SOME PROPERTIES OF THE MULTIPLICITY AND OF BLOW UPS AT EQUIMULTIPLE CENTERS. (I)

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Notation

(1) X Noetherian topological space.
(2) $(\Sigma, \geq)$ ordered.
(3) $f : X \to \Sigma$ upper-semi-continuos. ($\sigma \in \Sigma, F_\sigma = \{\ x \in X/f(x) \geq \sigma\}$ is closed.
(4) $Max(f)$ maximum value achieved.
(5) $\underline{Max}(f) = \{x \in X; f(x) = Max(f)\}$.

1. Introduction

Let X be an algebraic variety over a perfect field. The goal is to the study of the singular locus of X via the multiplicity. We shall explain how this leads to the resolution of singularities of X in the case of characteristic zero and describe some problems that arise in the positive characteristic case.

The multiplicity defines a function

$$\mathrm{mult}_X : X \to \mathbb{N},$$

which assigns to each point $x \in X$ the multiplicity of the local ring $\mathcal{O}_{X,x}$. This function is upper semi-continuous when $\mathbb{N}$ is given the natural order, and hence it stratifies X as a finite union of locally closed subsets, the level sets

$$F_n(X) := \{x \in X : \mathrm{mult}_X(x) = n\}.$$

The variety X is regular if and only if $\mathrm{mult}_X(x) = 1$ for all $x \in X$. If not, we denote by

$$n(X) : \text{ the maximum value of the function } \mathrm{mult}_X = Max\, \mathrm{mult}_X\, .$$

For $n = n(X)$, the level set

$$F_n(X) = \underline{Max}\, \mathrm{mult}_X$$

which is the highest multiplicity locus, is closed. We shall give a description of this closed set $F_n(X)$ which could lead, ultimately, to a simplification of it via blow-ups in a sense to be specified later on.

A remarkable fact, stated in the following theorem, is that the multiplicity does not increase when blowing up at equimultiple centers:

Theorem 1.1 (Dade). *If* $X \overset{\pi}{\longleftarrow} X_1$ *is the blow up at a smooth center included in a level set* $F_n(X)$*, then*

$$\mathrm{mult}_X(\pi(x_1)) \geq \mathrm{mult}_{X_1}(x_1) \quad \textit{for all} \quad x_1 \in X_1.$$

In particular, $n(X_1) \leq n(X)$. If $n(X_1) < n(X)$, we say shall that $X \overset{\pi}{\longleftarrow} X_1$ is *a reduction of the multiplicity* of X. More generally,

Definition 1.2. A reduction of the multiplicity of X is a sequence of morphisms

$$X \longleftarrow X_1 \longleftarrow \ldots \longleftarrow X_{r-1} \longleftarrow X_r \ ,$$

where $X_i \leftarrow X_{i+1}$ is a blow up at a smooth center included in the maximal multiplicity locus of X_i and $n(X) = n(X_1) = \cdots = n(X_{r-1}) > n(X_r)$.

Note that if any singular variety over a field k admits a reduction of the multiplicity, any variety over k will admit a resolution of singularities.

Theorem 1.3. *(Adv. 2014). Let* X *be a singular variety over a field* k *of characteristic zero, then* X *admits a reduction of the multiplicity.*

There are various ways in which one can present a singular variety X, and one of them is to view it as a ramified cover of a regular one. In this approach one defines a finite morphism say $X \to V$, where V is a smooth variety. Very classical notions, such as the discriminant, are defined in this context. It is natural to ask if we can extract information from the discriminant so as to simplify the singularities of X.

This is the perspective in this approach to resolution of singularities. For example, assume that V is affine, say with (regular) ring of functions S, and suppose that the ring of functions of X is given by $S[Z]/\langle Z^2 + a_1 Z + a_2 \rangle$, where Z is a variable over S. This is a particular case in which X is viewed as a two fold cover of V, and in this case the discriminant, namely $a_1^2 - 4a_2$ will allow us to describe the singular locus of X, and ultimately to resolve the singularity at least if the characteristic is zero.

Pairs, transformations of pairs and closed subsets.

Fix a smooth algebraic variety V over a perfect field k. Given a coherent ideal J on V ($\mathcal{O}_V$-ideal for short) and a positive integer b, we call (J, b) a *pair*. There is a closed subset of V associated with this pair, namely

$$\mathrm{Sing}(J, b) := \{x \in V : \nu_x(J) \geq b\} \subset V.$$

Here $\nu_x(J)$ denotes the order of J_x at the regular local ring $\mathcal{O}_{V,x}$. If $\mathrm{Sing}(J, b) \neq \emptyset$, a smooth subscheme included in $\mathrm{Sing}(J, b)$ is said to be *a permissible center for* (J, b). If

$$V \overset{\pi_Y}{\longleftarrow} V_1 \supset H_1$$

is the blow-up of V at a permissible center Y for (J, b), the fact that Y is included in $\mathrm{Sing}(J, b)$ ensures the existence of a factorization

$$J\mathcal{O}_{V_1} = I(H_1)^b J_1,$$

where J_1 is an $\mathcal{O}_{V_1}$-ideal and $H_1 \subset V_1$ is the exceptional hypersurface. V_1 is regular and the new pair (J_1, b) is called the *transform* of (J, b). Again, there is a closed subset $\mathrm{Sing}(J_1, b) \subset V_1$ associated with (J_1, b), and if this set is not empty, we can repeat the above construction again. This leads us to the following definition.

Definition 1.4. Fix a pair (J, b) on V such that $\mathrm{Sing}(J, b) \neq \emptyset$.

(1) A *permissible sequence of blow-ups* for (J,b) is a sequence of the form

$$\begin{array}{ccccccccc} V=V_0 & \xleftarrow{\pi_1} & V_1 & \xleftarrow{\pi_2} & \dots & \xleftarrow{\pi_{r-1}} & V_{r-1} & \xleftarrow{\pi_r} & V_r \\ (J,b)=(J_0,b) & & (J_1,b) & & \dots & & (J_{r-1},b) & & (J_r,b) \end{array},$$

where $V_i \leftarrow V_{i+1}$ is the blow up of V_i at a permissible centre for (J_i,b) and (J_{i+1},b) is the transform of (J_i,b).

(2) A permissible sequence as above is said to be a resolution for (J,b) if $\mathrm{Sing}(J_r,b)=\emptyset$.

A fundamental result for resolution of singularities is that over fields of characteristic zero every pair (J,b) admits a resolution. Moreover, there is an algorithm which produces such a resolution in the sense that given V and (J,b), it provides the first center to blow up and produces $V \leftarrow V_1$, and one obtains a transform (J_1,b) on V_1. So either $\mathrm{Sing}(J_1,b)$ is empty (in which case we have produced a resolution for (J,b)), or this set is not empty and the algorithm produces a new center to blow up, say $V_1 \leftarrow V_2$, and a transform (J_2,b), and so on. The point is that for some index r, $\mathrm{Sing}(J_r,b)=\emptyset$. Detailed proofs can be found in literature. Let us indicate that this algorithm has been implemented, and we refer here to

https://www.risc.uni-linz.ac.at/projects/basic/adjoints/blowup
`https://www.singular.uni-kl.de/Manual/4-0-3/sing_1454.htmSEC1529`

for a software that has been developed to implement this algorithm.

The crucial property of pairs is their role as an assignment of closed subsets. To clarify this assertion let us fix a smooth variety V over a perfect field k and a pair (J,b) on V. We already assigned to this pair the closed subset $\mathrm{Sing}(J,b)\subset V$. We also defined when a blow-up $V\leftarrow V_1$ is permissible for (J,b), and in such case we defined the transform (J_1,b) of (J,b), which is a pair on V_1. In particular, a closed subset $\mathrm{Sing}(J_1,b)$ in the variety V_1 is obtained from (J,b) and the permissible blow-up $V\leftarrow V_1$.

Note that if a sequence of transformations of V as above is permissible for a pair (J,b), say

(1.4.1) $$\begin{array}{ccccccccc} V & \xleftarrow{\pi_1} & V_1 & \xleftarrow{\pi_2} & \dots & \xleftarrow{\pi_{r-1}} & V_{r-1} & \xleftarrow{\pi_r} & V_r \\ (J,b) & & (J_1,b) & & & & (J_{r-1},b) & & (J_r,b) \end{array},$$

Then the pairs (J_i,b), $i=1,\dots,r$ are determined by the sequence (and by the first pair (J,b)). Hence so are the closed subsets $\mathrm{Sing}(J_i,b)\subset V_i$. We shall say that the sequence of sets

$$(\mathrm{Sing}(J_0,b),\mathrm{Sing}(J_1,b),\dots,\mathrm{Sing}(J_r,b))$$

are the closed subsets defined by (J,b) and the permissible sequence of transformations (1.10.3).

Different sequences of transformations define different closed subsets. We will draw special attention to the closed subsets defined by (J,b) via permissible sequences over V.

Let $V\times\mathbb{A}^1\to V$ be the projection, and note that a pair (J,b) on V induces, by pull-back, a pair $(J\mathcal{O}_{V\times\mathbb{A}^1},b)$ over $V\times\mathbb{A}^1$.

Definition 1.5. Two pairs (J,b) and (J',b') on a smooth variety V are said to be equivalent, say $(J,b)\sim(J',b')$, if:

(1) Every sequence of transformation of V that is permissible for one of them is also permissible for the other, and both define the same closed sets.
(2) The same holds for the pull-backs of the pairs (J,b) and (J',b') at $V\times\mathbb{A}^1$

Example 1.6. Fix a pair (J,b) over V and a positive integer s. We claim that the two pairs (J,b) and (J^s,bs) over V are equivalent. In fact, note that

$$\mathrm{Sing}(J,b)=\{x\in V:\nu_x(J)\geq b\}=\{x\in V:\nu_x(J^s)\geq br\}=\mathrm{Sing}(J^s,bs).$$

In addition, if Y is a regular center in $\mathrm{Sing}(J,b)=\mathrm{Sing}(J^s,bs)$ we get two transforms :

$$\begin{array}{ccc} V & \longleftarrow & V_1 \\ (J,b) & & (J_1,b) \end{array}$$

$$\begin{array}{ccc} V & \longleftarrow & V_1 \\ (J^s,bs) & & ((J^s)_1,bs) \end{array}$$

which define two pairs on V_1, and we finally check that $(J^s)_1=(J_1)^s$. In other words, the pairs (J_1,b) and $((J^s)_1,bs)$ are linked by the same relation as before. The same will happen if we multiply V by a line $\mathbb{A}^1$. This shows that any sequence of transformation of V which is permissible for (J,b) will be also permissible for (J^s,sb) and viceversa.

1.7. Fix an étale morphism $\alpha: V'\to V$. For any sequence (1.4.1) we get, by taking fiber products

$$\begin{array}{ccccccccc} V' & \xleftarrow{\pi'_1} & V'_1 & \xleftarrow{\pi'_2} & V'_2 & & \cdots & \xleftarrow{\pi'_r} & V'_r \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha_1} & & \downarrow{\scriptstyle\alpha_2} & & & & \downarrow{\scriptstyle\alpha_r} \\ V & \xleftarrow{\pi_1} & V_1 & \xleftarrow{\pi_2} & V_2 & & \cdots & \xleftarrow{\pi_r} & V_r \end{array} \tag{1.7.1}$$

and one can view the upper row as a sequence of transformations of the pullbacks of the pairs (J,b) and (J',b') to V'. However, there could be other sequences of transformations over these pullbacks defined over V' which do not arise in this way.

Definition 1.8. Two pairs (J,b) and (J',b') on a smooth variety V are strongly equivalent if

(1) they are equivalent over V (as in 1.5).
(2) The same holds for the pull-backs of the pairs (J,b) and (J',b') at $\alpha: V'\to V$ for any étale morphism.

1.9. On constructive resolution. For each integer $d\geq 1$ there is a totally ordered set $(\Gamma_d,\leq)$, and for any pair (J,b) over a smooth variety V of dimension d , or say for $\mathcal{F}=(V,(J,b))$, there are "functions " $\{f^{(0)},f^{(1)},\dots\}$ and sequences say:

- $f^{(0)}_{\mathcal{F}}: Sing(J,b)\to\Gamma_d$ such that $\underline{Max} f^{(0)}_{\mathcal{F}}$ is regular.
- Setting

$$\begin{array}{ccc} V & \longleftarrow & V_1 \\ (J,b) & & (J_1,b) \end{array}$$

  as above, then either $Sing(J_1,b)$ is empty or a function $f^{(1)}_{\mathcal{F}}: Sing(J,b)\to\Gamma_d$ is defined, and it is such that $\underline{Max}(f^{(1)}_{\mathcal{F}})$ is regular.
- Setting

$$\begin{array}{ccccc} V & \longleftarrow & V_1 & \longleftarrow & V_2 \\ (J,b) & & (J_1,b) & & (J_2,b) \end{array}$$

  as in the previous step then either $Sing(J_2,b)$ is empty or a function $f^{(2)}_{\mathcal{F}}: Sing(J_2,b)\to\Gamma_d$ is defined, and it is such that $\underline{Max}(f^{(2)}_{\mathcal{F}})$ is regular.
- ....

We say that the previous procedure is a constructive resolution of pairs in dimension d if

(1) For each pair (J,b) over a smooth scheme V of dimension d there is an integer r so that setting

$$\begin{array}{ccccccccc} V & \longleftarrow & V_1 & \longleftarrow & V_2 & \longleftarrow \dots \longleftarrow & V_{r-1} & \longleftarrow & V_r \\ (J,b) & & (J_1,b) & & (J_2,b) & & (J_{r-1},b) & & (J_r,b) \end{array}$$

then $Sing(J_r, b)$ is empty.
(2) If (J, b) and (J', b') are equivalent over V, set $\mathcal{F} = (V, (J, b))$ and $\mathcal{F}' = (V, (J', b'))$. We require that the functions $f^{(i)}_{\mathcal{F}} : Sing(J_i, b) \to \Gamma_d$ and $f^{(i)}_{\mathcal{F}'} : Sing(J'_i, b) \to \Gamma_d$ be the same.
(3) If (J, b) and (J', b') are strongly equivalent and if $\alpha : V' \to V$ is étale, then the statement in (2) holds for the pullbacks of the pairs over V'.

1.10. More on closed sets: Intersections. Fix (J, b) and (I, d) two pairs over V. Then there is a natural notion of intersection, denoted here by

$$(J, b) \cap (I, d) = (K, c)$$

where $K = \langle J^c, I^b\rangle$ and $c = bd$.

(1) $Sing(K, c) = Sing(J, b) \cap Sing(I, d)$ at V
(2) A sequence

$$V \xleftarrow{\pi_1} V_1 \xleftarrow{\pi_2} \dots \xleftarrow{\pi_{r-1}} V_{r-1} \xleftarrow{\pi_r} V_r \qquad (1.10.1)$$
$$(K, c) \quad (K_1, c) \quad (K_{r-1}, b) \quad (K_r, b)$$

induces two sequences

$$V \xleftarrow{\pi_1} V_1 \xleftarrow{\pi_2} \dots \xleftarrow{\pi_{r-1}} V_{r-1} \xleftarrow{\pi_r} V_r \qquad (1.10.2)$$
$$(J, b) \quad (J_1, c) \quad (J_{r-1}, b) \quad (J_r, b)$$

and

$$V \xleftarrow{\pi_1} V_1 \xleftarrow{\pi_2} \dots \xleftarrow{\pi_{r-1}} V_{r-1} \xleftarrow{\pi_r} V_r \qquad (1.10.3)$$
$$(I, d) \quad (I_1, d) \quad (I_{r-1}, d) \quad (I_r, d)$$

Moreover

$$Sing(K_i, c) = Sing(J_i, b) \cap Sing(I_i, d) \text{ at } V_i,\ i = 1, \dots, r$$

The role of pairs in the simplification of singularities. Hironaka's approach.

Pairs were introduced by Hironaka, and used to prove resolution of singularities in characteristic zero. He shows that if we know how to resolve pairs, in the sense of 1.4, one can improve the highest Hilbert Samuel function. He also proves that resolution is achieved by successive improvements of the Hilbert Samuel function. This will be formulated in Theorem 1.13.

Hironaka proves that a similar result holds for the Samuel stratification. Recall that a variety X can be stratified by the Hilbert-Samuel function

$$HS_X : X \to \mathbb{N}^{\mathbb{N}}.$$

$$\sigma \in \mathbb{N}^{\mathbb{N}} \text{ if and only if } \sigma : \mathbb{N} \to \mathbb{N}$$
$$HS_X(x)(n) = length(\mathcal{O}_{X,x}/m^n).$$

The value of this function at a closed point $x \in X$ is the Hilbert function of the local ring $\mathcal{O}_{X,x}$. When $\mathbb{N}^{\mathbb{N}}$ is ordered lexicographically, it turns out that HS_X is upper semi-continuous. Let

$$S = S(X) \in \mathbb{N}^{\mathbb{N}} = Max(HS_X)$$

denote the highest value achieved by HS_X, that is, the highest Hilbert function, and let

$$F_S(X) = \underline{Max} HS_X (\subset X)$$

be the stratum of highest value. Hironaka gives different characterizations for a regular center Y to be included in $F_S(X)$ (permissible centers in his notation). He also proves the following:

Theorem 1.11. *Let $X \leftarrow X_1$ be the blow up at a regular center $Y \subset F_S(X)$. Then $S(X) \geq S(X_1)$.*

It is therefore natural to formulate now a *reduction* in the following terms.

Definition 1.12. We say that a sequence of blow ups $X \leftarrow X_1 \leftarrow \dots X_r \leftarrow X_{r+1}$ defined as above is a *reduction of the Hilbert-Samuel function* if $S(X) = S(X_1) = \cdots = S(X_r) > S(X_{r+1})$.

Suppose now that $X \leftarrow X_1$ is any blow-up at a regular center Y (not necessarily in the conditions of Theorem 1.11), and fix a closed immersion $X \subset V$, where V is regular. Then one can blow up of V at Y, say $V \leftarrow V_1$, and there is a closed immersion, say $X_1 \subset V_1$, and the restriction of the latter blow-up to X_1 is $X \leftarrow X_1$. Usually, once we fix $Y \subset X \subset V$, it is said that $X_1(\subset V_1)$ is the *strict transform* of $X \subset V$. In other words, closed immersions $X \subset V$ are preserved by blowing up at regular centers $Y \subset X$. Therefore, once we fix $X \subset V$, a sequence of blow-ups at regular centers over X, say $X \leftarrow X_1 \leftarrow \dots X_s$ induces a sequence a blow-ups $V \leftarrow V_1 \leftarrow \dots V_s$, together with closed immersions $X_i \subset V_i$ $1 \leq i \leq s$. We are finally prepared to present the role of pairs within Hironaka's approach.

Theorem 1.13. *Fix a variety X over a perfect field with highest Hilbert-Samuel function $S = S(X)$. If $X \subset V$, where V is a regular variety, there is a pair (J, b) over V which describes the Hilbert-Samuel stratum $F_S(X)$ as follows:*

(1) $\mathrm{Sing}(J, b) = F_S(X)$ *in V.*

(2) *For any sequence*

$$X \longleftarrow X_1 \longleftarrow \dots \longleftarrow X_r \longleftarrow X_{r+1} \tag{1.13.1}$$

defined by blowing up regular centers $Y_i \subset F_S(X)$ and such that

$$S := S(X) = S(X_1) = \cdots = S(X_r) \geq S(X_{r+1})$$

we get a permissible sequence of blow-ups for the pair (J, b) in the sense of (1.4), say

$$\begin{array}{ccccccccc} V & \xleftarrow{\pi_1} & V_1 & \xleftarrow{\pi_2} & \dots & \xleftarrow{\pi_r} & V_r & \xleftarrow{\pi_r} & V_{r+1} \\ (J,b) & & (J_1,b) & & \dots & & (J_r,b) & & (J_{r+1},b) \end{array} \tag{1.13.2}$$

and

$$\mathrm{Sing}(J_i, b) = F_S(X_i) \subset X_i \subset V_i \text{ for } i = 1, \dots, r+1. \tag{1.13.3}$$

(3) *Conversely, for a permissible sequence of blow-ups for the pair (J, b), say (1.13.2), we get a sequence (1.13.1) and the equalities (1.13.3) hold.*

Remark 1.14. 1) The pair (J, b) is constructed in terms of the inclusion $X \subset V$ in a very explicit manner, Theorem 1.13 states that the closed sets defined by (J, b) corresponds naturally to a Hilbert-Samuel stratum. The construction of (J, b) is not unique, but if another pair (J', b') fulfills the previous conditions then (J, b) and (J', b') are strongly equivalent over V.

Conversely if (J, b) and (J', b') are strongly equivalent over V one can replace (J, b) by (J', b') in the theorem.

2) The theorem also shows that the following two conditions are equivalent:

- $\mathrm{Sing}(J_{r+1}, b) = \emptyset$
- $S = S(X_r) > S(X_{r+1})$

and that $\mathrm{Sing}(J_{r+1}, b) = F_S(X_{r+1})$ if $S = S(X_{r+1})$. This leads to the following:

Corollary 1.15. *1) A resolution of the pair (J, b) in Theorem 1.13 defines a reduction of the highest Hilbert-Samuel function of X (1.12).*

2) If (1.13.2) is the constructive resolution of (J, b), and if (J', b') is another pair that fulfills the theorem, then (1.13.2) is also the constructive resolution of (J', b').

3) If (1.13.1) is the reduction of the Hilbert Samuel function defined by the algorithmic resolution (1.13.2), then (1.13.1) does not depend on the choice of (J, b) in the Theorem.

Some comments are in order as regarding the formulation of Theorem 1.13. In the first place, the existence of a pair (J, b) over V (defined in terms of $X \subset V$) is only local. To be precise, one can cover X by finitely many open sets so that, after replacing X by such restrictions, there are closed immersions $X \subset V$ and pairs (J, b) which fulfils the conditions of the theorem. Moreover, these restrictions are, strictly speaking, restrictions in the sense of étale topology. As for the construction of (J, b), let us indicate that it is defined in terms of the equations defining $X \subset V$, using techniques of division at the henselization of $\mathcal{O}_{V,x}$, and finally descending the results to a suitable étale neighbourhood (a Theorem of Aroca).

The role of pairs in the simplification of the multiplicity.

Consider a finite a dominant morphism of varieties $\delta : X \to V$ such that V is regular. In the case that both varieties are affine, if S is the coordinate ring of V and B is that of X, the morphism δ induces a finite extension of rings $S \subset B$ such that S is a regular domain. If K denotes the quotient field of S, we call

$$n := [B \otimes_S K : K].$$

the generic rank of δ. This definition extends also to the case where V and X are not necessarily affine. Given an arbitrary variety X over a field of characteristic zero, the goal is to resolve its singularities by looking at the multiplicity as main invariant. The following two results introduce the roll played by finite and dominant morphisms in this task.

Theorem 1.16. *Let X be a variety over a perfect field, and fix a point $x \in X$ of multiplicity, say n. After restricting X to an (étale) neighbourhood of x we can define a finite dominant morphism $\delta : X \to V$ of generic rank n, where V is regular.*

The role of pairs in the study of the multiplicity.

Proposition 1.17. (A corollary of a theorem of Zariski.) *Let $\delta : X \to V$ be a finite and dominant morphism of generic rank n, where V is regular.*

(1) *Points of X have at most multiplicity n. Let $F_n(X) \subset X$ be the points of multiplicity n.*
(2) *If $F_n(X) \neq \emptyset$, then δ induces an homeomorphism $F_n(X) \cong \delta(F_n(X))$.*
(3) *Fix $Y \subset X$ included in $F_n(X)$, and assume that Y is irreducible.*
   (a) *Y is regular in X if and only if $\delta(Y)$ is regular in V.*
   (b) *If Y is regular there is a finite and dominant morphism $\delta_1 : X_1 \to V_1$ such that*

(1.17.1)
$$\begin{array}{ccc} X & \longleftarrow & X_1 \\ {\scriptstyle \delta}\downarrow & & \downarrow{\scriptstyle \delta_1} \\ V & \longleftarrow & V_1 \end{array}$$

is commutative, where $X \leftarrow X_1$ is the blow-up of X at Y and $V \leftarrow V_1$ is the blow-up of V at $\delta(Y)$. The generic rank of $\delta_1 : X_1 \to V_1$ is again n.

1.18. It follows from Proposition 1.17 that given $\delta : X \to V$ as above, of generic rank n, for any sequence

(1.18.1)
$$\begin{array}{ccccccc} X = X_0 & \xleftarrow{\pi_1} & X_1 & \xleftarrow{\pi_2} & \dots & \xleftarrow{\pi_r} & X_r \\ F_n(X_0) & & F_n(X_1) & & \dots & & F_n(X_r) \end{array}$$

where $X_i \leftarrow X_{i+1}$ is the blow-up at a regular center $Y_i \subset F_n(X_i)$, one has $n = n(X) = n(X_1) = \cdots = n(X_{r-1}) \geq n(X_r)$, and (1.18.1) induces

(1.18.2)
$$\begin{array}{ccccccc} X & \longleftarrow & X_1 & \longleftarrow \cdots \longleftarrow & X_{r-1} & \longleftarrow & X_r \\ \downarrow{\scriptstyle \delta} & & \downarrow{\scriptstyle \delta_1} & & \downarrow{\scriptstyle \delta_{r-1}} & & \downarrow{\scriptstyle \delta_r} \\ V & \longleftarrow & V_1 & \longleftarrow \cdots \longleftarrow & V_{r-1} & \longleftarrow & V_r, \end{array}$$

where all vertical morphisms are finite, and

(1.18.3) $$F_n(X_i) \cong \delta_i(F_n(X_i)) \text{ for i = 1, ..., r.}$$

Theorem 1.19. *Let $\delta : X \to V$ be a finite and dominant morphism, say of generic rank n, between affine varieties field of characteristic $p \geq 0$, where V is regular. If p does not divide n, one can attach to $\delta : X \to V$ a pair (K, d) on the regular variety V so that*

(1) $\mathrm{Sing}(K, d) = \delta(F_n(X))$.
(2) *For any sequence (1.18.1), the lower row of (1.18.2) induces*

(1.19.1)
$$\begin{array}{ccccccc} V = V_0 & \xleftarrow{\pi_1} & V_1 & \xleftarrow{\pi_2} & \ldots & \xleftarrow{\pi_r} & V_r \\ (K, d) & & (K_1, d) & & \ldots & & (K_r, d) \end{array}$$

(1.19.2) $$\mathrm{Sing}(K_i, d) = \beta_i(F_n(X_i)) \subset V_i \textit{ for } i = 1, \ldots, r.$$

(3) *Conversely, any sequence (1.19.1) induces a sequence (1.18.2) and the equalities in (1.19.2) hold.*

Theorem 1.20. *Let X' be a variety over a perfect field, and fix a point $x' \in X'$ of multiplicity n. There is an étale neighborhood, say $(X, x) \to (X', x')$, where we can define a finite dominant morphism $\delta : X \to V$ of generic rank n, where V is regular.*

The Claim. The locally defined reductions of the multiplicity given by the algorithmic resolution.

1.21. We remark that the construction of the pair (K, d) given in the theorem is quite explicit. For example in the hypersurface case: $S \subset B = S[\theta] \cong S[Z]/\langle f(Z) \rangle$,

$$f(Z) = X^n + a_1 X^{n-1} + \cdots + a_0 \in S[Z]$$

a monic polynomial. $\delta : \mathrm{Spec}(B) \to \mathrm{Spec}(S)$ has generic rank n.

We shall show how to construct a pair (K, d) by means of *universal* equations on the coefficients of f that lead to the pair (K, d).

Example 1.22. For $n = 2$,

$$f(Z) = X^2 + a_1 X + a_2 \in S[Z],$$

and we set $H(a_1, a_2) = a_1^2 - 4a_2$. Then the pair

$$(K, d) = (H, 2) = (\langle a_1^2 - 4a \rangle, 2)$$

fulfills the theorem. So for $n = 2$ we consider the discriminant with "weight 2".

Example 1.23. For $n = 3$,

$$f(Z) = X^3 + a_1 X^2 + a_2 X + a_3 \in S[Z],$$

we set $H(a_1, a_2, a_3) = 3a_2 - a_1^2$ and $G(a_1, a_2, a_3) = -9a_1 a_2 + 2a_1^3 + 27a_3$. In this case the pair

$$(K, d) = (\langle H(a_1, a_2, a_3) \rangle, 2) \cap (\langle G(a_1, a_2, a_3) \rangle, 3)$$

fulfills the theorem.