Generalized WF fixed points and anomalous dimensions from conformal multiplet recombination

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- Numerical conformal bootstrap methods based solely on Conformal Invariance and crossing symmetry provide accurate estimates of the low-lying spectrum of local operators of non-trivial quantum field theories describing critical systems like 3d Ising, $O(N)$-sigma models, Yang-Lee edge singularity, surface transitions and so on
- These results were traditionally obtained using Renormalization Group approach with diagrammatic expansions like ϵ expansion or Monte Carlo simulations
- How to explain the numerical agreement analytically?
  three main approaches:

① Large spin perturbation theory & Crossing Symmetry

② Mellin space approach to Conformal Bootstrap

③ Conformal multiplet recombination

Conformal multiplet recombination (Rychkov & Tan, 2015)

Consider the massless ϕ^4 theory in $d = 4 - \epsilon$ dimensions described by the action

$$S = \int d^d x \left[\frac{1}{2} (\partial\phi)^2 + \frac{1}{4!} g \phi^4 \right]$$

The lowest non-trivial order results for the anomalous dimensions of ϕ^4 theory in $d = 4 - \epsilon$ dimensions are reproduced assuming the following three Axioms

1. The WF fixed point is conformally invariant
2. Every local operator $\mathcal{O}$ of the WF fixed point reduces to a corresponding free field operator in the $\epsilon \to 0$ limit
3. ϕ^3 is a descendant of ϕ in the WF fixed point as a consequence of the e.o.m.

$$\partial^2 \phi = \frac{1}{3!} g \phi^3$$

* The Axiom 3 is too strong as it assumes e.o.m. which a priori have nothing to do with Conformal Symmetry
* In this talk:
  1. I wish to show (according to FG, A.Guerrieri, A. Petkou and C.Wen, PRL 118(2017)061601 and arXiv:1702.03938) that the "Axiom 3" is actually a Theorem of CFT, namely ❶ & ❷⇒"❸"
  2. define the Wilson-Fisher fixed point and its associated ϵ expansion using only CFT notions
  3. extend the analysis to other WF fixed points (ϕ^3 in $d = 6$, ϕ^6 in $d = 3, \dots$) and to *generalized free field theories*
  4. compute at the first non-trivial order in ϵ of the anomalous dimensions of scalar operators and OPE coefficients of $O(N)$-invariant theories
  5. compute the anomalous dimensions of infinite classes of spinning operators, including the non conserved ones (FG,arXiv:1711.05530)

- A CFT in d dimensions is defined by a set of local operators $\{\mathcal{O}_k(x)\}$ $x \in \mathcal{R}^d$ and their correlation functions

$$\langle \mathcal{O}_1(x_1) \ldots \mathcal{O}_n(x_n) \rangle$$

- Local operators can be multiplied. Operator Product Expansion:

$$\mathcal{O}_1(x)\mathcal{O}_2(0) \sim \sum_k \mathsf{c}_{12k}(x)\mathcal{O}_k(0)$$

- $\mathcal{O}_{\Delta,\ell,f}(x)$ are labelled by a scaling dimension Δ

$$\mathcal{O}_{\Delta,\ell,f}(\lambda x) = \lambda^{-\Delta}\mathcal{O}_{\Delta,\ell,f}(x)$$

  an $SO(d) \subset SO(d+1,1)$ representation ℓ (spin), and possibly a flavor index f
- among the local operators there are the identity and generally a (unique) energy -momentum tensor $T_{\mu\nu}(x) = \mathcal{O}_{d,2}(x)$

➡ a CFT has no much to do with Lagrangians, coupling constants or equations of motion.

- Acting with the $SO(d+1,1)$ Lie algebra $[J_{\mu,\nu}, P_\mu, K_\mu, D]$ on a state $|\Delta,\ell\rangle = \mathcal{O}_{\Delta,\ell}|0\rangle$ generates a whole representation of the conformal group. The local operator of minimal Δ (or $K_\nu|\Delta,\ell\rangle = 0$) is said a primary, the others are descendants
- Not all the primaries define irreducible representations:
- There are primaries admitting an invariant subspace: there is a descendant which is also primary. It corresponds to a null state i.e. a state of null norm
- ➾ Denoting with $[\Delta,\ell]$ a descendant primary and with $[\Delta',\ell']$ its parent primary (assumed to be described by a *symmetric traceless* tensor), in view of the fact that they belong to the same representation , they must share the eigenvalues $c_2, c_4, \ldots$ of all the Casimir operators $C_2, C_4, \ldots$

$$c_2(\Delta,\ell) = c_2(\Delta',\ell')\,;\, c_4(\Delta,\ell) = c_4(\Delta',\ell')\,;\ldots$$

- since $[\Delta,\ell]$ and $[\Delta',\ell']$ belong to the same rep.➾ $\Delta = \Delta' + n$ and the first two eq.s fix uniquely the possible pairs:

- There are three families of descendant primaries:

Parent Primary	Descendant Primary		
Δ'_k	Δ_k	ℓ	
$1-\ell'-k$	$1-\ell+k$	$\ell'+k$	$k=1,2,\dots$
$\frac{d}{2}-k$	$\frac{d}{2}+k$	ℓ'	$k=1,2,\dots$
$d+\ell'-k-1$	$d+\ell+k-1$	$\ell'-k$	$k=1,2,\dots,\ell$

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- There is a scalar parent primary $\mathcal{O}$ with the same scaling dimension $\Delta_{\mathcal{O}} = \frac{d}{2}-1$ of the canonical free scalar field ϕ_f
- $P^2\mathcal{O}(x)|0\rangle \equiv -\partial^2\mathcal{O}(x)|0\rangle$ is a null state
- If the theory is unitary ⇨ $\partial^2\mathcal{O}(x)=0$ ⇨ $\mathcal{O}(x)\equiv\phi_f(x)$

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- Similarly the scalar parent primary with $k>1$ describes a *generalized free conformal field theory* having a Lagrangian description with a ∂^{2k} kinetic term

A CFT in $d - \epsilon$ dimensions is a *smooth deformation* of the free field theory in d dimensions if

1. $\exists\, \mathcal{O}_i \leftrightarrow \mathcal{O}_i^f \,:\, \Delta_{\mathcal{O}_i} \equiv \Delta_{\mathcal{O}_i^f} + \gamma_i = \Delta_{\mathcal{O}_i^f} + \gamma_i^{(1)}\epsilon + \gamma_i^{(2)}\epsilon^2 + \dots$
2. $\mathcal{O}_i^f \times \mathcal{O}_j^f = \sum_k \mathrm{c}_{ijk}^f \mathcal{O}_k^f, \;\; \mathcal{O}_i \times \mathcal{O}_j = \sum_k (\mathrm{c}_{ijk}^f + O(\epsilon))\mathcal{O}_k$

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For generic d this deformation does not exist, since $\partial^2\phi$ with $\Delta_\phi = \Delta_{\phi_f} + \gamma_\phi^{(1)}\epsilon + \dots$ does not have a counterpart in the free theory *unless* there is a scalar ϕ_f^m with the same scaling dimensions of $\partial^2\phi_f$, i.e.
$m(\frac{d}{2}-1) = \frac{d}{2}+1$ ➩ $d, = 3\ m = 5;\ d = 4,\ m = 3;\ d = 6,\ m = 2$

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$m(\frac{d}{2}-1) = \frac{d}{2} + 1$ ➩ $d_m = 3,\; m = 5;\, d_m = 4,\; m = 3;\, d_m = 6,\; m = 2$
d_m is the *upper critical dimension* and the smooth deformation at $d = d_m - \epsilon$ is the Wilson-Fisher fixed point in the CFT approach.

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2. $\mathcal{O}_i^f \times \mathcal{O}_j^f = \sum_k \mathrm{c}_{ijk}^f \mathcal{O}_k^f, \ \ \mathcal{O}_i \times \mathcal{O}_j = \sum_k (\mathrm{c}_{ijk}^f + O(\epsilon))\mathcal{O}_k$

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$m(\frac{d}{2}-1) = \frac{d}{2}+1$ ➩ $d_m = 3,\ m = 5;\ d_m = 4,\ m = 3;\ d_m = 6,\ m = 2$
d_m is the *upper critical dimension* and the smooth deformation at $d = d_m - \epsilon$ is the Wilson-Fisher fixed point in the CFT approach.
For generalized free field theories (i. e. $k > 1$) $d_m = \frac{2k(m+1)}{m-1}$
Note that ϕ^m is a primary of the free theory and a descendant of the deformed (=interacting) theory

How to extract the anomalous dimensions γ_i in a WF fixed point?

1. Look for primaries $\mathcal{O}_f$ and $\mathcal{O}'_f$ such that in $d = d_m$ *(upper critical dimension)*
   $[\mathcal{O}_f] \times [\mathcal{O}'_f] = c_1 [\phi_f] + c_m [\phi_f^m] + \ldots$ (possible only if m is odd)
2. in $d = d_m - \epsilon$ the smoothly deformed CFT *(= interacting theory)*
   $[\phi^m]$ is absorbed by $[\phi]$
   $[\mathcal{O}] \times [\mathcal{O}'] = (c_1 + O(\epsilon))\,[\phi] + \ldots$
3. The matching conditions of these two fusion rules in the $\epsilon \to 0$ limit gives $\gamma_{\mathcal{O}}$ and $\gamma_{\mathcal{O}'}$ at the first non vanishing order in ϵ

❄ In particular we take $\mathcal{O}_f = \phi_f^p$ and $\mathcal{O}'_f = \mathcal{O}^f_{p,\ell}$ a spin ℓ primary made with $p + 1$ factors of ϕ_f and ℓ derivatives

➫ $\Delta_{\mathcal{O}^f_{p,\ell}} = (p+1)(\frac{d}{2} - 1) + \ell \quad (2p + 1 \geq m)$

Null states and poles

* Factorizing the 4-pt function in the [12]-channel in $d = d_m$

$$\langle \phi_f^p \, \mathcal{O}_{p,\ell}^f \mathcal{O}_{p,\ell}^f \phi_f^p \rangle = \sum_{\mathcal{O}} c_{\mathcal{O}}^2 \sum_{\alpha \in H_{\mathcal{O}}} \frac{\langle \phi_f^p \, \mathcal{O}_{p,\ell}^f | \alpha \rangle \langle \alpha | \mathcal{O}_{p,\ell} \phi_f^p \rangle}{\langle \alpha | \alpha \rangle} =$$

$$= c_1^2 \sum_{\alpha \in H_{\phi_f}} \frac{\langle \phi_f^p \, \mathcal{O}_{p,\ell}^f | \alpha \rangle \langle \alpha | \mathcal{O}_{p,\ell}^f \phi_f^p \rangle}{\langle \alpha | \alpha \rangle} + c_m^2 \sum_{\alpha \in H_{\phi_f^m}} \frac{\langle \phi_f^p \, \mathcal{O}_{p,\ell}^f | \alpha \rangle \langle \alpha | \mathcal{O}_{p,\ell}^f \phi_f^p \rangle}{\langle \alpha | \alpha \rangle} + \dots$$

with $\langle \phi_f | \phi_f \rangle = \langle \phi_f^m | \phi_f^m \rangle = 1$

* At the WF fixed point in $d = d_m - \epsilon$, ϕ^m and its descendants are absorbed by ϕ as a sub-representation H_χ with $\chi = P^2 \phi = -\partial^2 \phi$:

$$\langle \phi^p \, \mathcal{O}_{p,\ell} \mathcal{O}_{p,\ell} \phi^p \rangle =$$

$$c_1^2 \left(\sum_{\alpha \in H_{\phi_f}} \frac{\langle \phi_f^p \, \mathcal{O}_{p,\ell}^f | \alpha \rangle \langle \alpha | \mathcal{O}_{p,\ell}^f \phi_f^p \rangle}{\langle \alpha | \alpha \rangle} + \sum_{\beta \in H_\chi} \frac{\langle \phi^p \, \mathcal{O}_{p,\ell} | \beta \rangle \langle \beta | \mathcal{O}_{p,\ell} \phi^p \rangle}{\langle \beta | \beta \rangle} \right) + \dots$$

⇨ Matching condition: $c_m^2 \to c_1^2 \frac{\langle \phi^p \, \mathcal{O}_{p,\ell} | \chi \rangle \langle \chi | \mathcal{O}_{p,\ell} \phi^p \rangle}{\langle \chi | \chi \rangle}$

$$\langle \chi | \chi \rangle \equiv \langle \phi | K^2 P^2 | \phi \rangle = 8 d \Delta_\phi (\Delta_\phi - \Delta_{\phi_f})$$

⇨ The 4-pt function of the interacting theory has a pole at $\Delta_\phi = \Delta_{\phi_f}$

Computing $\langle \chi | \mathcal{O}_{p,\ell} | \phi^p \rangle$

- $\mathcal{O}_{p,\ell}$ is a symmetric traceless tensor with ℓ indices that can be represented as $\mathcal{O}_{p,\ell}(x,z) = \mathcal{O}_{\mu_1,\dots,\mu_\ell} z^{\mu_1} \cdots z^{\mu_\ell}$ $(z^\mu \in \mathbb{C}^\ell, z \cdot z = 0)$
- At $d = d_m - \epsilon$ we have the OPE
  $$\phi(x)\,\phi^p(0) = (c_1 + O(\epsilon))\,\frac{(x\cdot z)^\ell}{(x^2)^{\frac{\Delta_\phi + \Delta_{\phi^p} - \Delta_{p,\ell} + \ell}{2}}}[\mathcal{O}_{p,\ell}(0,z) + \text{descendants}]$$
- Applying ∂^2 to both sides $(\chi(x) = -\partial^2 \phi(x))$
  $$\chi(x)\,\phi^p(0) = (c_1 + O(\epsilon))\,\frac{\mathrm{M}_{\mathbf{p},\ell}\,(x\cdot z)^\ell}{(x^2)^{\frac{\Delta_\chi + \Delta_{\phi^p} - \Delta_{p,\ell} + \ell}{2}}}[\mathcal{O}_{p,\ell}(0,z) + \text{descendants}]$$

⇨ $\langle \chi | \mathcal{O}_{p,\ell} | \phi^p \rangle = c_1\, \mathrm{M}_{p,\ell}$

$$\mathrm{M}_{p,\ell} = \left(\Delta_\phi + \Delta_{\phi^p} - \Delta_{p,\ell} + \ell\right)\left(\Delta_{p,\ell} - \Delta_\phi - \Delta_{\phi^p} - 2 + d + \ell\right)$$

$$= \left(\gamma_\phi + \gamma_{\phi^p} - \gamma_{p,\ell}\right)(d - 2 + 2\ell) + O(\epsilon^2) = O(\epsilon)$$

➩ The matching conditions can be written more precisely in the form

$$\lim_{\epsilon\to 0} \frac{\mathrm{M}^2_{p,\ell}}{\langle\chi|\chi\rangle} = \lim_{\epsilon\to 0} \frac{\mathrm{M}^2_{p,\ell}}{4d(d-2)\gamma_\phi} = \left(\frac{c_m^2}{c_1^2}\right) \equiv \left(\frac{\langle\phi_f^m \phi_f^p \mathcal{O}^f_{p,\ell}\rangle}{\langle\phi_f \phi_f^p \mathcal{O}^f_{p,\ell}\rangle}\right)^2$$

$$\gamma_{\phi^p} = \gamma^{(1)}_{\phi^p}\epsilon + \gamma^{(2)}_{\phi^p}\epsilon^2 + \dots$$

$$\mathrm{M}^2_{p,\ell} = O(\epsilon^2) \Rightarrow \ \gamma^{(1)}_\phi = 0$$

* If $\ell = 0$ ➪ $\mathcal{O}_{p,0} = \phi^{p+1}$ ➪

$$\frac{\langle\phi_f^{m=2q+1}\phi_f^p\phi_f^{p+1}\rangle}{\langle\phi_f\phi_f^p\phi_f^{p+1}\rangle} = \binom{p}{q}\frac{\sqrt{(2q+1)!}}{(q+1)!}$$

Examples

In $d = 4$ and $m = 3$ (i.e. with a perturbing ϕ^4 potential) we get the recursion relation

$$\frac{\left(\gamma^{(1)}_{\phi^{p+1}} - \gamma^{(1)}_{\phi^p}\right)^2}{\gamma^{(2)}_\phi} = 12\, p^2$$

$$\Rightarrow \quad \gamma^{(1)}_{\phi^p} = \tfrac{\kappa_4}{2}\, p(p-1), \quad \kappa_4 = \pm\sqrt{12\gamma^{(2)}_\phi}$$

There is another way to calculate $\gamma^{(1)}_{\phi^3} = 3\kappa_4$: ϕ^3 is a primary descendant of ϕ_f of dimension $\Delta_{\phi_f} + 2$, then
$\Delta_{\phi^3} = 3\Delta_{\phi_f} + \gamma^{(1)}_{\phi^3}\epsilon = \Delta_{\phi_f} + 2, \Rightarrow \quad \gamma^{(1)}_{\phi^3} = 1$, then $\kappa_4 = \frac{1}{3}, \quad \gamma^{(2)}_\phi = \frac{1}{108}$

Similarly in $d = 3$ and $m = 5$ (multicritical Ising with a ϕ^6 potential)

$$\gamma^{(1)}_{\phi^p} \equiv \gamma^{(1)}_p = \frac{\kappa_3}{3}\, p(p-1)(p-2), \quad \kappa_3 = \pm\sqrt{10\gamma^{(2)}_\phi}$$

⇨ $\gamma^{(1)}_{\phi^5} = 20\kappa_3$, matching with the primary descendant of ϕ yields $\gamma^{(1)}_{\phi^5} = 2$, thus

$$\kappa_3 = \frac{1}{10}, \qquad \gamma^{(2)}_\phi = \frac{1}{1000}$$

- All these results in $d = 3$ and $d = 4$ coincide with those obtained with Feynman diagrams in quantum field theory

Generalizations

- ❋ For any generalized free field of dimension $\Delta_\phi = \frac{d}{2} - k$ in which ϕ^{2q+1} is a (null) descendant in the smoothly deformed theory
- ➩ in $d_{m=2q+1} - \epsilon$ there is a (generalized) WF critical point characterized by the following spectrum of anomalous dimensions

$$\gamma^{(1)}_{\phi^p} = \frac{q}{(q+1)_{q+1}}(p-q)_{q+1}\ , \quad (p>1)$$

$$\gamma^{(2)}_{\phi} = (-1)^{k+1} 2 \frac{(q+1)\left(\frac{k}{q}\right)_k}{k\left(\frac{(q+1)\,k}{q}\right)_k}(q)^2\left[\frac{((q+1)!)^2}{(2(q+1))!}\right]^3$$

$k = 1, 2, \ldots ; \quad q = 1, 2, \ldots ; \quad p = 1, 2, \ldots$

OPE coefficients in $d = 4$

Other results can be obtained by considering deformations of OPE free theories in which a ϕ_f^3 contribution on the RHS appears

$$[\phi_f] \times [\phi_f^4] = 2[\phi_f^3] + \sqrt{5}[\phi_f^5] + \text{spinning op.}$$

or

$$[\phi_f^2] \times [\phi_f^5] = \sqrt{10}[\phi_f^3] + 5\sqrt{2}[\phi_f^5] + \sqrt{21}[\phi_f^7] + \text{spinning op.}$$

the ϕ_f^3 contribution should be replaced by the conformal block of ϕ in the interacting theory.

$$c^2_{\phi\phi\phi^4} = 2\gamma_\phi^{(2)}\epsilon^2 + O(\epsilon^3) = \frac{1}{54}\epsilon^2 + O(\epsilon^3)$$
$$c^2_{\phi\phi^2\phi^5} = 5\gamma_\phi^{(2)}\epsilon^2 + O(\epsilon^3) = \frac{5}{108}\epsilon^2 + O(\epsilon^3);$$

OPE coefficients in generalized free field theories

In the generalized WF fixed point at $d = d_{2q+1} - \epsilon$ with $d_{2q+1} = \frac{2k(q+1)}{q}$

$$c^2_{\phi\,\phi^p\phi^{p+2r+1}} = \epsilon^2 \left[\frac{((q+1)!)^2}{(2(q+1))!}\right]^3 \frac{q^2\,(2r+p+1)!p!}{((q-r)!)^2((q+1+r)!)^2((r+p-q)!)^2}$$

$$\times \frac{\left(\frac{k}{q}\right)_k ((k-1)!)^2 \left(\frac{k}{q}\right)_{2k}}{\left(\frac{(q+1)k}{q}\right)_k \left[\left(\frac{(r+1)k}{q}\right)_k \left(\frac{-r\,k}{q}\right)_k\right]^2} + O(\epsilon^3)$$

$k = 1, 2, \ldots;\quad q = 1, 2, \ldots;\quad p = 1, 2, \ldots;\quad r = 1, 2, \ldots$

Spinning operators

$$\gamma^{(1)}_{p,\ell} = \gamma^{(1)}_{\phi^p} + 4\gamma^{(2)}_{\phi} \frac{\langle \phi_f^{2q+1} \phi_f^p \mathcal{O}^f_{p,\ell}\rangle}{\langle \phi_f \phi_f^p \mathcal{O}^f_{p,\ell}\rangle} \frac{1+q}{q(1+q\ell)} \frac{\left(\sqrt{(2q+1)!}\right)^3}{((q+1)!)^2}$$

- ❋ Difficult to apply when $\mathcal{O}^f_{p,\ell}$ is degenerate (then both p and ℓ large
- ❋ $p = 1$ corresponds to higher-spin conserved currents $\mathcal{O}_{1,\ell} \equiv \mathcal{J}_\ell$.
- ❋ $\langle \phi_f^{2q+1} \phi_f^p \mathcal{J}^f_\ell \rangle = 0$ ➪ $\gamma^{(1)}_{1,\ell} \equiv \gamma^{(1)}_\ell = 0$

d	$\gamma^{(1)}_{p,2}$	$\gamma^{(1)}_{p,3}$	$\gamma^{(1)}_{2,\ell}$
4	$\frac{(p-1)(4+3p)}{18}$	$\frac{p^2-3}{6}$	$\frac{1}{3} + \frac{2\,(-1)^\ell}{3(\ell+1)}$
3	$\frac{(p-1)(p-2)(5p+18)}{150}$	$\frac{(p-2)(7p^2+11p-90)}{210}$	0

Higher-spin conserved currents $\mathcal{J}_\ell$

- Conservation law: $\frac{\partial}{\partial x} \cdot D_z \, \mathcal{J}_\ell = 0$

  $D_z^\mu = \left(\frac{d+2}{2} + z \cdot \frac{\partial}{\partial z}\right) \frac{\partial}{\partial z_\mu} - \frac{1}{2} z^\mu \frac{\partial^2}{\partial z \cdot \partial z}$ (Todorov operator)

- Applying $\frac{\partial}{\partial x} \cdot D_z$ to both sides of the OPE

$$\mathcal{J}_\ell(x)\, \mathcal{O}(0) = \frac{c_{\mathcal{J}_\ell \mathcal{O} \mathcal{O}'} \, (-x \cdot z)^\ell}{(x^2)^{\frac{\Delta_{\mathcal{J}_\ell} + \Delta_{\mathcal{O}} - \Delta_{\mathcal{O}'} + \ell}{2}}} \mathcal{O}'(0) + \dots$$

⇒ $c_{\mathcal{J}_\ell \mathcal{O} \mathcal{O}'}(d - 4 + 2\ell)(\Delta_{\mathcal{J}_\ell} + \Delta_{\mathcal{O}} - \Delta_{\mathcal{O}'} - d + 2 - \ell) = 0$

⇒ $\Delta_{\mathcal{J}_\ell} = d - 2 + \ell$; If $\Delta_{\mathcal{O}} \neq \Delta_{\mathcal{O}'}$ ⇒ $c_{\mathcal{J}_\ell \mathcal{O} \mathcal{O}'} = 0$

Weakly broken HS currents $\mathcal{J}_\ell$

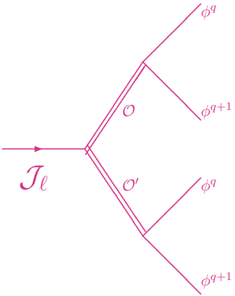
- useful tool: the five-point function $\langle\phi^q\phi^{q+1}\mathcal{J}_\ell\,\phi^q\phi^{q+1}\rangle =$ $\sum_{\mathcal{O}}\sum_{\mathcal{O}'} c_{\mathcal{O}} c_{\mathcal{O}'} c_{\mathcal{J}_\ell} \sum_{\alpha\in H_{\mathcal{O}}}\sum_{\beta\in H_{\mathcal{O}'}} \frac{\langle\phi^q\phi^{q+1}|\alpha\rangle\langle\alpha|\mathcal{J}_\ell|\beta\rangle\langle\beta|\phi^q\phi^{q+1}\rangle}{\langle\alpha|\alpha\rangle\langle\beta|\beta\rangle}$
- $\mathcal{J}_\ell$ is conserved in the free theory. If $\Delta_{\mathcal{O}}\neq\Delta_{\mathcal{O}'}$ ➩ $\langle\mathcal{O}|\mathcal{J}_\ell|\mathcal{O}'\rangle = 0$

- The 5-pt function of the interacting theory has a double pole at $\Delta_{\mathcal{O}}=\Delta_{\mathcal{O}'}=\Delta_{\phi_f}$
- Matching condition:

$$\lim_{\epsilon\to 0}\frac{(2\gamma_\phi-\gamma_\ell)(d-4+2\ell)}{\gamma_\phi\,2d(d-2)} = 2q+1 \equiv \frac{d+2}{d-2}$$

➩ $$\gamma_\ell^{(2)} = 2\gamma_\phi^{(2)}\left(1-\frac{(\nu+1)(\nu+2)}{(\ell+\nu-1)(\ell+\nu)}\right)\quad \nu=\frac{d}{2}-1$$

➩ $\gamma_2^{(2)} = 0$ as expected by stress tensor conservation



$O(N)$- invariant models

- generalized free theories with scalar fields $\phi_i, i = 1, 2, \ldots, N$ transforming as vectors under $O(N)$
- $\gamma^{(i)}_{p,s} \equiv$ anomalous dimensions of symmetric traceless rank-s tensors $\phi^{2p}\,\phi_{i_1}\phi_{i_2}\ldots\phi_{i_s} - \text{traces}$

➩ for $d_u = 4k$ $\gamma^{(1)}_{p,s} = \frac{s(s-1)+p(N+6(p+s)-4)}{N+8}$, $\gamma^{(2)}_{\phi} = \frac{(-1)^{k+1}(k)_k(N+2)}{2k(2k)_k(N+8)^2}$

➩ for $d_u = 3k$

$$\gamma^{(1)}_{p,s} = \frac{(2p+s-2)(s(s-1)+p(3N+10(p+s)-8))}{3(3N+22)}$$

$$\gamma^{(2)}_{\phi} = \frac{(-1)^{k+1}(k/2)_k(N+2)(N+4)}{8k(3k/2)_k(3N+22)^2}$$

O(N)-invariant models, symmetric traceless weakly broken HS currents

$d = 4 - \epsilon$ matching condition:

$$\lim_{\epsilon\to 0} \frac{(2\gamma_\phi - \gamma_\ell)(d-4+2\ell)}{\gamma_\phi\, 2d(d-2)} = \frac{N+6}{N+2}$$

➯ $$\gamma_\ell^{(2)} = 2\gamma_\phi^{(2)}\left(1 - \frac{2(N+6)}{\ell(\ell+1)(N+2)}\right)$$

$d = 3 - \epsilon$ matching condition:

$$\lim_{\epsilon\to 0} \frac{(2\gamma_\phi - \gamma_\ell)(d-4+2\ell)}{\gamma_\phi\, 2d(d-2)} = \frac{(N+1)(N+8)}{N^2+2N+4}$$

➯ $$\gamma_\ell^{(2)} = 2\gamma_\phi^{(2)}\left(1 - \frac{6(N+4)}{(2\ell-1)(2\ell+1)(N^2+2N+4)}\right)$$

Conclusions

1. Wilson-Fisher fixed points in $d - \epsilon$ can be seen as smooth deformations of free-field theories only using CFT notions, with no reference to Lagrangians, coupling constants or equations of motion
2. The anomalous dimensions of scalar and spinning operators at the first non vanishing order are easily obtained
3. $O(N)$ symmetric models and generalized free fields allow to define a more general class of WF fixed points
4. Higher order calculations require more constraints from conformal bootstrap equations.