SOME PROPERTIES OF THE MULTIPLICITY AND OF BLOW UPS AT EQUIMULTIPLE CENTERS (II).

DIEGO A. SULCA. AND ORLANDO E. VILLAMAYOR U.

Integral closure of ideals

Definition 0.1. Fix a ring R and a proper ideal $I \subset R$. An element $r \in R$ is said to be integral over I if there are elements $c_1 \in I,\ c_2 \in I^2,\ \ldots, c_n \in I^n$ (for some n) such that

$$r^n + c_1 r^{n-1} + \ldots + c_n = 0.$$

The set $\overline{I} \subset R$ of all those $r \in R$ which are integral over I is called the integral closure of I. An extension of I, say $I \subset J$, is said to be integral over I if $J \subset \overline{I}$. When the previous condition holds we also say that I is a reduction of J.

(1) W is a variable over R,

$r \in \overline{I}$ if and only if $rW \in R[W]$ is integral over the Rees ring $R[IW] := R \oplus IW \oplus I^2W^2 \oplus \ldots$

(2) Given $I \subset J$. Then I is a reduction of J iff $R[IW] \subset R[JW]$ is finite.

(3) $I \subseteq \overline{I} \subseteq \sqrt{I}$.

Lemma 0.2. *Let $R \subset R'$ be an integral extension of rings and I an ideal of R. Then $\overline{I} = \overline{IR'} \cap R$.*

Proof. $R[IW] \subset R'[(IR')W]$ integral. Use (1) for rW, and $r \in R$. $\circlearrowleft$

Remark 0.3. An inclusion of ideals $I \subset J$ of R defines an inclusion of Rees rings: $R[IW] \subset R[JW]$, which in general does **not define** a morphism $Proj(R[JW]) \to Proj(R[IW])$.

Remark 0.4. If I is a reduction of J the inclusion of Rees rings: $R[IW] \subset R[JW]$ is finite, and morphism $Proj(R[JW]) \to Proj(R[IW])$ is defined and finite,

Remark 0.5. If $S \subset R$ is finite and K is an ideal in S, then $S[KW] \subset R[(KR)W]$ is finite. In particular there is a well defined morphism at the Projective schemes.

Remark 0.6. Suppose $S \subset R$ is finite. $K \subset S$, $J \subset R$, and $KR \subset J$ is a reduction in R. Then

$$S[KW] \subset R[JW] \text{ is finite.}$$

In fact

$$S[K] \subset R[(KR)W] \subset R[JW]$$

In particular

$$\begin{array}{ccc} Spec(R) & \longleftarrow & X_1 = Proj(R[JT]) \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta_1} \\ Spec(S) & \longleftarrow & V_1 = Proj(S[K]) \end{array}$$

Multiplicity and integral closure of ideals

Given a local ring (R, M) and a primary ideal J for M, we will denote by $e_R(J)$ the multiplicity of J.

$$\text{Samuel: } length(R/J^{n+1}) \equiv \frac{e_R(J)}{d!}n^d + b_1 n^{d-1} + \cdots .$$

When $J = M$, $e_R(M)$ will be also called the multiplicity of the local ring R.

When R is regular, $e_R(M) = 1$ and the converse is true if we require that $\hat{R}$, the completion of R at M, is equidimensional and has no embedded components. The latter conditions holds, for example, if R is an excellent local domain (EGA IV, (7.8.3)).

Theorem 0.7 (Nagata)**.** *If (R, M) is an equidimensional excellent local ring, and if Q is a prime ideal, then $e_{R_Q}(QR_Q) \leq e_R(M)$.*

The following result of Rees draws a first connection between the theory of multiplicities and that of integral closure of ideals.

Theorem 0.8 (Rees)**.** *If $I \subset J$ are primary ideals for the maximal ideal in a formally equidimensional local ring (R, M), then J is integral over I if and only if $e_R(I) = e_R(J)$.*

Zariski's Theorem on finite morphisms and the class $\mathscr{T}$

The following result of Zariski describes the behaviour of the multiplicity with respect to finite extension of rings.

Theorem 0.9. *Let (A, M) be a local domain, say with fraction field K. Let B be a finite extension of A and set $L := B \otimes_A K$. Let $Q_1, \dots, Q_r$ denote the maximal ideals of the semi-local ring B, and assume that $\dim B_{Q_i} = \dim A$, $i = 1, \dots, r$. Then*

$$e_A(M)[L : K] = \sum_{1 \leq i \leq r} e_{B_{Q_i}}(MB_{Q_i})[k_i : k],$$

where k_i is the residue field of B_{Q_i}, k is the residue field of (A, M).

We start by defining a class of finite extensions where we can apply Theorems 0.9 and 0.8.

Definition 0.10. We denote by $\mathscr{T}$ the class of finite extensions of rings $S \subset B$ that fulfill the following two conditions:

(1) S is an excellent regular domain and all saturated chains of prime ideals of S have the same length, equal to $\dim S$
(2) all saturated chain of prime ideals in B have the same same length, equal to $\dim B$, and $\dim B = \dim S$.
(3) B has not zero divisors in S.

Fix $S \subset B$ in the class $\mathscr{T}$. The following properties follow easily from the definition

(a) Given $\mathfrak{p} \in \text{Spec}(S)$ and $Q \in \text{Spec}(B)$, both extensions $S_{\mathfrak{p}} \subset B_{\mathfrak{p}}$ and $S/(S \cap Q) \subset B/Q$ belong to the class $\mathscr{T}$ if $S \cap Q$ is regular en S.
(b) Any subextension $S \subset B_1 (\subset B)$ belongs to the class.

The dimension $[B \otimes_S K : K]$ will be called the **generic rank** of the extension.

Corollary 0.11. *Let $S \subset B$ be an extension of rings in the class $\mathscr{T}$, let K denote the fraction field of of S and set $n := [B \otimes_S K : K]$. Then for any $P \in \operatorname{Spec}(B)$, one has*

$$e_{B_P}(PB_P) \leq n.$$

The equality holds if and only if, setting $\mathfrak{p} := P \cap S \in \operatorname{Spec}(S)$, the following three conditions are satisfied.

1. *P is the only prime of B lying above $\mathfrak{p}$, namely $B_P = B_{\mathfrak{p}}$.*
2. *The induced inclusion $S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}} \subset B_P/PB_P$ is an equality.*
3. *$PB_{\mathfrak{p}}$ is integral over $\mathfrak{p}B_{\mathfrak{p}}$.*

Proof.

$$e_A(M)[L : K] = \sum_{1 \leq i \leq r} e_{B_{Q_i}}(MB_{Q_i})[k_i : k],$$

The hypothesis on the extension $S \subset B$ ensures that Theorem 0.9 can be applied to the extension $S_{\mathfrak{p}} \subset B_{\mathfrak{p}}$. Since $S_{\mathfrak{p}}$ is regular, it has multiplicity one, and from the formula

$$n \geq e_{B_P}(\mathfrak{p}B_P)[B_P/PB_P : S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}}] \geq e_{B_P}(PB_P)[B_P/PB_P : S_{\mathfrak{p}}/\mathfrak{p}S_{\mathfrak{p}}] \geq e_{B_P}(PB_P),$$

and that the equalities hold if and only if conditions (1) and (2) above hold and $e_{B_P}(\mathfrak{p}B_P) = e_{B_P}(PB_P)$. Finally, the hypothesis on $S \subset B$ also ensures that B_P is excellent and equidimensional and hence formally equidimensional so we can apply Theorem 0.8 that says that $e_{B_P}(\mathfrak{p}B_P) = e_{B_P}(PB_P)$ is equivalent to the condition (3) of the corollary. ◌

Given $S \subset B$ in the class $\mathscr{T}$, of generic rank n, let $F_n(B) \subset \operatorname{Spec}(B)$ denote the set of those P such that B_P has multiplicity equal to the generic rank of the extension. The previous corollary says that $F_n(B)$ is either empty or it is is the maximal multiplicity locus of the scheme $\operatorname{Spec}(B)$. The induced morphism $\delta : \operatorname{Spec}(B) \to \operatorname{Spec}(S)$ will be said to be **transversal** if $F_n(B) \neq \emptyset$. As a consequence of the previous corollary we obtain:

Corollary 0.12. *Fix $S \subset B$ in the class $\mathscr{T}$, of generic rank n. If the morphism $\delta : \operatorname{Spec}(B) \to \operatorname{Spec}(S)$ is transversal, then $F_n(B) = \delta^{-1}(\delta(F_n(B)))$ and δ induces a homeomorphism*

$$F_n(B) \cong \delta(F_n(B)),$$

and two points in correspondence have the same residue field.

Proposition 0.13. *Fix $S \subset B$ in the class $\mathscr{T}$, of generic rank n. If $\delta : \operatorname{Spec}(B) \to \operatorname{Spec}(S)$ is transversal, then:*

1. *An irreducible scheme $Y \subset F_n(B)$ is regular if and only if $\delta(Y)$ is regular.*
2. *I the previous setting the induced finite morphism $\delta : Y \to \delta(Y)$ is an isomorphism of affine schemes.*

Proof. A) If $\delta(Y)$ is regular, then as $\delta : Y \to \delta(Y)$ is birational and finite $\delta : Y \to \delta(Y)$ must be an isomorphism.

B) Assume that Y is regular and choose $x \in \delta(Y) \subset \delta(F_n(B))$. There is a unique $z \in Y$ dominating x, say $\mathcal{O}_{\delta(Y),x} \subset \mathcal{O}_{Y,z}$, and moreover: $z \in F_n(B)$ so both have the same residue field and $m_x\mathcal{O}_{Y,z}$ is a reduction of the maximal ideal m_z. So Th 0.8 (Rees) + Zariski says that $\mathcal{O}_{\delta(Y),x}$ is regular, and hence the finite birational morphism $\delta : Y \to \delta(Y)$ is an isomorphism over x. As this holds for any point $\delta : Y \to \delta(Y)$ is an isomorphism.

Remark 0.14. The last Proposition is crucial in our discussion: We will show that if P is the generic point of the regular center Y in $\mathrm{Spec}(B)$, and if $\mathfrak{p}$ is the generic point of the regular center $\delta(Y)$ in $\mathrm{Spec}(S)$, then

$$S[\mathfrak{p}W] \subset B[PW] \tag{0.14.1}$$

is finite. In particular we get a diagram of blow ups:

$$\begin{array}{ccc} Spec(B) & \longleftarrow & X_1 \\ \delta \downarrow & & \downarrow \delta_1 \\ Spec(S) & \longleftarrow & V_1 \end{array} \tag{0.14.2}$$

0.15. The case $\delta : X \to V$ defined by $S \subset B = S[Z]/\langle Z^n + a_1 Z^{n-1} + \cdots + a_n\rangle$, and the Tchirnhausen pair of a monic polynomial.

Fix a monic polynomial, say $f(Z) = Z^n + a_1 Z^{n-1} + \ldots + a_1 Z + a_0 \in S[Z]$, over a regular k-algebra S. Assume here that S is a domain, with quotient field K. As S is regular it is a normal ring. Considered $B = S[Z]/\langle f(Z)\rangle$ and the finite morphism $\delta : \mathrm{Spec}(B) \to \mathrm{Spec}(S)$. It is natural to expect that there be significant information concerning this morphism, or say of $S \subset B$, which is encoded in the coefficients of $f(Z)$. Such is the case with the discriminant. On the other hand, if we let $Z_1 = Z - \lambda$ for some $\lambda \in S$, then $f(Z) = g(Z_1) \in S[Z] = S[Z_1]$ for some $g(Z_1) = Z_1^n + b_1 Z_1^{n-1} + \ldots + b_{n-1} Z_1 + b_n$, and $B = S[Z]/\langle f(Z)\rangle = S[Z_1]/\langle g(Z_1)\rangle$.

In particular, if there is information of $\delta : \mathrm{Spec}(B) \to \mathrm{Spec}(S)$ in the coefficients of the polynomial it is reasonable to expect that it will not distinguish coefficients of $f(Z)$ from those of $g(Z_1)$.

To clarify these claims we bring the problem to a *universal context.*

Fix a field k and consider the polynomial ring in n variables $k[Y_1, \ldots, Y_n]$. The *universal polynomial* of degree n, is

$$F_n(Z) = (Z - Y_1) \cdots (Z - Y_n) = Z^n - s_{n,1} Z^{n-1} + \ldots + (-1)^n s_{n,n} \in k[Y_1, \ldots, Y_n, Z],$$

where for $i = 1, \ldots, n$, $s_{n,i} \in k[Y_1, \ldots, Y_n]$ is the i-th symmetric polynomial. The diagram

$$\begin{array}{ccc} \mathrm{Spec}(k[s_{n,1}, \ldots, s_{n,n}][Z]) & \hookleftarrow & \mathrm{Spec}(k[s_{n,1}, \ldots, s_{n,n}][Z]/\langle F_n(Z)\rangle) \\ & \searrow^{\pi} & \downarrow \alpha \\ & & \mathrm{Spec}(k[s_{n,1}, \ldots, s_{n,n}]) \end{array} \tag{0.15.1}$$

illustrates the universal situation. In fact, if $B = S[Z]/\langle f(Z)\rangle$ and $f(Z) = Z^n + a_1 Z^{n-1} + \ldots + a_1 Z + a_0 \in S[Z]$: $\delta : \mathrm{Spec}(B) \to \mathrm{Spec}(S)$ is obtained from the specialization:

$$\begin{array}{rccc} \Delta : & k[s_{n,1}, \ldots, s_{n,n}] & \longrightarrow & S \\ & (-1)^i s_{n,i} & \to & a_i. \end{array} \tag{0.15.2}$$

In other words, if $B = S[Z]/\langle f(Z)\rangle$ and $f(Z) = Z^n + a_1 Z^{n-1} + \ldots + a_1 Z + a_0 \in S[Z]$, there is a commutative diagram

$$\begin{array}{ccc} \mathrm{Spec}(k[s_{n,1}, \ldots, s_{n,n}][Z]/\langle F_n(Z)\rangle) & \longleftarrow & \mathrm{Spec}(B) \\ \alpha \downarrow & & \downarrow \delta \\ \mathrm{Spec}(k[s_{n,1}, \ldots, s_{n,n}]) & \longleftarrow & \mathrm{Spec}(S) \end{array} \tag{0.15.3}$$

which, in addition, is a fiber product. Here

$$k[Y_1, \dots, Y_n]^{S_n} = k[s_{n,1}, \dots, s_{n,n}] \tag{0.15.4}$$

is a polynomial ring, in particular it is smooth over k, and $\Delta : k[s_{n,1}, \dots, s_{n,n}] \to S$ is a morphism of k-algebras. Consider the subring

$$k[Y_i - Y_j]_{1 \le i,j \le n} \subset k[Y_1, \dots, Y_n]$$

and note that the permutation group S_n also acts on this subring. So there is an inclusion

$$k[Y_i - Y_j]_{1 \le i,j \le n}^{S_n} \subset k[Y_1, \dots, Y_n]^{S_n} = k[s_{n,1}, \dots, s_{n,n}]. \tag{0.15.5}$$

As S_n is a finite group the algebra in the left hand side is finitely generated. Set

$$k[G_{m_1}, \dots, G_{m_r}] := k[Y_i - Y_j]_{1 \le i,j \le n}^{S_n}, \tag{0.15.6}$$

and, since S_n acts linearly in $k[Y_1, \dots, Y_n]$ (preserving the degree of this graded ring), we can take each generator G_{m_i} as an homogeneous polynomial in $k[Y_1, \dots, Y_n]$. Let

$$m_i = \text{degree } G_{m_i} \tag{0.15.7}$$

where $k[Y_1, \dots, Y_n]$ is graded in the usual way. The inclusion (0.15.5) yields an expression

$$G_{m_i} = G_{m_i}(s_{n,1}, \dots, s_{n,n}) \tag{0.15.8}$$

where $G_{m_i}(s_{n,1}, \dots, s_{n,n})$ is weighted homogeneous of degree m_i in $k[s_{n,1}, \dots, s_{n,n}] (\subset k[Y_1, \dots, Y_n])$.

This latter assertion says that there is a polynomial, say $G_{m_i}(V_1, \dots, V_n)$, which is homogeneous of degree m_i in $\mathbb{Z}[V_1, \dots, V_n]$, when this ring is graded so that each V_i is given weight i. Here $G_{m_i}(s_{n,1}, \dots, s_{n,n})$ is obtained by setting $V_i = s_{n,i}$, $i = 1, \dots, n$.

The morphism $\Delta : k[s_{n,1}, \dots, s_{n,n}] \to S$ maps $G_{m_i}(s_{n,1}, \dots, s_{n,n})$ to the element $G_{m_i}(a_1, \dots, a_n) \in S$. Fix $\lambda \in S$ and set $S[Z] = S[Z_1]$ where $Z_1 = Z - \lambda$. Let

$$f(Z) = Z^n + a_1 Z^{n-1} + \ldots + a_{n-1} Z + a_n = g(Z_1) = Z_1^n + b_1 Z_1^{n-1} + \ldots + b_{n-1} Z_1 + b_n.$$

$$G_{m_i}(a_1, \dots, a_n) = G_{m_i}(b_1, \dots, b_n) \text{ in } S. \tag{0.15.9}$$

Remark 0.16. Let $k[F_1, \dots, F_s]$ be a graded ring generated by homogeneous elements F_i, and set $m_i = \deg(F_i)$, $i = 1, \dots, s$. Let (R, M) be a local regular ring and a k-algebra, and let $\Delta : k[F_1, \dots, F_s] \to R$ be an homomorphisms of k-algebras. If $\Delta(F_i)$ has order $\geq m_i$, for $i = 1, \dots, s$, then, for any homogeneous element $G \in k[F_1, \dots, F_s]$, $\Delta(G)$ has order $\geq d$ at R, where d denotes the degree of G.

Theorem 0.17 (Adv. 2014, Theorem 1.16)**.** *Let k be a field of characteristic zero, and let S be a regular excellent k-algebra. Fix $f(Z) = Z^n + a_1 Z^{n-1} + \ldots + a_{n-1} Z + a_n \in S[Z]$, and*

$$\begin{array}{ccc} \operatorname{Spec}(S[Z]) & \hookleftarrow & \operatorname{Spec}(S[Z]/\langle f(Z) \rangle) \\ & \searrow & \downarrow \delta \\ & & \operatorname{Spec}(S). \end{array} \tag{0.17.1}$$

Let $F_n(S[Z]/\langle f(Z) \rangle)$, or simply F_n, denote the set of n-fold points of $\{f(Z) = 0\} \subset \operatorname{Spec}(S[Z])$. Consider the morphism obtained by specialization, say $\Delta : k[s_{n,1}, \dots, s_{n,n}] \longrightarrow S$, $s_{n,i} \to (-1)^i a_i$. Then:

$$\delta(F_n) = \bigcap_{1 \le j \le r} \{x \in \operatorname{Spec}(S) : \nu_x(G_{m_j}(a_1, \dots, a_n)) \geq m_j\}, \tag{0.17.2}$$

for G_{m_j} as in (0.18.4) and m_j as in (0.15.7).

Proof. Recall the description of the universal polynomial in 0.15 where the coefficients are the generators of $k[Y_1,\dots,Y_n]^{S_n}$ (see (0.15.4)). If the characteristic is zero (or if the characteristic does not divide n), one can check that

$$k[Y_1,\dots,Y_n] = (k[Y_i - Y_j]_{1\le i,j\le n})[s_{n,1}],$$

where $s_{n,1} = Y_1 + Y_2 + \dots + Y_n$. As $s_{n,1}$ is an invariant by the action of S_n, we conclude that

$$k[Y_1,\dots,Y_n]^{S_n} = (k[Y_i - Y_j]^{S_n}_{1\le i,j\le n})[s_{n,1}],$$

or say

$$k[s_{n,1},\dots,s_{n,n}] = k[G_{m_1},\dots,G_{m_r}][s_{n,1}]. \tag{0.17.3}$$

This gives an expression of this algebra by two different collection of homogeneous generators. Therefore each G_{m_i} is a weighted homogeneous polynomial in $s_{n,1},\dots,s_{n,n}$, and conversely, each $s_{n,i}$ is a weighted homogeneous in $G_{m_1},\dots,G_{m_r}, s_{n,1}$.

A) $\supset$ in 0.17.2. Fix a prime $\mathfrak{p} \in S$ and replace S by $S_{\mathfrak{p}}$. $\Delta : k[s_{n,1},\dots,s_{n,n}] \longrightarrow S_{\mathfrak{p}}$, $\Delta(s_{n,i}) = (-1)^i a_i$.

Recall that given $\lambda \in S$, $S[Z] = S[Z_1]$ where $Z_1 = Z - \lambda$, and let

$$f(Z) = Z^n + a_1 Z^{n-1} + \dots + a_{n-1}Z + a_n = g(Z_1) = Z_1^n + b_1 Z_1^{n-1} + \dots + b_{n-1}Z_1 + b_n.$$

Then, the following are equivalent:

(1) There is $\lambda \in S$ such that $\nu_{\mathfrak{p}}(b_i) = i($ which is the degree of $s_{n,i})$.
(2) There is $\lambda \in S$ such that $f(Z) \in \langle Z-\lambda,\ \mathfrak{p}S\rangle^n$ in $S[Z]$.
(3) $\nu_{\mathfrak{p}}(G_{m_i}(a_1,\dots,a_n)) \ge m_i$ $i = 1,\dots,r$.

For (3) $\Rightarrow$ (1) $\quad k[s_{n,1}, s_{n,2},\dots,s_{n,n}] = k[G_{m_1},\dots,G_{m_r}][s_{n,1}]$.

Set $\overline{f}(Z) = Z^n + \overline{a}_1 Z^{n-1} + \dots + \overline{a}_{n-1}Z + \overline{a}_n \in k(\mathfrak{p})[Z]$, there is $\overline{\lambda} \in k(\mathfrak{p})$ such that $\overline{b}_1 = 0$ after changing Z. Choose λ such that $\nu_{\mathfrak{p}}(b_1) \ge 1$.

B) $\subset$ in 0.17.2.

$S \subset B = S[Z]/\langle f(Z)\rangle$ is an inclusion in the class $\mathscr{T}$, finite of generic rank n.

Fix $\mathfrak{p} \in Spec(S)$ with residue field $k(\mathfrak{p})$, set

$$\overline{f}(Z) = Z^n + \overline{a}_1 Z^{n-1} + \dots + \overline{a}_{n-1}Z + \overline{a}_n = h_1^{n_1}\cdot h_2^{n_2}\cdots h_r^{n_r} \text{ with } h_i \text{ irreducible at } k(\mathfrak{p})[Z].$$

(1) There is a unique prime $P \in Spec(B)$ dominating $\mathfrak{p} \in Spec(S)$ iff $r = 1$.
(2) There is a unique prime $P \in Spec(B)$ dominating and rational over $\mathfrak{p} \in Spec(S)$ iff $r = 1$ and $h_1 = (Z - \lambda)$.

Fix $P \in F_n(B)$ and set $\mathfrak{p} = P \cap S$. After localization of S at $\mathfrak{p}$, and according to Zariski $B/P = S/\mathfrak{p}$, so there is $\lambda \in S$ such that $Z - \lambda \in S[Z]$ maps to zero in B/P.

Set $Z_1 = Z - \lambda$ and $f(Z) = g(Z_1) = Z_1^n + b_1 Z_1^{n-1} + \dots + b_{n-1}Z_1 + b_n$. As P is the only prime in B dominating $\mathfrak{p}$ in S, and $B/P = S/\mathfrak{p}$, we see that all $b_i \in \mathfrak{p}$, $i = 1,\dots,n$, so

$$f(Z) \in \langle Z-\lambda, \mathfrak{p}S[Z]\rangle. \tag{0.17.4}$$

Here P in B is the class of $\langle Z-\lambda, \mathfrak{p}S[Z]\rangle$: so the multiplicity of B at P is n if and only in

$$f(Z) \in \langle Z-\lambda, \mathfrak{p}S[Z]\rangle^n.$$

We finally apply the equivalence in A) to show that A, 3) holds at $\mathfrak{p}$.

Remark 0.18. 1) In our previous discussion $S \subset B = S[Z]/\langle f(Z)\rangle$ and given a prime P in B we have localized S at $\mathfrak{p} = P \cap S$ in order to apply Zariski's theorem. If $P \in F_n(B)$, and after localizing S at $\mathfrak{p}$ we showed that P in B is the class of $\langle Z-\lambda, \mathfrak{p}S[Z]\rangle$ in $S[Z]$, and hence P is the ideal $\langle \mathfrak{p}B, \Theta - \lambda \rangle$ in B, and $\Theta - \lambda$ is in the integral closure of $\mathfrak{p}B$. We finally claim that $S[\mathfrak{p}W] \subset B[PW]$ is finite, in fact

$$S[\mathfrak{p}W] \subset B[\mathfrak{p}BW] \subset B[\mathfrak{p}BW, (\Theta - \lambda)W] = B[PW]$$

are all finite.

2) We claim that localization of S at $\mathfrak{p}$ is not necessary in 1) if B/P is regular. To this end recall Prop 0.13: *Fix $S \subset B$ in the class $\mathscr{T}$. If $\delta : \mathrm{Spec}(B) \to \mathrm{Spec}(S)$ is transversal of generic rank n, then:*

a) *An irreducible scheme $Y \subset F_n(B)$ is regular if and only if $\delta(Y)$ is regular.*
b) *In the setting of 1) the induced finite morphism $\delta : Y \to \delta(Y)$ is an isomorphism of affine schemes.*

which we apply in the particular setting

$$0 \to \langle f(Z) \rangle \to S[Z] \to B \to 0$$

Take $\mathfrak{p}$ to be the generic point of $\delta(Y)$ (i.e., $\mathfrak{p} = P \cap S$). After a change of variable $Z - \lambda$ we may assume that $P \subset B$ corresponds to the *prime ideal* $\langle Z, \mathfrak{p}S[Z] \rangle$ via $Spec(B) \subset Spec(S[Z])$, and moreover

$$f(Z) = Z^n + a_1 Z^{n-1} + \cdots + a_n \in \langle Z, \mathfrak{p}S[Z] \rangle^n. \tag{0.18.1}$$

so P is the class of $\langle Z, \mathfrak{p}S[Z] \rangle$ and $\nu_{\mathfrak{p}}(a_i) \geq i$ and we claim that:

$$S[\mathfrak{p}W] \subset B[PW]$$

is finite. In fact, we get

$$S[\mathfrak{p}W] \subset B[\mathfrak{p}BW] \subset B[\mathfrak{p}BW, \Theta W] = B[PW] \tag{0.18.2}$$

In particular we get a diagram of blow ups:

$$\begin{array}{ccc} Spec(B) & \longleftarrow & X_1 \\ \delta \downarrow & & \downarrow \delta_1 \\ Spec(S) & \longleftarrow & V_1 \end{array} \tag{0.18.3}$$

3) Recall that

$$G_{m_i} = G_{m_i}(s_{n,1}, \ldots, s_{n,n}) \tag{0.18.4}$$

is weighted homogeneous of degree m_i, and each $s_{n,i}$ has weight i. Set, as before

$$G_{m_i}(a_1, \ldots, a_n) \in S$$

and assume that $t^i a_i' = a_i$ for some t and a_i' in S. Then

$$G_{m_i}(ta_1', t^2 a_2' \ldots, t^n a_n') = t^{m_i} G_{m_i}(a_1', \ldots, a_n').$$

4) Consider an affine open chart $U \subset V_1$, say $U = Spec(S_1)$ so that $\mathfrak{p}S_1 = tS_1$ for some element t. We analyze the previous diagram over U, namely

$$\begin{array}{ccc} Spec(B) & \longleftarrow & X_1 \supset \delta_1^{-1}(U) \\ \delta \downarrow & & \downarrow \delta_1 \\ Spec(S) & \longleftarrow & V_1 \supset U \end{array} \tag{0.18.5}$$

Assume that the conditions in (0.18.1) holds for $f(Z) \in S[Z]$ in the presentation of B, so $\nu_{\mathfrak{p}}(a_i) \geq i$, and note that setting $\delta_1^{-1}(U) = Spec(B_1) \to U = Spec(S_1)$, then

$$f(Z) = Z^n + a_1 Z^{n-1} + \cdots + a_n = Z^n + ta_1' Z^{n-1} + \cdots + t^n a_n' \in S_1[Z]$$

and

$$(\frac{Z}{t})^n + a'_1(\frac{Z}{t})^{n-1} + \cdots + a'_n \in S_1[(\frac{Z}{t})]$$

gives a presentation of B_1. In particular

$$\delta(F_n(B_1)) = \bigcap_{1\leq j\leq r} \{x \in \mathrm{Spec}(S_1) : \nu_x(\frac{G_{m_j}(a_1,\dots,a_n)}{t^{m_j}}) \geq m_j\} \tag{0.18.6}$$

and

$$\frac{G_{m_j}(a_1,\dots,a_n)}{t^{m_j}} = G_{m_j}(a'_1,\dots,a'_n)$$

5) Concerning (0.17.2)

$$\delta(F_n) = \bigcap_{1\leq j\leq r} \{x \in \mathrm{Spec}(S) : \nu_x(G_{m_j}(a_1,\dots,a_n)) \geq m_j\},$$

If M is a multiple of m_i of all $i = 1, 2, \dots, r$, then

$$\delta(F_n) = Sing(K, M)$$

$$K = \langle G_{m_i}^{\frac{M}{m_i}}(a_1,\dots,a_n), i = 1,\dots,r\rangle$$

and, for $B = S[Z]/\langle Z^n + a_1Z^{n-1} + \cdots + a_n\rangle$

$$\begin{array}{ccc} X_0 = Spec(B) & \longleftarrow & X_1 \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta_1} \\ V_0 = Spec(S) & \longleftarrow & V_1 \\ & & \\ (K, M) & & (K_1, M) \end{array} \tag{0.18.7}$$

where (K_1, M) is the transform of (K, M), $Sing(K, M) = \delta(F_n(X_0))$, and $Sing(K_1, M) = \delta(F_n(X_1))$

Proposition 0.19. *Fix $\delta : X \to V$ finite of generic rank n, for any sequence*

$$\begin{array}{ccccccc} X = X_0 & \overset{\pi_1}{\longleftarrow} & X_1 & \overset{\pi_2}{\longleftarrow} & \dots & \overset{\pi_r}{\longleftarrow} & X_r \\ F_n(X_0) & & F_n(X_1) & & \dots & & F_n(X_r) \end{array} \tag{0.19.1}$$

where $X_i \leftarrow X_{i+1}$ is the blow-up at a regular center $Y_i \subset F_n(X_i)$, one has $n = n(X) = n(X_1) = \cdots = n(X_{r-1}) \geq n(X_r)$, and (0.19.1) induces

$$\begin{array}{ccccccccc} X & \longleftarrow & X_1 & \longleftarrow & \cdots \longleftarrow & X_{r-1} & \longleftarrow & X_r \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta_1} & & & \downarrow{\scriptstyle \delta_{r-1}} & & \downarrow{\scriptstyle \delta_r} \\ V & \longleftarrow & V_1 & \longleftarrow & \cdots \longleftarrow & V_{r-1} & \longleftarrow & V_r, \end{array} \tag{0.19.2}$$

where all vertical morphisms are finite, and

$$F_n(X_i) \cong \delta_i(F_n(X_i)) \text{ for } i = 1, \dots, r. \tag{0.19.3}$$

Theorem 0.20. *Let $\delta : X \to V$ be a finite and dominant morphism, say of generic rank n, between affine varieties field of characteristic $p \geq 0$, where V is regular. If p does not divide n, one can attach to $\delta : X \to V$ a pair (K, M) on the regular variety V so that*

(1) *$\mathrm{Sing}(K, M) = \beta(F_n(X))$.*

(2) *For any sequence (0.19.1), the lower row of (0.19.2) induces*

(0.20.1)
$$\begin{array}{ccccccc} V = V_0 & \xleftarrow{\pi_1} & V_1 & \xleftarrow{\pi_2} & \dots & \xleftarrow{\pi_r} & V_r \\ (K, M) & & (K_1, M) & & \dots & & (K_n, M) \end{array}$$

(0.20.2)
$$\mathrm{Sing}(K_i, M) = \beta_i(F_n(X_i)) \subset V_i \text{ for } i = 1, \dots, r+1.$$

(3) *Conversely, any sequence (0.20.1) induces a sequence (0.19.2) and the equalities in (0.20.2) hold.*

Corollary 0.21. *A resolution of (K, M) over V defines a reduction of the multiplicity of X.*

Case $\delta : X \to V$ defined by $S \subset B = S[Z]/\langle Z^n + a_1 Z^{n-1} + \dots + a_n\rangle$. If M is a multiple of m_i of all $i = 1, 2, \dots, r$, then

$$K = \langle G_{m_i}^{\frac{M}{m_i}}(a_1, \dots, a_n), i = 1, \dots, r\rangle$$

$$\delta(F_n(X)) = Sing(K, M)$$

and, given a blow up on a regular center in the n-fold points

(0.21.1)
$$\begin{array}{ccc} X_0 = Spec(B) & \longleftarrow & X_1 \\ \delta \downarrow & & \downarrow \delta_1 \\ V_0 = Spec(S) & \longleftarrow & V_1 \end{array}$$

$$(K, M) \qquad\qquad (K_1, M)$$

where (K_1, M) is the transform of (K, M), $Sing(K, M) = \delta(F_n(X_0))$, and $Sing(K_1, M) = \delta(F_n(X_1))$

(K, M) is the Tchirnhausen pair of the monic polynomial.