

Bilocal effective action for tensor models

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Introduction

Recently, the melonic limit has been rediscovered in the context of *AdS/CFT*:

- In the SYK model and in models of tensor quantum mechanics, the melonic limit leads to interesting features for a holographic description of extremal black holes
- However, SYK and tensor models have important differences

This talk:

In the SYK model, a bilocal action formulation plays a key role, but an analog formulation was missing for its tensorial cousins

⇒ Introduce the two-particle irreducible (2PI) effective action for tensor models

Overview

- SYK model
- Tensor models in $d = 1$
- 2PI effective action

SYK model

The Sachdev-Ye-Kitaev (SYK) model

A model of N Majorana fermions in $d = 1$: [Sachdev, Ye - 1992; Kitaev - 2015]

$$\mathbf{S}_{\text{SYK}}[\psi] = \int dt \left(\frac{1}{2} \sum_{a=1}^N \psi_a \partial_t \psi_a + \frac{i^{q/2}}{q!} \sum_{a_1, \dots, a_q} J_{a_1 \dots a_q} \psi_{a_1} \dots \psi_{a_q} \right)$$

where $J_{a_1 \dots a_q}$ is a random tensorial coupling, with Gaussian distribution:

$$P[J_{a_1 \dots a_q}] \propto \exp \left\{ -\frac{N^{q-1} (J_{a_1 \dots a_q})^2}{2(q-1)! J^2} \right\}$$

Randomness \Rightarrow **quenched** average of intensive quantities,
e.g. the free energy:

$$\overline{F} = -\frac{1}{N} \overline{\ln Z} = -\frac{1}{N} \int \prod_{a_1 < a_2 < \dots < a_q} [dJ_{a_1 \dots a_q}] P[J_{a_1 \dots a_q}] \ln \int [d\psi] e^{-\mathbf{S}_{\text{SYK}}[\psi]}$$


Quenched average can be represented with new lines in connected fermionic graphs

At large- N , after the quenched average on $J_{a_1 \dots a_q}$, the leading diagrams are melonic

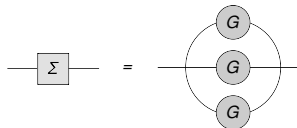
[SYK], [Bonzom, Lionni, Tanasa - 2017]

Conformal limit

Because of the melonic dominance, the large- N Schwinger-Dyson equations form a closed equation for the 2-point function, e.g. for $q = 4$:

$$G(\omega) = (-i\omega - \Sigma(\omega))^{-1}$$


$$\Sigma(\omega) = J^2 \int \frac{d\omega_1 d\omega_2}{(2\pi)^2} G(\omega_1) G(\omega_2) G(\omega - \omega_1 - \omega_2)$$



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- In the UV limit $\omega \rightarrow \infty$ the self energy $\Sigma(\omega)$ can be neglected: the theory is asymptotically free
- In the IR limit $\omega \rightarrow 0$ the free inverse propagator (“ $-i\omega$ ”) can be neglected and one obtains a conformal invariant solution:

$$G(\omega) \sim \omega^{2\Delta-1}, \quad \Delta = 1/q$$

Replica trick

- ④ A standard method for dealing with quenched average: replica trick, i.e.

$$\begin{aligned}\overline{\ln Z} &= \lim_{n \rightarrow 0} \partial_n \overline{Z^n} \\ &= \lim_{n \rightarrow 0} \partial_n \int \prod_{a_1 < a_2 < \dots < a_q} [dJ_{a_1 \dots a_q}] P[J_{a_1 \dots a_q}] \int \prod_{\alpha=1}^n [d\psi^\alpha] e^{-\mathbf{S}_{\text{SYK}}[\psi^\alpha]} \\ &= \lim_{n \rightarrow 0} \partial_n \int \left(\prod_{\alpha=1}^n [d\psi^\alpha] \right) e^{-\frac{1}{2} \sum_{\alpha} \int_t \psi_a^\alpha \partial_t \psi_a^\alpha + \frac{i^q J^2}{2qN^{q-1}} \sum_{\alpha, \beta} \int_t \psi_{a_1}^\alpha \dots \psi_{a_q}^\alpha \int_{t'} \psi_{a_1}^\beta \dots \psi_{a_q}^\beta}\end{aligned}$$

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- ② Introduce bilocal variables by a Faddeev-Popov type of trick: insert

$$\begin{aligned}1 &= \int \prod_{\alpha\beta}^{1\dots n} [dG^{\alpha\beta}] \delta \left(NG^{\alpha\beta}(t, t') - \sum_a \psi_a^\alpha(t) \psi_a^\beta(t') \right) \\ &= \int \prod_{\alpha\beta}^{1\dots n} [dG^{\alpha\beta}] [d\Sigma^{\alpha\beta}] e^{-\frac{1}{2} \int_{t, t'} \Sigma^{\alpha\beta}(t, t') \left(NG^{\alpha\beta}(t, t') - \sum_a \psi_a^\alpha(t) \psi_a^\beta(t') \right)}\end{aligned}$$

inside functional integral, use the constraint in the interaction part ($\sim G^q$), and integrate out the fermions (now Gaussian)

Bilocal effective action

$$\overline{Z}^n = \int \left(\prod_{\alpha\beta}^{1\dots n} [dG^{\alpha\beta}][d\Sigma^{\alpha\beta}] \right) e^{-N\mathbf{S}_{\text{eff}}[G,\Sigma]}$$

where

$$\mathbf{S}_{\text{eff}}[G, \Sigma] = -\frac{1}{2}\widehat{\text{Tr}} \ln(\partial_t - \Sigma) + \frac{1}{2} \sum_{\alpha\beta}^{1\dots n} \int_{t,t'} \left(\Sigma^{\alpha\beta}(t, t') G^{\alpha\beta}(t, t') - \frac{J^2}{q} (G^{\alpha\beta}(t, t'))^q \right)$$

- N -dependence factors in front of the action: large- N limit is given by saddle-point approximation
- For SYK there is no replica symmetry breaking at large N ($T_{\text{RSB}} \sim e^{-\sqrt{N}}$)
[Georges, Parcollet, Sachdev - 2001], hence we can use a replica-symmetric ansatz:

$$G^{\alpha\beta}(t, t') = \delta^{\alpha\beta} \underline{G}(t, t') + \frac{1}{N^{1/2}} g^{\alpha\beta}(t, t')$$

with $\underline{G}(t, t')$ solving the saddle point equation (= Schwinger-Dyson equation)

- At leading and subleading order in $\frac{1}{N}$, $n = 0$ coincides with $n = 1$ (quenched = annealed)

Leading order result:

$$\overline{\ln Z}_{LO} = N \left(\frac{1}{2} \text{Tr}[\ln \underline{G}^{-1}] + \frac{1}{2} \text{Tr}[\partial_t \underline{G}(t, t')] + \frac{J^2}{2q} \int_{t,t'} \underline{G}(t, t')^q \right)$$

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again conformal invariant in the IR

Bilocality and holography

The CFT_1 is expected to have an AdS_2 dual, and thus be useful to better understand extremal black holes

Bilocal action formulation is a useful tool in the construction of the holographic dual:
bilocality \sim extra dimensions [Das, Jevicki - '03]

Basic idea:

$$\phi(x, y) = \sigma(u, v) = \sum_{l, m} \sigma_{l, m}(u, r) Y_{l, m}(\theta)$$

where $x, y \in \mathbb{R}^d$, $\theta \in S^{d-1}$, and

$$u = \frac{1}{2}(x + y), \quad v = \frac{1}{2}(x - y), \quad r^2 = v \cdot v$$

and $Y_{l, m}(\theta)$ are the spherical harmonics on S^{d-1} .

$$\sigma_{l, m}(\vec{u}, r) \sim \text{higher spin fields in } d + 1 \text{ dimensions}$$

From the bilocal action to Jackiw-Teitelboim gravity

Bilocal action has also played an important role in identifying the gravitational part of the bulk dual: [Kitaev - '15; Maldacena, Stanford - '16]

- Fluctuations $g^{\alpha\beta}(t, t')$ around saddle point have an exact zero mode in the IR/conformal limit:

$$\overline{\ln \bar{Z}_{NLO}} = -\frac{1}{2} \text{Tr}[\ln(I_- - \tilde{\mathcal{K}}I_-)]$$

with the ladder kernel:

$$\tilde{\mathcal{K}}(t_1, t_2; t_3, t_4) = -J^2(q-1) |\underline{G}(t_1, t_2)|^{\frac{q-2}{2}} \underline{G}(t_1, t_3) \underline{G}(t_2, t_4) |\underline{G}(t_3, t_4)|^{\frac{q-2}{2}}$$

Zero mode: $\tilde{\mathcal{K}} = 1$ eigenmode

\Rightarrow non-conformal corrections lead to an effective action for the zero mode, a.k.a. the Schwarzian action

$$S_{\text{Sch}} = -N \frac{\alpha}{J} \int d\tau \left(\frac{f'''}{f'} - \frac{3}{2} \left(\frac{f''}{f'} \right)^2 \right)$$

Effective action for pseudo-Golstone modes of reparametrization invariance ($\tau \rightarrow f(\tau)$)

From the bilocal action to Jackiw-Teitelboim gravity

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- Jackiw-Teitelboim gravity

$$S_{JT} = -\frac{1}{16\pi G} \left[\int_M d^2x \sqrt{g} (R - 2\Lambda)\phi + 2 \int_{\partial M} dt \phi K \right]$$

has similar phenomenon: singularity at conformal AdS boundary

→ cut out near-boundary region

→ Schwarzian action from on-shell evaluation of the boundary action

⇒ Same pattern of symmetry breaking, controlled by the same effective action

Tensor models in $d = 1$

Tensor quantum mechanics

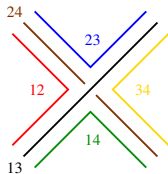
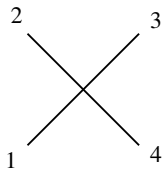
The same melonic limit, and hence conformal symmetry in the IR, can be obtained without disorder, but with q fermions in $(q-1)$ -fundamental representation of $O(N)^{q-1}$

[Witten - '16]

Gurau-Witten model: $d = 1$ real fermionic generalization of colored tensor model

$$\begin{aligned} \mathbf{S}_{\text{GW}}[\psi] &= \frac{1}{2} \sum_{c=1}^q \int_t \psi_{\mathbf{a}_c}^{(c)}(t) \partial_t \psi_{\mathbf{a}_c}^{(c)}(t) \\ &+ \frac{i^{q/2} \lambda}{N^{(q-1)(q-2)/4}} \int_t \prod_{c=1}^q \psi_{\mathbf{a}_c}^{(c)}(t) \prod_{c_1 < c_2} \delta_{a_{c_1}^{c_1} a_{c_2}^{c_2}} \end{aligned}$$

where $\mathbf{a}_c = (a_{c_1}^c | c_1 \in \{1, \dots, q\} \setminus \{c\})$



Quenched vs annealed models

Advantages of annealed tensor models:

- No quenched disorder, which is unnatural for *AdS/CFT*
- In SYK the $O(N)$ symmetry only emerges after quenching; in tensor models it is there from the beginning, so it can be gauged:
 - ⇒ Gauging gets rid of the non-singlet states, which are an obstacle in the search for the gravity dual
- Subleading corrections better understood (several years of results from tensor models and no tricky issues with replica limit)

Important differences:

- In non-gauged version, global symmetry becomes almost local in IR: new soft modes besides the Schwarzian mode [Minwalla et al. - '17]
- Many more invariants (singlets) in tensor models: complicated bulk dual
- No bilocal action formulation (...until last February)

2PI effective action

2PI effective action – general definition

[Cornwall, Jackiw, Tomboulis - '74]

Define the generating functional:

$$\mathbf{W}[j, k] = \ln \int [d\varphi] \exp \left\{ -\mathbf{S}[\varphi] + j_{\mathbf{a}}\varphi_{\mathbf{a}} + \frac{1}{2}\varphi_{\mathbf{a}}k_{\mathbf{ab}}\varphi_{\mathbf{b}} \right\}$$

The 2PI effective action is the double Legendre transform with respect to the sources:

$$\Gamma[\phi, G] = -\mathbf{W}[\mathbf{J}, \mathbf{K}] + \mathbf{J}_{\mathbf{a}}\phi_{\mathbf{a}} + \frac{1}{2}\phi_{\mathbf{a}}\mathbf{K}_{\mathbf{ab}}\phi_{\mathbf{b}} + \frac{1}{2}\text{Tr}[\mathbf{G}\mathbf{K}]$$

with \mathbf{J} , and \mathbf{K} such that:

$$\begin{aligned} \frac{\delta \mathbf{W}}{\delta j_{\mathbf{a}}}[\mathbf{J}, \mathbf{K}] &= \langle \varphi_{\mathbf{a}} \rangle_{\mathbf{J}, \mathbf{K}} = \phi_{\mathbf{a}} \\ \frac{\delta \mathbf{W}}{\delta k_{\mathbf{ab}}}[\mathbf{J}, \mathbf{K}] &= \frac{1}{2} \langle \varphi_{\mathbf{a}}\varphi_{\mathbf{b}} \rangle_{\mathbf{J}, \mathbf{K}} = \frac{1}{2} (G_{\mathbf{ab}} + \phi_{\mathbf{a}}\phi_{\mathbf{b}}) \end{aligned}$$

and

$$\begin{aligned} \frac{\delta \Gamma}{\delta \phi_{\mathbf{a}}}[\phi, G] &= \mathbf{J}_{\mathbf{a}}[\phi, G] + \mathbf{K}_{\mathbf{ab}}[\phi, G]\phi_{\mathbf{b}} \\ \frac{\delta \Gamma}{\delta G_{\mathbf{ab}}}[\phi, G] &= \frac{1}{2}\mathbf{K}_{\mathbf{ba}}[\phi, G] \end{aligned}$$

$$\mathbf{J} = \mathbf{K} = 0 \Rightarrow \underline{\phi} = \langle \varphi \rangle, \underline{G} = 2\langle \varphi\varphi \rangle_c$$

2PI effective action – loop expansion

[Cornwall, Jackiw, Tomboulis - '74]

It is not hard to prove that

$$\Gamma[\phi, G] = \underbrace{\mathbf{S}[\phi]}_{\text{tree level}} + \underbrace{\frac{1}{2}\text{Tr}[\ln G^{-1}] + \frac{1}{2}\text{Tr}[\mathbf{S}_{\phi\phi}[\phi]G]}_{\text{one loop}} + \underbrace{\Gamma_2[\phi, G]}_{\text{two or more loops}}$$

with the following Feynman rules:

vertices:	$\mathbf{S}_{\text{int}}[\phi, \varphi] = \mathbf{S}[\phi + \varphi]$ starting at cubic order in φ
propagator:	$G(x, y)$

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$\mathbf{\Gamma}_2[\phi, G]$ is given by the sum of all the ($n \geq 2$)-loops **two-particle irreducible** vacuum graphs

Hint:

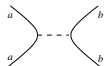
$$\frac{\delta \Gamma}{\delta G_{\mathbf{ab}}} = 0 \quad \Rightarrow \quad G^{-1} = \mathbf{S}_{\phi\phi}[\phi] + 2 \frac{\delta \mathbf{\Gamma}_2}{\delta G} = G_0^{-1} - \Sigma$$

Large- N limit of the 2PI effective action

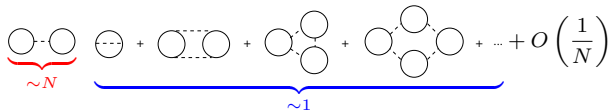
– vector $O(N)$ model –

$$\mathbf{S}[\varphi] = \frac{1}{2} \int_{x,y} \varphi_a(x) C^{-1}(x,y) \varphi_a(y) + \frac{\lambda}{4!N} \int_x (\varphi_a(x) \varphi_a(x))^2$$

vertex ($\phi = 0$) :



2PI vacuum graphs:



$$\Gamma_2[\phi, G] = N \frac{\lambda}{4!} \int_x G(x,x)^2 + \frac{1}{2} \text{Tr}[\ln(\mathbf{1} + \frac{\lambda}{6} G(x,y)^2)] + O\left(\frac{1}{N}\right)$$

Large- N limit of the 2PI effective action

– vector $O(N)$ model –

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Loop structure of $1/N$ expansion: from same trick as in SYK

$$\begin{aligned} Z &= \int [d\varphi] e^{-\mathbf{S}[\varphi]} \\ &= \int [d\varphi][d\tilde{G}][d\tilde{\Sigma}] e^{-\mathbf{S}[\varphi] - \frac{1}{2} \int_{x,y} \tilde{\Sigma}(x,y) (N\tilde{G}(x,y) - \sum_a \varphi_a(x) \varphi_a(y))} \\ &= \int [d\tilde{G}][d\tilde{\Sigma}] e^{-N \left\{ \frac{1}{2} \text{Tr}[(C^{-1} - \tilde{\Sigma})\tilde{G}] + \frac{1}{2} \text{Tr}[\ln(\tilde{\Sigma})] + \frac{\lambda}{4!} \int_x \tilde{G}(x,x)^2 \right\}} \\ &\equiv \int [d\tilde{G}][d\tilde{\Sigma}] e^{-N \mathbf{S}_{\text{eff}}[\tilde{G}, \tilde{\Sigma}]} \end{aligned}$$

Large- N limit of the 2PI effective action

– colored tensor model –

$$\mathbf{S}_{\text{GW}}[\psi] = \frac{1}{2} \sum_{c=1}^q \int_t \psi_{\mathbf{a}_c}^{(c)}(t) \partial_t \psi_{\mathbf{a}_c}^{(c)}(t) + \frac{i^{q/2} \lambda}{N^{(q-1)(q-2)/4}} \int_t \prod_{c=1}^q \psi_{\mathbf{a}_c}^{(c)}(t) \prod_{c_1 < c_2} \delta_{a_{c_1 c_2} a_{c_2 c_1}}$$

Introduce a bilocal source for each color and perform the double Legendre transform:

$$\begin{aligned} \Gamma[\Psi_{\mathbf{a}_c}^{(c)}(t), G_{\mathbf{a}_c \mathbf{b}_c}^{(c)}(t, t')] &= \mathbf{S}_{\text{GW}}[\Psi^{(c)}] + \frac{1}{2} \sum_{c=1}^q \text{Tr}[\ln(G^{(c)})] - \frac{1}{2} \sum_{c=1}^q \text{Tr}[(G_0^{(c)})^{-1} G^{(c)}] \\ &\quad + \Gamma_2[\Psi^{(c)}, G^{(c)}] \end{aligned}$$

with

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Large- N limit of the 2PI effective action

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In the symmetric phase

$$\underline{\Psi}_{\mathbf{a}_c}^{(c)}(t) = 0, \quad \underline{G}_{\mathbf{a}_c \mathbf{b}_c}^{(c)}(t, t') = G(t, t') \prod_{c' \neq c} \delta_{a_{c'} b_{c'}}^{c'}$$

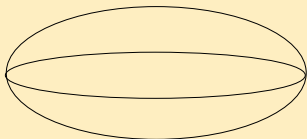
⇒ each trace on a color index counts as a factor N

Leading-order 2PI effective action for colored tensor model

Large N :

Counting traces = counting faces in stranded graph \Rightarrow **leading order is given by melons**

There is only one melon graph which is also 2PI: the fundamental melon



$$\begin{aligned}\Gamma_2^{\text{LO}}[\Psi^{(c)} = 0, G^{(c)} = G] &= -\frac{\lambda^2}{2N^{(q-1)(q-2)/2}} \int_{t,t'} \prod_{c=1}^q G_{\mathbf{a}\mathbf{c}\mathbf{b}\mathbf{c}}^{(c)}(t, t') \prod_{c_1 < c_2} \delta_{a_{c_2}^{c_1} a_{c_1}^{c_2}} \delta_{b_{c_2}^{c_1} b_{c_1}^{c_2}} \\ &= -\frac{\lambda^2 N^{q-1}}{2} \int_{t,t'} G(t, t')^q\end{aligned}$$

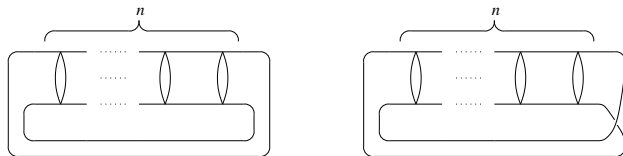
\Rightarrow Same result as in the SYK model

Next three subleading orders

Leading order is $\sim N^{q-1}$, while subleading orders start at $\sim N^2$:

$$\Gamma_2[G] = \Gamma_2^{(q-1)}[G] + \Gamma_2^{(2)}[G] + \Gamma_2^{(1)}[G] + \Gamma_2^{(0)}[G],$$

Subleading diagrams are an infinite family of ladders:



where colors and twist lead to different scalings with N : [Bonzom, Lionni, Tanasa - 2017]

- Order N^2 \Rightarrow rails: alternating colors without twist
- Order N \Rightarrow rails: alternating colors with twist
- Order 1 \Rightarrow rails: non-alternating colors with or without twist

Resummation of subleading contributions

Subleading diagrams can be resummed:

$$\begin{aligned}\Gamma_2^{(2)}[G] &= N^2 \frac{1}{4} \binom{q}{2} \text{Tr} \left[I^= \ln \left(1 - \lambda^4 \hat{\mathcal{K}}^2 \right) \right] \\ \Gamma_2^{(1)}[G] &= N \frac{1}{4} \binom{q}{2} \text{Tr} \left[(-I^\times) \ln \left(1 - \lambda^4 \hat{\mathcal{K}}^2 \right) \right] \\ \Gamma_2^{(0)}[G] &= \frac{1}{2} \text{Tr} \left[I_- \ln \left(1 - (q-1) \lambda^2 \hat{\mathcal{K}} \right) \right] \\ &\quad + \frac{q-1}{2} \text{Tr} \left[I_- \ln \left(1 + \lambda^2 \hat{\mathcal{K}} \right) \right] - \frac{1}{2} \binom{q}{2} \text{Tr} \left[I_- \ln \left(1 - \lambda^4 \hat{\mathcal{K}}^2 \right) \right]\end{aligned}$$

where $\hat{\mathcal{K}}$ is a ladder kernel, and $I_\pm = (I^= \pm I^\times)/2$ are projectors on (anti-)symmetric bilocal functions

Free energy

Putting all together we have the free energy (for $q > 4$):

$$\begin{aligned} -\ln Z = \Gamma[0, \underline{G}] = & N^{q-1} \frac{q}{2} \text{Tr}[\ln(\underline{G}^{(0)})] - N^{q-1} \frac{q}{2} \text{Tr}[\partial_t \underline{G}^{(0)}] - N^{q-1} \frac{\lambda^2}{2} \int_{t,t'} \underline{G}^{(0)}(t, t')^q \\ & + \left[\frac{N(N-1)}{2} \binom{q}{2} \right] \frac{1}{2} \text{Tr} \left[\ln \left(1 - \lambda^4 [\hat{\underline{K}}^{(0)}]^2 I_+ \right) \right] \\ & + \left[\left(\frac{N(N-1)}{2} + (N-1) \right) \binom{q}{2} \right] \frac{1}{2} \text{Tr} \left[\ln \left(1 - \lambda^4 [\hat{\underline{K}}^{(0)}]^2 I_- \right) \right] \\ & + (q-1) \frac{1}{2} \text{Tr} \left[\ln \left(1 + \lambda^2 [\hat{\underline{K}}^{(0)}] I_- \right) \right] + \frac{1}{2} \text{Tr} \left[\ln \left(1 - (q-1) \lambda^2 [\hat{\underline{K}}^{(0)}] I_- \right) \right] \end{aligned}$$

where $\underline{G}^{(0)}$ and $\hat{\underline{K}}^{(0)}$ are the leading-order on-shell two-point function and its ladder kernel (subleading corrections to $\underline{G}^{(0)}$ contribute the free energy at order N^{5-q})

We have rearranged subleading contributions to highlight a peculiar structure...

Interpretation in terms of an auxiliary theory

The trace-log terms can be interpreted as the one loop correction to the leading-order action

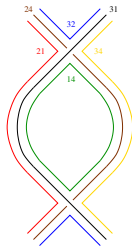
$$N^{q-1} \frac{q}{2} \text{Tr}[\ln(G)] - N^{q-1} \frac{q}{2} \text{Tr}[\partial_t G] - N^{q-1} \frac{\lambda^2}{2} \int_{t,t'} G(t,t')^q$$

Hint:

Expanding around the saddle point

$$G_{\mathbf{a}_c \mathbf{b}_c}^{(c)}(t, t') = \underline{G}^{(0)}(t, t') \delta_{\mathbf{a}_c \mathbf{b}_c} + g_{\mathbf{a}_c \mathbf{b}_c}^{(c)}(t, t')$$

the one loop correction gives the $\det(\mathbb{I} - \lambda^2 \mathbb{K})^{-1/2}$, where the operator \mathbb{K} is a matrix in color space built out of kernels $\mathcal{K}^{(c_1 c_2)}$, such as



Interpretation in terms of an auxiliary theory

The trace-log terms can be interpreted as the one loop correction to the leading-order action

$$N^{q-1} \frac{q}{2} \text{Tr}[\ln(G)] - N^{q-1} \frac{q}{2} \text{Tr}[\partial_t G] - N^{q-1} \frac{\lambda^2}{2} \int_{t,t'} G(t,t')^q$$

⇒ fluctuations decompose into symmetric traceless matrices, antisymmetric matrices and scalars:

$$g_{\mathbf{a}_c \mathbf{b}_c}^{(c)}(t, t') = g^{(c)}(t, t') \prod_{i \neq c} \delta_{a_i^c b_i^c} + \sum_{i \neq c} g_{a_i^c b_i^c}^{(ci)}(t, t') \prod_{j \neq i, c} \delta_{a_j^c b_j^c} + \hat{g}_{\mathbf{a}_c \mathbf{b}_c}^{(c)}(t, t')$$

and $g_{a_{ci} b_{ci}}^{(ci)}(t, t')$ decomposed in symmetric traceless and antisymmetric parts

Taking into account the matrix structure of \mathbb{K} , we find exactly the free energy above

Conclusions and outlook

Conclusions and outlook

- The SYK model has brought the melonic limit of tensor models under the spotlight
- Tensor models offer a number of advantages over the SYK model (in particular because of no disorder)
- They also present some differences and extra challenges (new light modes, many more invariants, ...)
- We advocated the use of the 2PI formalism to bypass the lack of a bilocal reformulation of the path integral, showing that for SYK it reproduces the bilocal effective action
- Large- N of colored tensor model: only one 2PI diagram at leading order; infinite but summable families of 2PI diagrams at first three subleading orders
- Surprisingly, the $1/N$ expansion of the 2PI effective action of the colored tensor model suggests the existence of an effective bilocal reformulation, at least up to order N^0
- Many open directions and questions, in particular concerning the holographic interpretation of tensor models