Resol. of surf. using IFP; monomial case (after Benito -Villamayor)

IFP $\mathrm{inv} = (\cdots, \mu^\sim)$ $\rightarrow$ $\mu^\sim = \infty$ ---- • center $= V(\mathbb{I})$ by NonSingularity principle
• after blup, inv. decreases

$\searrow$ $\mu^\sim = 0$ (monomial case) ---- need further analysis

@ This is joint work with K. Matsuki (Purdue)

Today: analysis for monomial case in 3 dim ambient sp (resol. of surf)

Reference: Adv. Stud. Pure Math. 70 p115-214

§1 Setup

W: nonsing 3 fold $/k = \bar{k}$, char $= p > 0$

$\mathbb{I}$: IF on W $P \in V(\mathbb{I}) \subset W$ $\mathbb{H}$: LGS at P

Assumption: $\begin{cases} \cdot \text{ in monomial case } (\Leftrightarrow \mu^\sim = 0) \\ \cdot\ \# \text{LGS} = 1. \quad \text{Put } \mathbb{H} = \{(f, p^e)\} \end{cases}$

Note: $\#\mathbb{H} = 1$ case is essential.

- $\#\mathbb{H} = 3 \rightarrow$ monom. case does not occur
- $\#\mathbb{H} = 2 \rightarrow V(\mathbb{I}) = \{P\}$. after pt blup, easy to see the improvement
- $\#\mathbb{H} = 0 \rightarrow \mathbb{I} = \langle (z^4, 1) \rangle$, z: exc. After blup along $V(z)$, u decreases.

Notation: • (x, y, z): coord. at P

• $f = z^{p^e} + a_1 z^{p^e - 1} + \cdots + a_{p^e}$, $a_i \in k[\![x, y]\!]$

exc. div $\mathcal{E} = \{H_x\}$ or $\{H_x, H_y\}$

• (monom. case) $\longrightarrow$ $\exists\, M_u = x^\alpha y^\beta$: usual monom. $\begin{pmatrix} \alpha, \beta \\ \in \mathbb{Q} \end{pmatrix}$

s.t. $\begin{cases} \cdot\ \alpha + \beta \geq 1, \ \alpha > 0, \ \beta \geq 0 \\ \cdot\ \forall s, \ \mathbb{I}_s \subset (f) + (x^{\lceil s\alpha \rceil} y^{\lceil s\beta \rceil}) \\ \cdot\ \exists m, \ x^{m\alpha} y^{m\beta} \in \mathbb{I}_m \end{cases}$

Denote $\mu(*) := \mathrm{ord}_*(M_u)$ e.g., $\begin{cases} \mu(P) = \alpha + \beta \\ \mu(\xi_{H_x}) = \alpha \end{cases}$

Assumption • $\mathrm{I\!I} : \{ \frac{\partial^t}{\partial z^t} \mid t \in \mathbb{Z}_{\geq 0} \}$-saturated

$(\Rightarrow x^{\lceil \alpha i \rceil} y^{\lceil \beta i \rceil} \mid a_i$ for $0 < i < p^e)$

• z : well-adapted (WA) at $*$ $(* = P, \xi_H \ (H \in \mathcal{E}))$

$\Leftrightarrow$ i.e. $\frac{1}{p^e} \mathrm{ord}_* \, a_{p^e} \geq Sl(*) := \min\{ \frac{1}{p^e} \mathrm{ord}_*^{(p^e)} a_{p^e}, \mu(*) \}$

(where $\mathrm{ord}_*^{(p^e)}$ = ord at $*$ modulo p^e-th power terms)

e.g. z : not WA at ξ_{Hx} $\Leftrightarrow$ $\left[\begin{array}{l} a_{p^e} - x^s g(y) \in (x^{s+1}), \quad \alpha > \frac{s}{p^e} \in \mathbb{Z} \\ \text{and} \quad g \in k[[y^p]] \end{array} \right]$

Def $h_x = Sl(\xi_{Hx}), \ h_y = Sl(\xi_{Hy})$. $M_t = x^{h_x} y^{h_y}$: tight monom.

Rem • $M_t \mid M_u$, $M_t^{p^e} \mid a_{p^e}$

• $M_t = M_u$ is easy case. (strong monom. case)

§2 Center

Set $C = V(\mathrm{I\!I})$.

Claim 1

$$C = \begin{cases} V(z, xy) & \text{if } \ h_x \geq 1 \quad h_y \geq 1 \\ V(z, x) & \qquad \geq 1 \qquad < 1 \\ V(z, y) & \qquad < 1 \qquad \geq 1 \\ V(z, x, y) = P & \qquad < 1 \qquad < 1 \end{cases}$$

$\because$ Note $C \subset V(M_u, 1) \subset V(xy)$. Set $C' = C \cap H_x$.

Show : $C' \subset \begin{cases} V(z) & \text{if } h_x \geq 1 \\ V(z, y) & \qquad < 1 \end{cases}$ $\left(\begin{array}{l} \text{Note:} \\ x^{\lceil h_x i \rceil} \mid a_i \\ \text{for } 1 \leq i \leq p^e \end{array} \right)$

Case $h_x > 0$ $\quad C' \subset V(f|_{x=0}, p^e) = V(z^{p^e}, p^e) = V(z)$

Thus $C' \subset V(z)$. if $h_x < 1$, set $r = h_x \cdot p^e$ $(0 < r < p^e)$.

Then $\partial_{x^r} a_{p^e}|_{x=0} = (\text{unit}) \cdot y^{*>0}$, and hence

$C' \subset V(y^*, \underbrace{p^e - r}_{>0}) \subset V(y)$.

Case $h_x = 0$ Let $i_0 = \min\{i \mid x + a_i\} \le p^e$. If $i_0 = p^e$, set $\partial = \partial_y$. Otherwise, set $\partial = \partial_{z^{p^e - i_0}}$. Then,

$(\partial f)|_{x=0} = y^{*>0}$ (unit), and hence $C' \subset V(y)$.

$f|_{x=y=0} = z^{p^e}$ implies $C' \subset V(z)$. ▨

Claim 2 If $h_x \geq 1$, blowup along $V(x, z)$ improves the situation.

∵ $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x' \\ y' \\ x'(z' + \overset{0}{\overset{\|}{c}}) \end{pmatrix} \rightsquigarrow \begin{cases} \cdot \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} : \text{WA at } P \text{ and } \xi_{H_x} \\ \cdot\ \mu(\xi_x) \text{ and } h_x \text{ decrease by } 1. \end{cases}$ ▨

@ We always blup a curve component of $V(\mathrm{I})$ if possible, so that we have only to consider the case

$[\, C = V(\mathrm{II}) = P,\ h_x < 1 \text{ and } h_y < 1 \,]$.

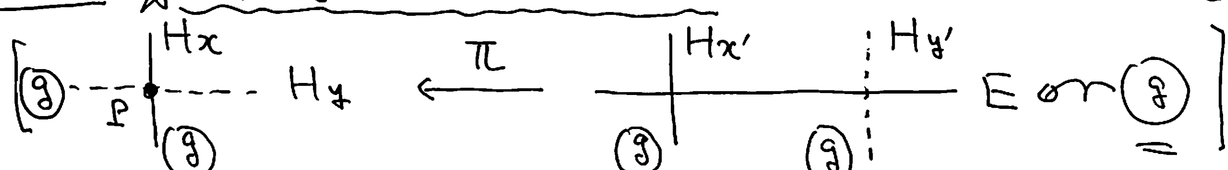
§3 Good / Bad hypersurf

Denote $\pi : W' \to W$ blowup at P with $E = \pi^{-1}(P) \subset W'$: exc. div:

Def H_x: good $\iff h_x = \mu(\xi_{H_x})$ $(\iff x^{\lceil a p^e \rceil} \mid a_{p^e})$

(otherwise bad)

Claim 3 ✡ H: good $(\forall H \in \mathcal{E}) \Rightarrow E$: good on $V(z)$



∵ Since $M_t = M_u$, $M_u^{p^e} \mid a_{p^e}$. It is preserved by blup ▨

Claim 4 Once ✡ holds, can resolve II keeping ✡.

∵ As $M_t = M_u$, $(z, 1)$: integral / II.

$\rightsquigarrow V(z)$ plays a role of HMC. ▨

@ we have only to consider the case with at least one bad exc. div.

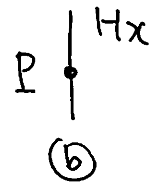
§4 invariant

May assume H_x : bad

Def Set $r = p^e \cdot h_x$, $a_{p^e} - x^r g(y) \in (x^{r+1})$ and

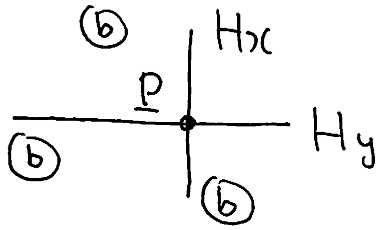
$\rho_x(P) = \frac{1}{p^e}\{\mathrm{ord}_P^{(p^e)}(x^r g(y)) - r\}$. we define :

Case ①



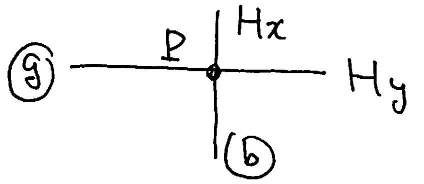
$\mathrm{inv}(P) = (\rho_x, 0, \mu(\xi_{H_x}))$

case ②



$\mathrm{inv}(P) = (\min(\rho_x, \rho_y), \max(\rho_x, \rho_y), 0)$

case ③

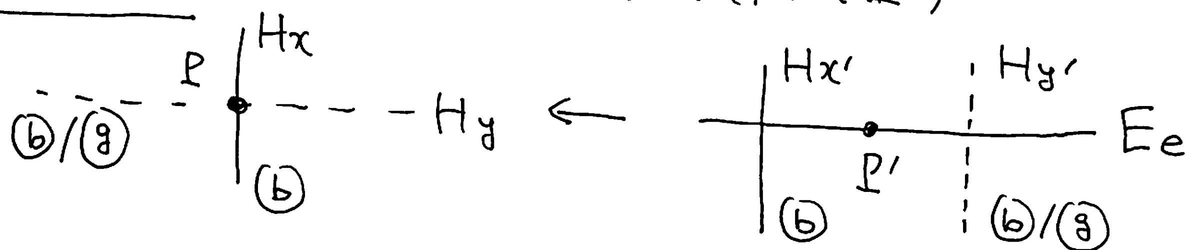


$\mathrm{inv}(P) = (\min(\rho_x, \mu(\xi_{H_x})), \max(\rho_x, \mu(\xi_{H_x})), 0)$

§5 Behavior

Let $P' \in E \cap V(\mathbb{I}')$

Prop 1



i) if $P' \in H_{x'}$ $\Rightarrow$ $\rho_x(P) > \rho_{x'}(P')$ (decrease)

ii) if E : bad $\Longrightarrow$ $\rho_x(P) \geq \rho_e(P')$ (non-increase)

(∵) i) y-chart $P' \in H_{x'}$. $P' \in V(\mathbb{I}')$ implies $P' \in V(z)$.

Thus monomial blowup. we can see :
$\begin{cases} \cdot\ f' : \text{WA at } \xi_{H_{x'}} \text{ (not nec. at } P') \\ \cdot\ \rho_x : \text{decrease} \end{cases}$

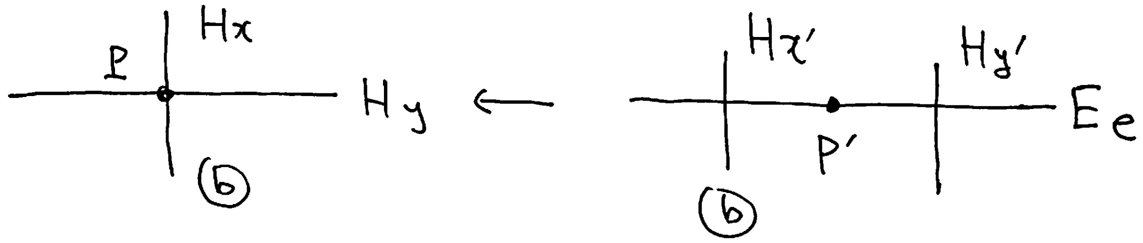
need some cleaning

ii) x-chart $E_e = E_{x'}$, $P' \in V(x') = E$

we can see
$\begin{cases} \cdot\ f' : \text{WA at } \xi_{H_{x'}} \text{ (not nec. at } P) \\ \cdot\ \rho_x : \text{non-increase}. \end{cases}$

☒

Prop 2



i) if H_y : bad and $P' \in H_{x'}$ $\Rightarrow$ $\rho_y(P) > \mu(\xi_{H_{x'}})$

ii) if H_y : good and $P' \notin H_{y'}$ $\Rightarrow$ $\mu(\xi_{H_x}) > \rho_e(P')$

(∵) i) Denote $a_{pe} - y^r \cdot g(x) \in (y^{r+1})$. Then we have

$$\rho_y + \frac{r}{pe} \geq \frac{1}{pe} \operatorname{ord} a_{pe} \underset{(E:\text{good})}{\geq} \mu(\xi_{H_x}) + \mu(\xi_{H_y}) \underset{(H_y:\text{bad})}{>} \frac{r}{pe} + \underbrace{\mu(\xi_{H_x})}_{= \mu(\xi_{H_{x'}}) \ (y\text{-chart})}$$

Thus $\rho_y > \mu(\xi_{H_{x'}})$.

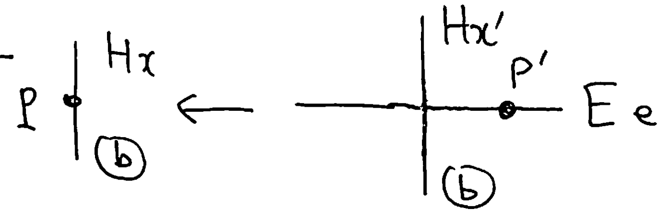
ii) y-chart $y' = e$. Put $d = \operatorname{ord}_P(a_{pe})$. We can write $a'_{pe} - e^{d-pe}\,\psi(x') \in (e)^{d-pe+1}$. We have

$$\rho_y + \frac{1}{pe}\deg\psi \leq \frac{d}{pe} \underset{(E:\text{bad})}{\lneq} \mu(P) = \mu(\xi_{H_x}) + \underbrace{\mu(\xi_{H_y})}_{= \rho_y \ (H_y:\text{good})}$$

Thus $\rho_e \leq \frac{1}{pe}\deg\psi < \mu(\xi_{H_x})$ ▨

Analysis

①



i) $P' \in H_{x'}$ (① ← ② or ③)

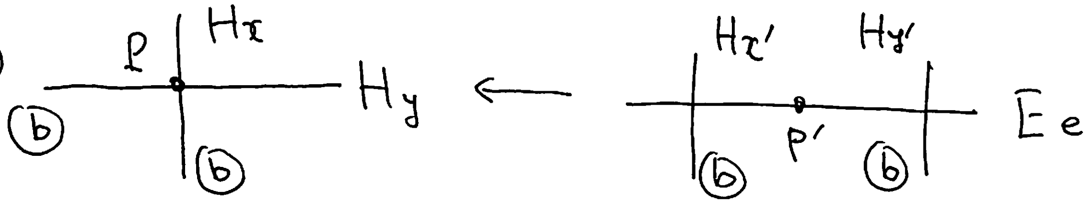
$\rho_x(P) > \rho_{x'}(P')$ ⇝ inv. decreases.

ii) $P' \notin H_{x'}$, E : bad (① ← ①)

$\rho_x(P) \geq \rho_e(P')$. $M_u = x^{\alpha} \underset{x\text{-chart}}{\longmapsto} e^{\alpha - 1}$ $\mu(\xi_{H_x}) > \mu(\xi_E)$.

⇝ inv. decreases.

②



i) $P' \in H_{x'}$, E: bad (② ← ②)

$\rho_x > \rho_{x'}$, $\rho_y \geq \rho_e$ ⇝ decrease.

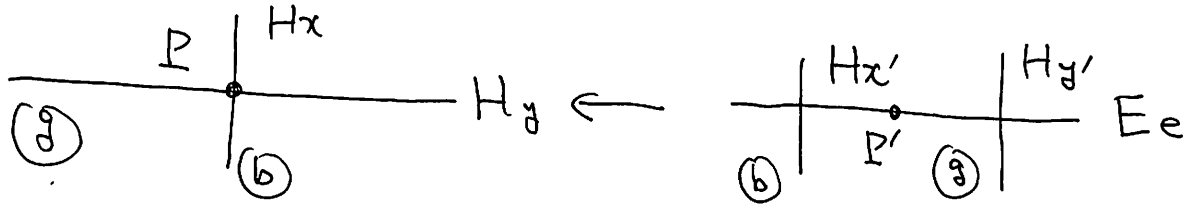
ii) $P' \in H_{x'}$, E: good (② ← ③)

$\rho_x > \rho_{x'}$. By Prop 2 i), $\rho_y > \mu(\xi_{H_{x'}})$. ⇝ decrease.

iii) $P' \notin H_{x'} \cup H_{y'}$, E: bad (② ← ①).

$\rho_x \geq \rho_e$ and $\rho_y \geq \rho_e$. We see $\rho_x > 0$. As H_x: bad, we can write $a_{p^e} - x^r g(y) \in (x^{r+1})$, $r = p^e \rho_x < p^e$. Thus $\rho_x > 0$. Similarly, $\rho_y > 0$. ⇝ decrease.

③



i) $P' \in H_{x'}$, E: good (③ ← ③)

y-chart $\rho_x > \rho_{x'}$. $x^\alpha y^\beta \mapsto x^\alpha y^{\alpha+\beta-1}$ implies $\mu(\xi_{H_x}) = \mu(\xi_{H_{x'}})$. ⇝ decrease.

ii) $P' \in H_{x'}$, E: bad (③ ← ②)

$\rho_x > \rho_{x'}$. By Prop 2 ii), $\rho_e < \mu(\xi_{H_x})$. ⇝ decrease.

iii) $P' \notin H_{x'} \cup H_{y'}$, E: bad (③ ← ①)

$\rho_x \geq \rho_e$. By Prop 2 ii), $\mu(\xi_{H_x}) > \rho_e$ ⇝ decrease.

iv) $P' \in H_{y'}$, E: bad (③ ← ③)

x-chart $\rho_x \geq \rho_e$. $x^\alpha y^\beta \mapsto e^{\alpha+\beta-1} y^\beta$ and $\beta = \rho_y < 1$ (Hy: good) implies $\mu(\xi_E) = \alpha + \beta - 1 < \alpha = \mu(\xi_{H_x})$. ⇝ decrease.