SOME PROPERTIES OF THE MULTIPLICITY AND OF BLOW UPS AT EQUIMULTIPLE CENTERS (III).

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Proof of Proposition 0.1 and Theorem 0.2 in the general case.

Proposition 0.1. *Fix a morphism $\delta : X \to V$ finite of generic rank n, defined by an extension $S \subset B$ in the class $\mathscr{T}$. For any sequence*

$$\begin{array}{ccccccc} X = X_0 & \xleftarrow{\pi_1} & X_1 & \xleftarrow{\pi_2} & \dots & \xleftarrow{\pi_r} & X_r \\ F_n(X_0) & & F_n(X_1) & & \dots & & F_n(X_r) \end{array} \tag{0.1.1}$$

where $X_i \leftarrow X_{i+1}$ is the blow-up at a regular center $Y_i \subset F_n(X_i)$, one has that

$$n = n(X) = n(X_1) = \cdots = n(X_{r-1}) \geq n(X_r)$$

and (0.1.1) induces

$$\begin{array}{ccccccccc} X & \longleftarrow & X_1 & \longleftarrow & \cdots & \longleftarrow & X_{r-1} & \longleftarrow & X_r \\ \downarrow \delta & & \downarrow \delta_1 & & & & \downarrow \delta_{r-1} & & \downarrow \delta_r \\ V & \longleftarrow & V_1 & \longleftarrow & \cdots & \longleftarrow & V_{r-1} & \longleftarrow & V_r, \end{array} \tag{0.1.2}$$

where all vertical morphisms are finite, and

$$F_n(X_i) \cong \delta_i(F_n(X_i)) \text{ for } i = 1, \dots, r. \tag{0.1.3}$$

Theorem 0.2. *Let $\delta : X \to V$ be a finite and dominant morphism, say of generic rank n, between affine varieties field of characteristic $p \geq 0$, where V is regular. If p does not divide n, one can attach to $\delta : X \to V$ a pair (K, b) on the regular variety V so that*

1. $\operatorname{Sing}(K, b) = \delta(F_n(X))$.
2. *For any sequence (0.1.1), the lower row of (0.1.2) induces*

$$\begin{array}{ccccccc} V = V_0 & \xleftarrow{\pi_1} & V_1 & \xleftarrow{\pi_2} & \dots & \xleftarrow{\pi_r} & V_r \\ (K, b) & & (K_1, b) & & \dots & & (K_n, b) \end{array} \tag{0.2.1}$$

$$\operatorname{Sing}(K_i, b) = \delta_i(F_n(X_i)) \subset V_i \text{ for } i = 1, \dots, r+1. \tag{0.2.2}$$

3. *Conversely, any sequence (0.2.1) induces a sequence (0.1.2) and the equalities in (0.2.2) hold.*

Corollary 0.3. *A resolution of (K, b) over V defines a reduction of the multiplicity of X.*

The case $\delta : X \to V$ defined by $S \subset B = S[Z]/\langle Z^n + a_1 Z^{n-1} + \cdots + a_n \rangle$, and the Tchirnhausen pair of a monic polynomial.

Set $Z_1 = Z - \lambda$, $\lambda \in S$ then $Z^n + a_1 Z^{n-1} + \cdots + a_n = Z_1^n + b_1 Z_1^{n-1} + \ldots + b_{n-1} Z_1 + b_n$

$$G(a_1, \dots, a_n) = G(b_1, \dots, b_n)$$

$$\delta(F_n(X)) = \bigcap_{1 \leq j \leq r} \{x \in \operatorname{Spec}(S) : \nu_x(G_{m_j}(a_1, \dots, a_n)) \geq m_j\}, \tag{0.3.1}$$

$$S[G_{m_1}(a_1,\dots,a_n)W^{m_1},\dots,G_{m_r}(a_1,\dots,a_n)W^{m_r}](\subset S[W])$$
$$Sing(G)=\delta(F_n(X)).$$

If b is a multiple of m_i of all $i=1,2,\dots,r$, then

$$K=\langle G_{m_i}^{\frac{b}{m_i}}(a_1,\dots,a_n), i=1,\dots,r\rangle$$

$$\delta(F_n(X))=Sing(K,b)$$

Proposition 0.4. *Fix $S\subset B$ in the class $\mathscr{T}$, and let K denote the fraction field of S. Given $\theta\in B$ $(\subset B\otimes_S K)$, let $f(Z)\in K[Z]$ denote its minimal polynomial, that is, the monic polynomial of lowest degree in $K[Z]$ such that $f(\theta)=0$ $(\in B\otimes_S K)$. Then $f(Z)\in S[Z]$ and the evaluation map $S[Z]\to S[\theta](\subset B\otimes_S K)$ induces an isomorphism $S[\theta]\cong S[Z]/\langle f(Z)\rangle$.*

Proof. $f(Z)\in S[Z]$ follows essentially from the fact that S is normal. Set

$$0\to J\to S[Z]\to S[\theta]\to 0$$

where J is the kernel of the morphism $S[Z]\to S[\theta]$ given by evaluation at θ. Its kernel includes $\langle f(Z)\rangle$. If $g(Z)$ is any other element in the kernel, the division algorithm in $S[Z]$ shows that $f(Z)$ divides $g(Z)$ in $S[Z]$, that is, $g(Z)\in\langle f\rangle$. Thus, $S[\theta]=S[Z]/\langle f(Z)\rangle$. ⥀

Recall Zariski's criterion: $S\subset B$ generic rank n. Fix P prime in B, and $\mathfrak{p}=P\cap S$. The following are equivalent:

a) $P\in F_n(B)$.

b)

- b1) P is unique dominating $\mathfrak{p}$.
- b2) $S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}}=B_P/PB_P$
- b3) $\mathfrak{p}B_P\subset PB_P$ is integral.

Given $f(Z)$ monic of degree n, $S\subset B=S[Z]/\langle f(Z)\rangle$ $\rho: S[Z]\to B\to 0$ and fix $P\in F_n(B)$, after localizing S at $\mathfrak{p}$:

$$\rho^{-1}(P)=\langle Z-\lambda,\mathfrak{p}S[Z]\rangle$$

in $S[Z]$. Moreover, setting $T=Z-\lambda$, then

$$f(Z)=g(T)=T^n+b_1T^{n-1}+\ldots+b_{n-1}T+b_n$$

and $\nu_{\mathfrak{p}}(b_i)\geq i$.

In particular

1. P is the ideal $\langle\mathfrak{p}B,\Theta-\lambda\rangle$ in B, and $\Theta-\lambda$ is in the integral closure of $\mathfrak{p}B$.
2. $k(\mathfrak{p})\otimes_S B=k(\mathfrak{p})[Z]/\langle Z-\overline{\lambda}\rangle^n$.

Proposition 0.5. *Let $S\subset B=S[\Theta_1,\dots,\Theta_m]$ be a morphism in the class $\mathscr{T}$, of generic rank n. Consider intermediate subrings $S\subset B_i=S[\Theta_i](\subset B)$, $i=1,\dots,m$, of generic rank d_i, such that $B=B_1B_2\dots B_m$, and let $\delta:\operatorname{Spec}(B)\to\operatorname{Spec}(S)$ and $\delta_i:\operatorname{Spec}(B_i)\to\operatorname{Spec}(S)$ denote the respective induced morphisms. Then*

$$\delta(F_n(B))=\bigcap_{i=1}^m\delta_i(F_{d_i}(B_i)).$$

In particular, if δ is transversal, each δ_i is also transversal.

Proof. 1) $\delta(F_n(B)) \subset \bigcap_{i=1}^m \delta_i(F_{d_i}(B_i))$.

2) $\delta(F_n(B)) \supset \bigcap_{i=1}^m \delta_i(F_{d_i}(B_i))$.
We analyze $C := B_1 \otimes \cdots \otimes_S B_m$ over a prime $p \in \bigcap_{i=1}^m \delta_i(F_{d_i}(B_i))$.

i) Set $k(p) = S_p/pS_p$

$$C \otimes_S k(p) = (B_1 \otimes_S B_2 \otimes \cdots \otimes_S B_m) \otimes_S k(p) = (B_1 \otimes_S k(p)) \otimes (B_2 \otimes_S k(p)) \cdots (B_m \otimes_S k(p)) =$$
$$= k(\mathfrak{p})[Z_1]/\langle Z_1 - \overline{\lambda}_1\rangle^{d_1} \otimes \cdots \otimes k(\mathfrak{p})[Z_m]/\langle Z_m - \overline{\lambda}_m\rangle^n = k(\mathfrak{p})[Z_1, \dots, Z_m]/\langle (Z_1 - \overline{\lambda}_1)^{d_1}, \dots, (Z_m - \overline{\lambda}_m)^{d_m}\rangle$$

($k(p)$ rational local ring).

ii) Let P be the unique prime in C over p. There is $S[Z_1, \dots, Z_M] \to C \to 0$ and P is in correspondence with $\langle \mathfrak{p}, Z_1 - \lambda_1, \dots, Z_M - \lambda_M\rangle$, so

$$P = \langle \mathfrak{p}C, \Theta_1 - \lambda_1, \dots, \Theta_M - \lambda_M\rangle \text{ and } \Theta_i - \lambda_i \text{ integral over } \mathfrak{p}C.$$

Setting $T_i = Z_i - \lambda_i$, then

$$f_i(Z_i) = g_i(T_i) = T_i^{d_i} + b_1 T_i^{d_i - 1} + \ldots + b_{d_i - 1} T_i + b_{d_i}$$

and $\nu_{\mathfrak{p}}(b_i) \geq i$.

iii) Finally as $C \to B$ is surjective: there is a unique prime Q of B over p, and Q is rational and integral over pB.

$\circlearrowleft$

0.6. Proof of Prop 0.1.

Fix an extension $S \subset B$ in the class $\mathscr{T}$, of generic rank n, and an epimorphism of S-algebras

$$\rho : S[Z_1, \dots, Z_m] \to B \to 0$$

where m is an integer and $Z_1, \dots, Z_m$ are variables over S.

Set $\theta_i := \rho(Z_i)$ so

$$S \subset B = S[\theta_1, \dots, \theta_m],$$

and $B_i := S[\theta_i] \subset B$ and

Let

$$f_i(Z_i) \in S[Z_i]$$

be the minimal polynomial of θ_i so $B_i \cong S[Z_i]/\langle f_i(Z_i)\rangle$ (Proposition 0.4).

Proposition 0.7. *(Recall) Fix $S \subset B$ in the class $\mathscr{T}$, of generic rank n. If $\delta : \operatorname{Spec}(B) \to \operatorname{Spec}(S)$ is transversal, then:*

1. *An irreducible scheme $Y \subset F_n(B)$ is regular if and only if $\delta(Y)$ is regular.*
2. *I the previous setting the induced finite morphism $\delta : Y \to \delta(Y)$ is an isomorphism of affine schemes.*

Proposition 0.8. *Set $S \subset B = S[\theta_1, \dots, \theta_m]$, and $\rho : S[Z_1, \dots, Z_m] \to B \to 0$ as before,*

$$f_i(Z_i) \in S[Z_i]$$

be the minimal polynomial of Θ_i. Let Q be a prime ideal of B such that $Q \in F_n(B)$ and B/Q is regular, and set $\mathfrak{p} = Q \cap S$. Then there are elements $\lambda_1, \dots, \lambda_m \in S$ such that the following hold.

1. *$\rho^{-1}(Q) = \langle \mathfrak{p}S[Z_1, \dots, Z_m], Z_1 - \lambda_1, \dots, Z_m - \lambda_m\rangle$ and $Q = \langle \mathfrak{p}B, \theta_1 - \lambda_1, \dots, \theta_m - \lambda_M\rangle$.*
2. *The expression of $f_i(Z_i)$ at $T_i = Z_i - \lambda_i$ is of the form*

$$f_i(Z_i) = h_i(T_i) = T_i^{d_i} + a_1^{(i)} T_i^{d_i - 1} + \ldots + a_{d_i - 1}^{(i)} T_i + a_{d_i}^{(i)}, \quad a_j^{(i)} \in \mathfrak{p}^j,$$

where $T_i = Z_i - \lambda_i$, and $h_i(T_i) \in S[T_i]$ is the minimal polynomial of $\theta_i - \lambda_i$.

Proof. (1) Follows from $\rho : S[Z_1, \dots, Z_m] \to B \to 0$. Note that $\rho^{-1}(Q)$ in (1) is a regular prime.

(2) Given $Q \in F_n(B)$ as above, we set $Q_i := Q \cap B_i$.
By Prop 0.4 $S[\theta_i] = S[Z]/\langle f_i(Z)\rangle$, $f_i(Z)$ monic of degree, say d_i.
Applying Prop 0.5: $Q_i \in F_{d_i}(S[\theta_i])$, and therefore, there is $\lambda_i \in S$ so that setting $T_i = Z_i - \lambda_i$

$$f_i(Z_i) = h_i(T_i) = T_i^{d_i} + a_1^{(i)} T_i^{d_i - 1} + \dots + a_{d_i-1}^{(i)} T_i + a_{d_i}^{(i)}, \quad a_j^{(i)} \in \mathfrak{p}^j,$$

where $T_i = Z_i - \lambda_i$, and $h_i(T_i) \in S[T_i]$ is the minimal polynomial of $\theta_i - \lambda_i$. ⥀

Corollary 0.9. *$S[\mathfrak{p}W] \subset B[QW]$ is a finite extension of Rees rings, in fact:*

$$S[\mathfrak{p}W] \subset B[\mathfrak{p}BW] \subset B[\mathfrak{p}BW, (\theta_1 - \lambda_1)W, \dots, (\theta_M - \lambda_M)W]$$

are all finite. In particular there is a square diagram of blow ups:

(0.9.1)
$$\begin{array}{ccc} Spec(B) & \longleftarrow & X_1 \\ {\scriptstyle \delta}\downarrow & & \downarrow{\scriptstyle \delta_1} \\ Spec(S) & \longleftarrow & V_1 \end{array}$$

Point (1) of the last Prop implies that $Q = \langle \mathfrak{p}B, \theta_1 - \lambda_1, \dots, \theta_m - \lambda_m\rangle$, and each $\theta_i - \lambda_i$ satisfies the equation $h_i(\theta_i - \lambda_i) = 0$, which is an equation of integral dependence over $\mathfrak{p}B$. Thus, Q is integral over $\mathfrak{p}B$.

Corollary 0.10. *The setting and notation being as in Proposition 0.8, let $x_1, \dots, x_r$ be generators for $\mathfrak{q}$ in S, and set*

$$S_t := S\left[\frac{x_1}{x_t}, \dots, \frac{x_r}{x_t}\right] \subset K,$$

where K is the fraction field of S, so that if $\operatorname{Spec}(S) \overset{\pi'}{\longleftarrow} V_1$ *is the blow-up of* $\operatorname{Spec}(S)$ *at $V(\mathfrak{p})$. V_1 is covered by the affine charts* $\operatorname{Spec}(S_t)$, $t = 1, \dots, r$.

1. *If* $\operatorname{Spec}(B) \overset{\pi}{\longleftarrow} X_1$ *is the blow-up of* $\operatorname{Spec}(B)$ *at $V(Q)$, then X_1 is obtained by glueing in the natural way the spectrum of the following subrings of $B \otimes_S K$:*
$$S_t\left[\frac{\theta_1 - \lambda_1}{x_t}, \dots, \frac{\theta_m - \lambda_m}{x_t}\right] \subset B_{x_t} \subset B \otimes_S K.$$
2. *Each extension $S_t \subset S_t\left[\frac{\theta_1-\lambda_1}{x_t}, \dots, \frac{\theta_m - \lambda_m}{x_t}\right]$ belongs to the class $\mathscr{T}$, and the minimal polynomial of $\frac{\theta_i - \lambda_i}{x_t}$ is*
$$h_i^{(t)}(\frac{T_i}{x_t}) := (\frac{T_i}{x_t})^{d_i} + \frac{a_1^{(i)}}{x_t}(\frac{T_i}{x_t})^{d_i-1} + \dots + \frac{a_{d_i-1}^{(i)}}{x_t^{d_i-1}}\frac{T_i}{x_t} + \frac{a_{d_i}^{(i)}}{x_t^{d_i}} \in S_t[\frac{T_i}{x_t}].$$
3. *The morphisms of affine schemes induced by the extensions $S_t \subset S_t\left[\frac{\theta_1-\lambda_1}{x_t}, \dots, \frac{\theta_m-\lambda_m}{x_t}\right]$ glue together into a finite and dominant morphism $\delta_1 : X_1 \to V_1$, and the following diagram is commutative. Moreover, δ_1 is the unique morphism from X_1 to V_1 making the diagram*

(0.10.1)
$$\begin{array}{ccc} Spec(B) & \longleftarrow & X_1 \\ {\scriptstyle \delta}\downarrow & & \downarrow{\scriptstyle \delta_1} \\ Spec(S) & \longleftarrow & V_1 \end{array}$$

commutative.

(4) *For each t, the ring $S_t[\frac{Z_1-\lambda_1}{x_t},\dots,\frac{Z_m-\lambda_m}{x_t}]\subset K[Z_1,\dots,Z_m]$ is a polynomial ring and the K-epimorphism $K[Z_1,\dots,Z_m]\to B\otimes_S K$ given by evaluating Z_i at θ_i induces an S_t-epimorphism*
$$\rho_t : S_t\left[\frac{Z_1-\lambda_1}{x_t},\dots,\frac{Z_m-\lambda_m}{x_t}\right]\to S_t\left[\frac{\theta_1-\lambda_1}{x_t},\dots,\frac{\theta_m-\lambda_m}{x_t}\right].$$

Proof of Theorem 0.2.

Proposition 0.11. *(Recall) Let $S\subset B$ be a morphism in the class $\mathscr{T}$ of generic rank n. Consider intermediate subrings $S\subset B_i(\subset B)$ of generic rank d_i, $i=1,\dots,m$, such that $B=B_1B_2\dots B_m$, and let $\delta:\mathrm{Spec}(B)\to\mathrm{Spec}(S)$ and $\delta_i:\mathrm{Spec}(B_i)\to\mathrm{Spec}(S)$ denote the respective induced morphisms. Then*
$$\delta(F_n(B))=\bigcap_{i=1}^{m}\delta_i(F_{d_i}(B_i)).$$
In particular, if δ is transversal, each δ_i is also transversal.

Set $B=S[\Theta_1,\dots,\Theta_m]$. For each $1\le j\le m$, set $S\subset B_j=S[\Theta_j]$ of generic rank d_j. There is a pair, say $(K^{(j)},b_j)$ over $Spec(S)$ representing the image of points of multiplicity d_i.
$$\delta(F_n(B))=\bigcap_i\delta_i(F_{d_i}(S[\Theta_i]))=\bigcap Sing(K^{(i)},b_i)$$

Given

(0.11.1)
$$\begin{array}{ccc} Spec(B) & \longleftarrow & X_1\\ \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta_1}\\ Spec(S) & \longleftarrow & V_1\end{array}$$
$$\delta(F_n(X_1))=\bigcap Sing(K_1^{(i)},b_i)$$

0.12. Given $\delta:X\to V$ by $S\subset B$ be a morphism in the class $\mathscr{T}$, of generic rank n, for any sequence

(0.12.1)
$$\begin{array}{ccccccc} X=X_0 & \xleftarrow{\pi_1} & X_1 & \xleftarrow{\pi_2} & \dots & \xleftarrow{\pi_r} & X_r\\ F_n(X_0) & & F_n(X_1) & & \dots & & F_n(X_r)\end{array}$$

where $X_i\leftarrow X_{i+1}$ is the blow-up at a regular center $Y_i\subset F_n(X_i)$, one has $n=n(X)=n(X_1)=\dots=n(X_{r-1})\ge n(X_r)$, and (0.12.1) induces

(0.12.2)
$$\begin{array}{ccccccccc} X & \longleftarrow & X_1 & \longleftarrow & \cdots & \longleftarrow & X_{r-1} & \longleftarrow & X_r\\ \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta_1} & & & & \downarrow{\scriptstyle\delta_{r-1}} & & \downarrow{\scriptstyle\delta_r}\\ V & \longleftarrow & V_1 & \longleftarrow & \cdots & \longleftarrow & V_{r-1} & \longleftarrow & V_r,\end{array}$$

where all vertical morphisms are finite, and

(0.12.3) $$F_n(X_i)\cong\delta_i(F_n(X_i))\text{ for i = 1, \dots, r.}$$

Theorem 0.13. *Let $\delta:X\to V$ be a finite and dominant morphism, say of generic rank n, between affine varieties field of characteristic $p\ge 0$, where V is regular. If p does not divide n, one can attach to $\delta:X\to V$ a pair (K,b) on the regular variety V so that*

(1) $\mathrm{Sing}(K,b)=\beta(F_n(X))$.

(2) *For any sequence (0.12.1), the lower row of (0.12.2) induces*

(0.13.1)
$$\begin{array}{cccccccc} V=V_0 & \stackrel{\pi_1}{\longleftarrow} & V_1 & \stackrel{\pi_2}{\longleftarrow} & \dots & \stackrel{\pi_r}{\longleftarrow} & V_r \\ (K,b) & & (K_1,b) & & \dots & & (K_r,b) \end{array}$$

(0.13.2)
$$\mathrm{Sing}(K_i,m)=\beta_i(F_n(X_i))\subset V_i \text{ for } i=1,\dots,r+1.$$

(3) *Conversely, any sequence (0.13.1) induces a sequence (0.12.2) and the equalities in (0.13.2) hold.*

Corollary 0.14. *A resolution of (K,b) over V defines a reduction of the multiplicity of X.*

Proof. **Of the Theorem.** Recall that for each $1\leq j\leq m$, set $S\subset B_j=S[\Theta_j]$ of generic rank d_j. There is a pair, say $(K^{(j)},D_j)$ over $Spec(S)$ representing the image of points of multiplicity d_i.

Take
$$(K,b)=\cap_{i=1,\dots,m}(K^{(i)},b_i)$$

As for (4) recall that it holds for the hypersurface case (i.e., for each $(K^{(i)},b_i)$ over $Spec(S)$).