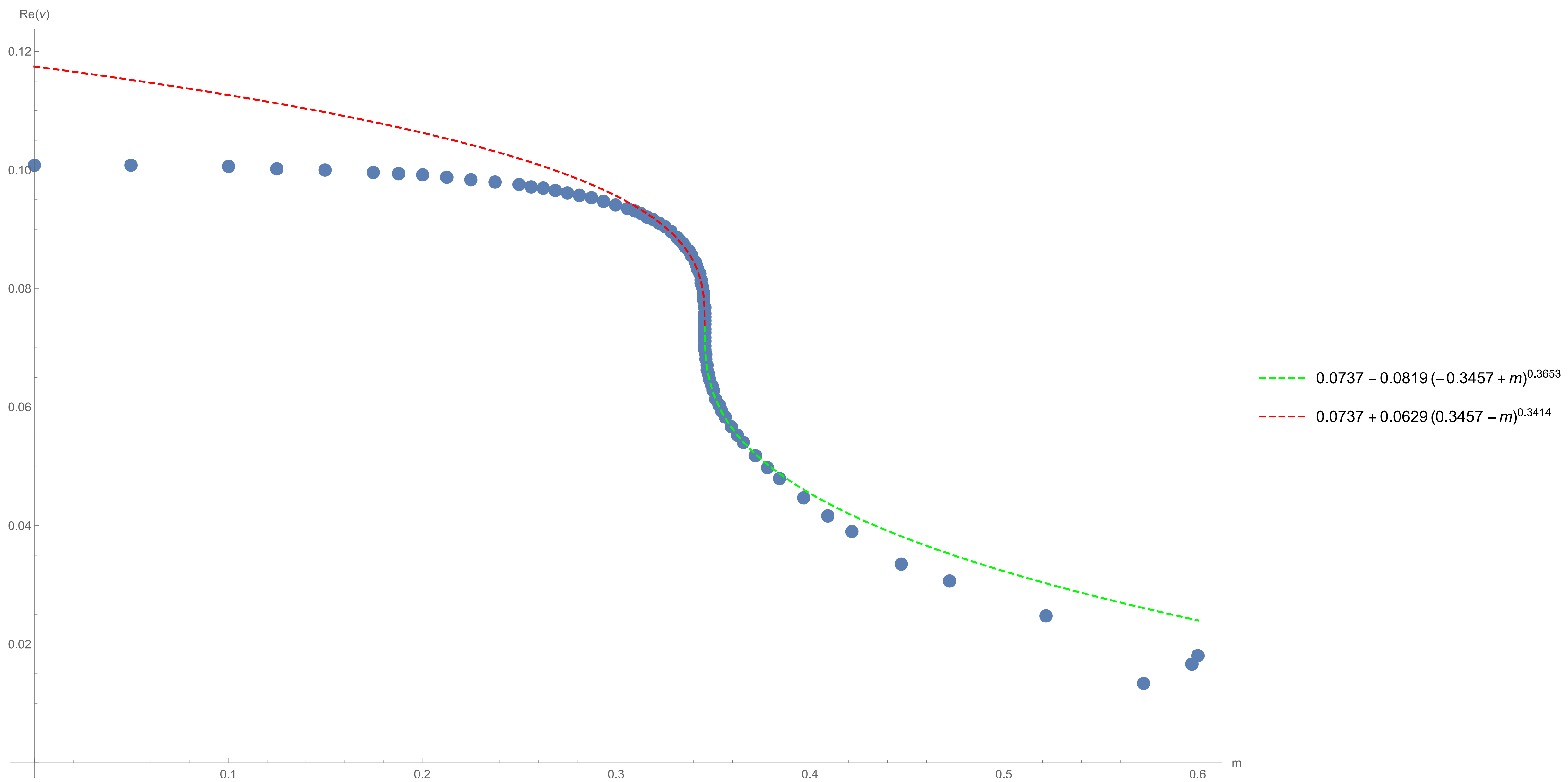


$$H_2 = ND\,\mathrm{tr}\Big(m\psi_\mu^\dagger\psi_\mu + \frac{1}{2}\sqrt{D}\big(\lambda\,\psi_\mu\psi_\nu\psi_\mu\psi_\nu^\dagger + \mathrm{H.c.}\big)\Big)$$

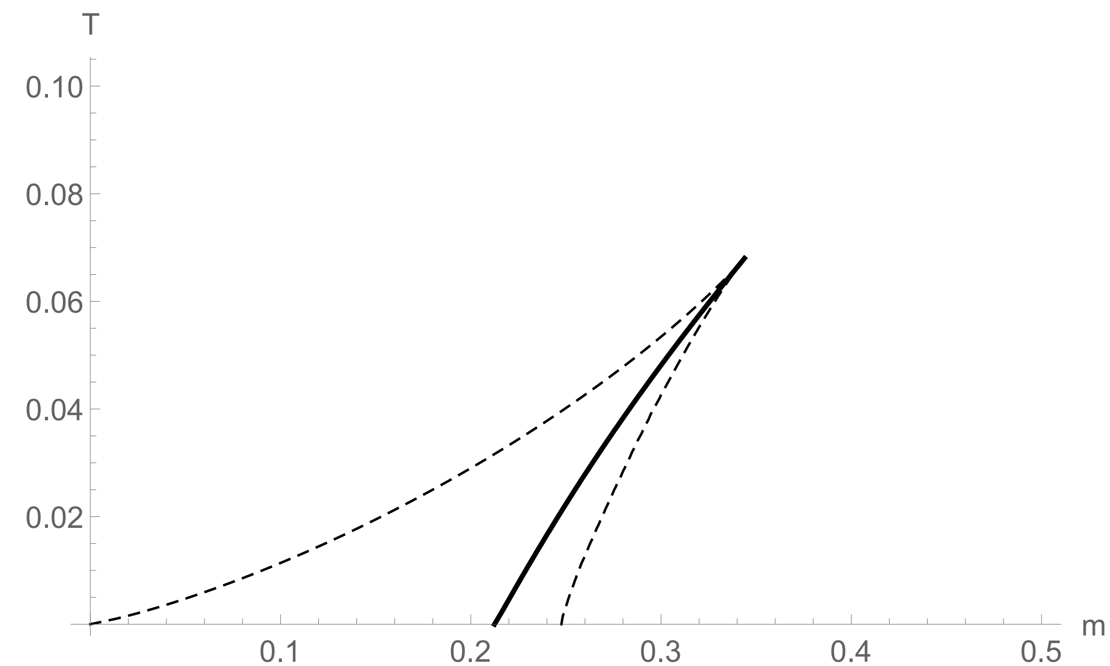


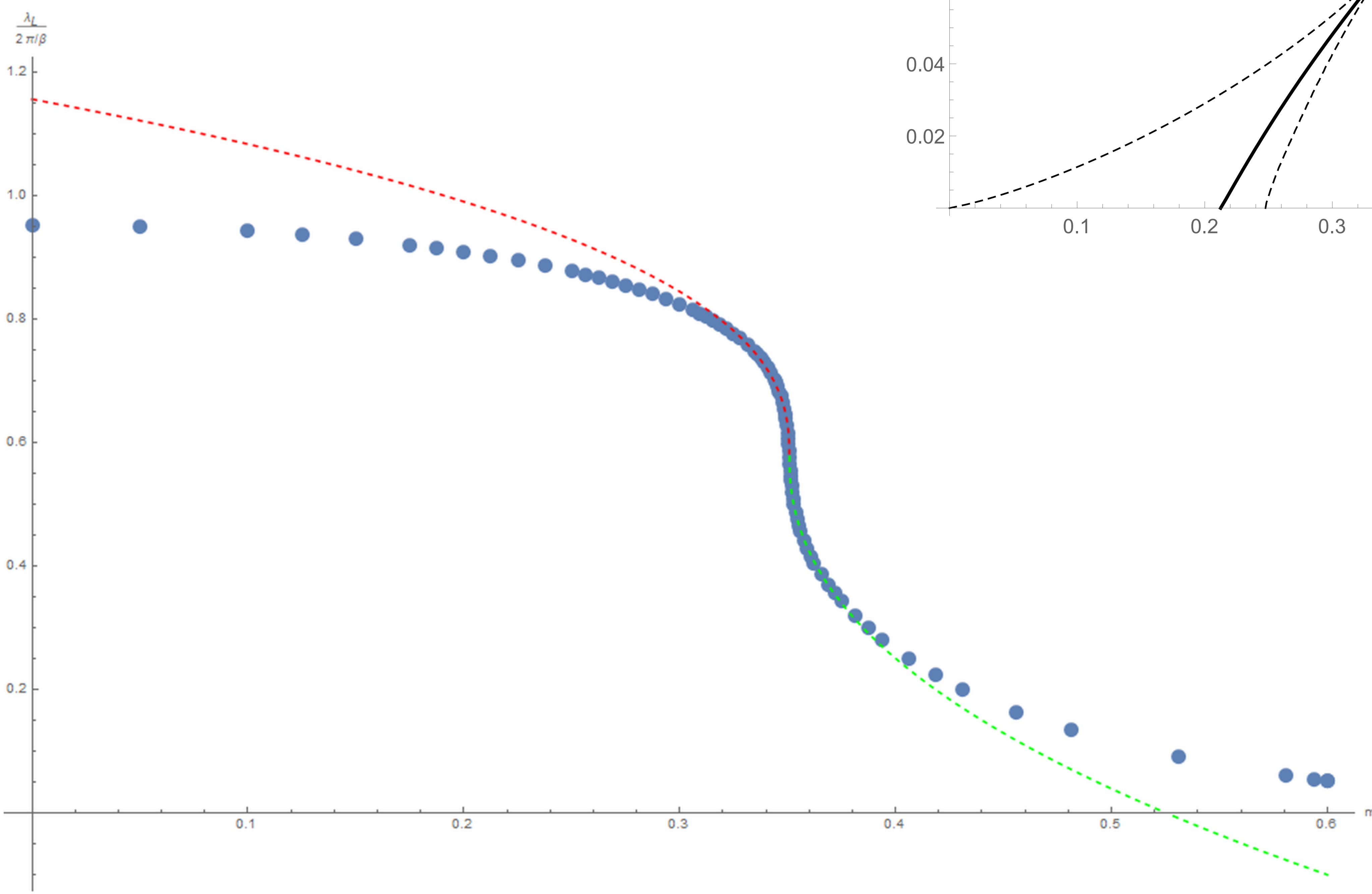


The quasi normal frequencies have a remarkable behaviour at the critical point

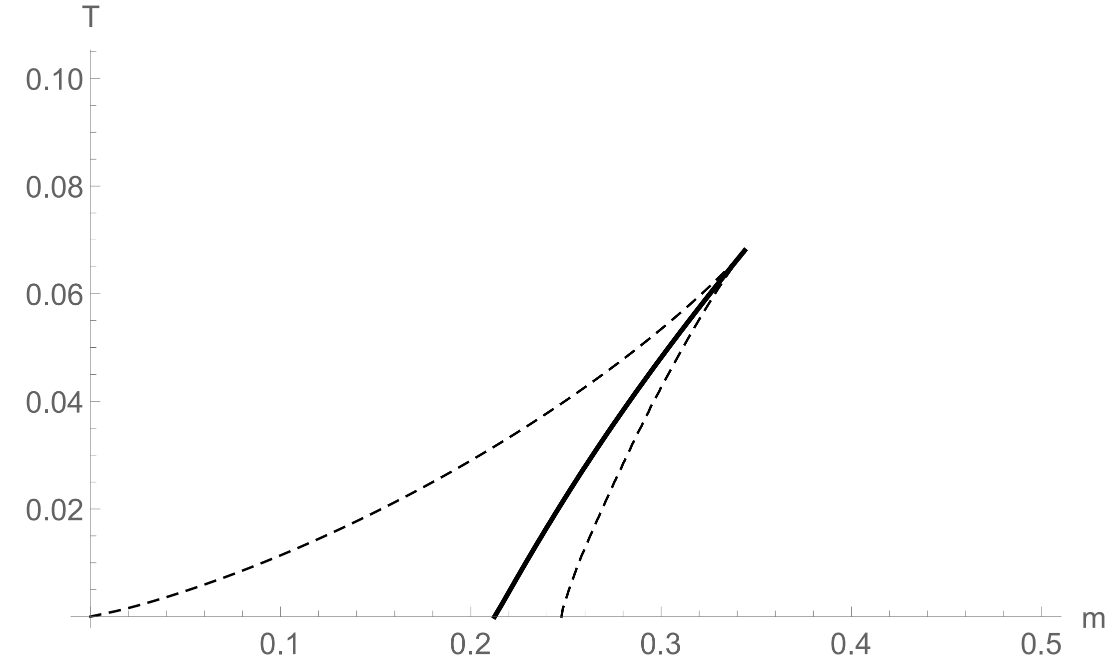


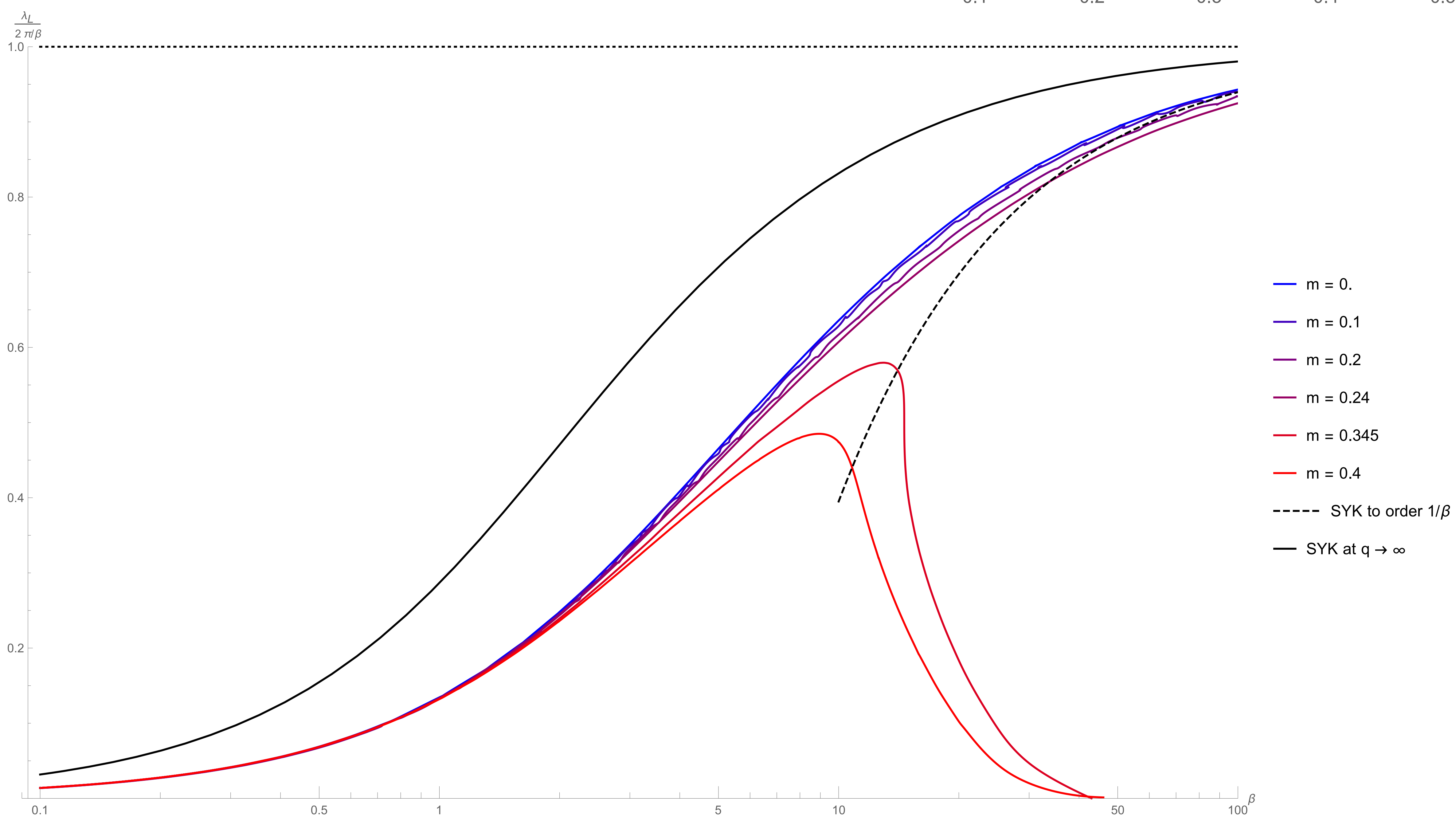
The Lyapunov exponents at the critical point



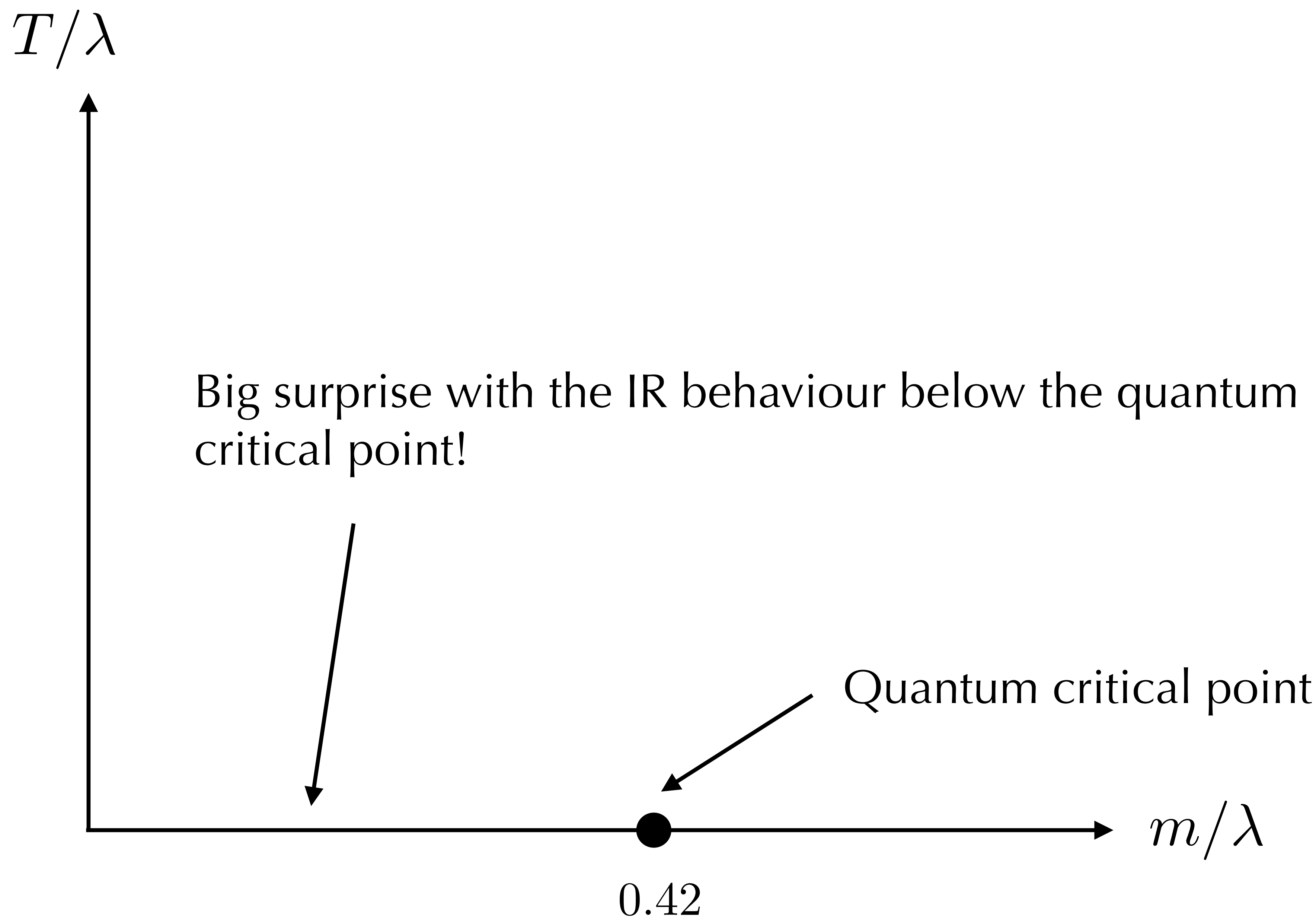


Lyapunov exponent for various masses





$$H_2 = N D\,\mathrm{tr}\Big(m\psi_\mu^\dagger\psi_\mu + \frac{1}{2}\sqrt{D}\big(\lambda\,\psi_\mu\psi_\nu\psi_\mu\psi_\nu^\dagger + \mathrm{H.c.}\big)\Big)$$



(r,s) model, r+s = q

$$\Sigma(t) = (-1)^r \lambda^2 \left[rG(t)^s G(-t)^{r-1} + sG(t)^r G(-t)^{s-1} \right]$$

Standard ansatz:

$$G(\tau) \sim \begin{cases} \dfrac{b_+}{|\tau|^{2\Delta}} & \text{when } \tau \to +\infty \\ -\dfrac{b_-}{|\tau|^{2\Delta}} & \text{when } \tau \to -\infty\,. \end{cases}$$

Works with $\Delta = \dfrac{1}{q}$

But the numerical solution shows that this is not realised even below the critical mass!

It is actually logically possible to consider a more general ansatz with asymmetric dimensions:

Required for consistency

$$G(\tau) \sim \begin{cases} \dfrac{b_+}{|\tau|^{2\Delta_+}} + \dfrac{c_+}{|\tau|^{2\Delta_-}} & \text{when } \tau \to +\infty \\ -\dfrac{c_-}{|\tau|^{2\Delta_-}} & \text{when } \tau \to -\infty \,, \end{cases}$$

with $\Delta_+ < \Delta_-$ (and a similar ansatz for $\Delta_+ > \Delta_-$).

This ansatz is possible because we are in one dimension. In dimension two or higher, it would be forbidden by rotational invariance.

A careful low energy analysis including subleading terms yields

$$(r+1)\Delta_+ + (s-1)\Delta_- = 1$$

$$\frac{\Gamma(2-2\Delta_+)\Gamma(1-2\Delta_+)}{\Gamma(1-2\Delta_-)\Gamma(2+2\Delta_- - 4\Delta_+)} = \frac{s}{r}\frac{\sin(2\pi\Delta_-)}{\sin(2\pi\Delta_+)}$$

This is the solution that is realized.

We have a spontaneous breaking of the conformal SL(2,R) invariance without the creation of a mass gap (since we still have a power-law behaviour!).

This is a totally new possibility in CFT1, much remains to be understood both from the QM side and from the dual holographic interpretation.

Numerically, we find that there is a macroscopic non-zero zero temperature entropy (like for black holes) but the Lyapunov exponent does not seem to saturate the bound.

The (r,s) models are even richer! The space of solutions of the Schwinger-Dyson equations turn out to have an extremely complex structure.
To discover it, one can study the following limit:

$$r = \frac{1+u}{2}q\,, \quad s = \frac{1-u}{2}q\,, \quad 0 \leq u < 1$$

$$q \to \infty$$

Parameterizing

$$\big(2G(t)\big)^q = e^{a(t)+b(t)}$$

where a is even and b is odd, the Schwinger-Dyson equations are then equivalent to a simple-looking, but actually quite non-trivial, non-linear system of coupled ODE:

$$\ddot{a} = 2\lambda^2 e^a \cosh(ub)$$

$$\ddot{b} = -2\lambda^2 u e^a \sinh(ub)$$

$$\ddot{a} = 2\lambda^2 e^a \cosh(ub)$$
$$\ddot{b} = -2\lambda^2 u e^a \sinh(ub)$$

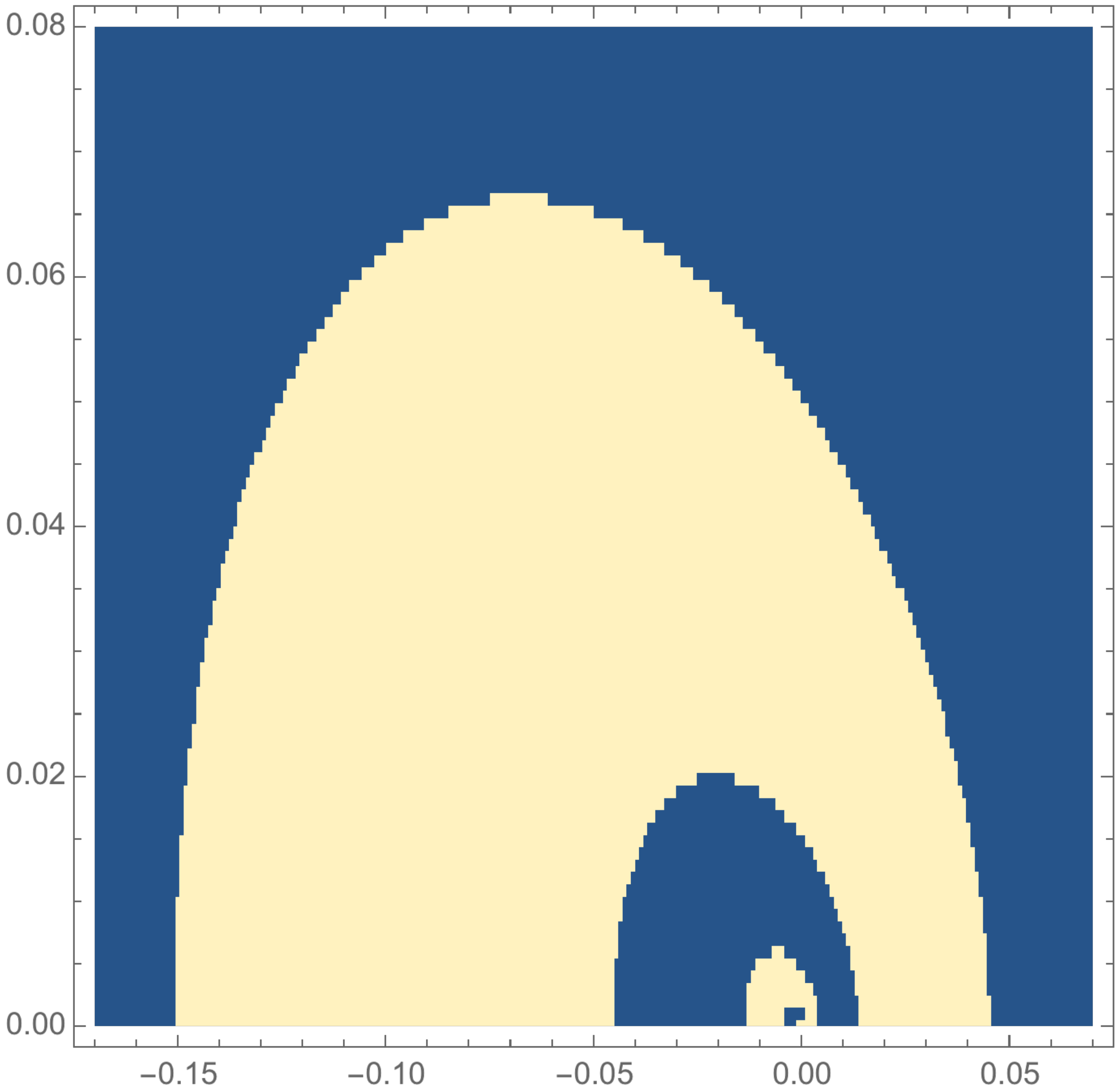
This system must be supplemented with the boundary conditions

$$a(t+\beta) = a(t)\,, \quad b(t+\beta) = b(t)\,, \quad a(0) = 0\,, \quad \dot{b}(0) = -m$$

A simple application is to find the solutions with asymmetric dimensions described above at finite q.

The most interesting aspect is that, because the equations are highly non-linear, one can find many solutions with given boundary conditions!

A typical "phase portrait"



I have teamed up with an expert on non-linear ODE and PDE to explore this highly non-trivial structure (Gregory Kozyreff).

Thank you for your attention!

And thank you for allowing me to make this phase transition (maybe not quantum, but certainly critical):

Brussels on a cold winter day

Okinawa on a nice summer day