Are locality and renormalisation reconcilable ?

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Brain teaser 1

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$$S := "1 + \frac{1}{2} + \cdots + \frac{1}{n} + \cdots" = \sum_{k=1}^{\infty} k^{-1}"$$

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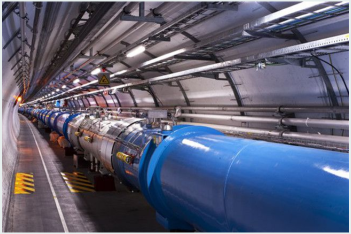

Brain teaser 2

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Example

$$\left(f_1(z) = z \,\wedge\, f_2(z) = \tfrac{1}{z}\right) \Longrightarrow f_1^{\mathrm{reg}}(0)\, f_2^{\mathrm{reg}}(0) = 0 \neq 1 = (f_1\, f_2)^{\mathrm{reg}}(0).$$

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while preserving locality / multiplicativity.

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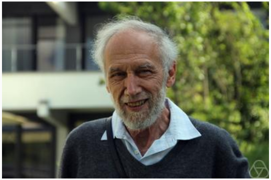

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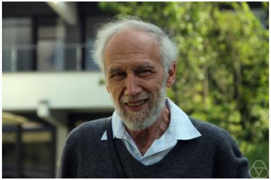

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A third multivariate approach

(with P. Clavier, L. Guo and B. Zhang)

using algebraic **locality**

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Independence of measurements

Observable $\mathcal{O} \longrightarrow$ Measurement $\langle \mathcal{O} \rangle \in \mathbb{C}$

$$\underbrace{\mathcal{O}_1 \text{ and } \mathcal{O}_2}_{\text{independent}} \quad \underbrace{\Longrightarrow}_{\text{locality}} \quad \underbrace{\langle \mathcal{O}_1 \star \mathcal{O}_2 \rangle = \langle \mathcal{O}_1 \rangle \langle \mathcal{O}_2 \rangle}_{\text{multiplicativity}}.$$

Locality in quantum field theory

Independence of events in QFT

An object is only directly influenced by its immediate surroundings. Two events situated in different locations do not influence each other.

Independence of measurements

Observable $\mathcal{O} \longrightarrow$ Measurement $\langle \mathcal{O} \rangle \in \mathbb{C}$

$$\underbrace{\mathcal{O}_1 \text{ and } \mathcal{O}_2}_{\text{independent}} \quad \underbrace{\Longrightarrow}_{\text{locality}} \quad \underbrace{\langle \mathcal{O}_1 \star \mathcal{O}_2 \rangle = \langle \mathcal{O}_1 \rangle \langle \mathcal{O}_2 \rangle}_{\text{multiplicativity}}.$$

Analogy: separation of variables ($n = n_1 + n_2$)

$$\underbrace{\int_{\mathbb{R}^n} f_1(x_1)\, f_2(x_2)\, dx_1\, dx_2}_{x_1 \text{ and } x_2 \text{ independent}} = \underbrace{\left(\int_{\mathbb{R}^{n_1}} f_1(x_1)\, dx_1 \right) \cdot \left(\int_{\mathbb{R}^{n_2}} f_2(x_2)\, dx_2 \right)}_{\text{multiplicativity}}.$$

A locality multivariate setup

A locality multivariate setup

Our plan

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LOCALITY

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Algebraic locality

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Let $U \subset \mathbb{R}^n$ be an open subset and $\epsilon \geq 0$. Two functions $\phi, \psi \in \mathcal{D}(U)$ are independent i.e., $\phi \top \psi$ whenever $d\left(\mathrm{Supp}(\phi), \mathrm{Supp}(\psi)\right) > \epsilon$.

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For $\epsilon = 0$, this amounts to disjointness of supports, otherwise to ϵ-separation of supports.

Locality in geometry (inspired by work of V. Deng)

Geometric data

- $\pi : E = E_+ \oplus E_- \to M$ a $\mathbb{Z}_2$-graded vector bundle on a closed manifold M;
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Counterexample

Equip $\mathbb{R}$ with the locality relation $x \top y \Longleftrightarrow x + y \notin \mathbb{Z}$.
$(\mathbb{R}, \top, +)$ is NOT a locality semi-group: for $U = \{1/3\}$ we have $(1/3, 1/3) \in (U^\top \times U^\top) \cap \top$ but $1/3 + 1/3 = 2/3 \notin U^\top$.

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$$f_1 \top f_2 \Longrightarrow \mathcal{S}(f_1 \underbrace{+}_{m_G} f_2) = \mathcal{S}(f_1) \underbrace{\cdot}_{m_{\mathcal{A}}} \mathcal{S}(f_2).$$

MULTIVARIATE GERMS

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Evaluating a fraction with a linear pole at zero

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In our approach, a given choice of locality fixes the value 0.

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Back to the brain teaser

$\ell := z_1 \perp z_2 =: L \Longrightarrow \frac{z_1}{z_2} \in \mathcal{M}_-(\mathbb{C}^2)$

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$$\begin{array}{rcl} \mathcal{M}(\mathbb{C}^\infty) \times \mathcal{M}(\mathbb{C}^\infty) \supset \top & \longrightarrow & \mathcal{M}(\mathbb{C}^\infty) \\ \left(f = \frac{h(\vec{\ell})}{\vec{L}^{\vec{s}}}, \widetilde{f} = \frac{\widetilde{h}(\vec{\widetilde{\ell}})}{\vec{\widetilde{L}}^{\vec{\widetilde{s}}}} \right) & \longmapsto & f \cdot \widetilde{f} = \frac{h(\vec{\ell}) \cdot \widetilde{h}(\vec{\widetilde{\ell}})}{\vec{L}^{\vec{s}} \cdot \vec{\widetilde{L}}^{\vec{\widetilde{s}}}}. \end{array}$$

Partial product on multivariate meromorphic germs

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$$\begin{array}{ccc} \mathcal{M}(\mathbb{C}^\infty) \times \mathcal{M}(\mathbb{C}^\infty) \supset \top & \longrightarrow & \mathcal{M}(\mathbb{C}^\infty) \\ \left(f = \frac{h(\vec{\ell})}{\vec{L}^{\vec{s}}}, \widetilde{f} = \frac{\widetilde{h}(\vec{\widetilde{\ell}})}{\vec{\widetilde{L}}^{\vec{\widetilde{s}}}} \right) & \longmapsto & f \cdot \widetilde{f} = \frac{h(\vec{\ell}) \cdot \widetilde{h}(\vec{\widetilde{\ell}})}{\vec{L}^{\vec{s}} \cdot \vec{\widetilde{L}}^{\vec{\widetilde{s}}}}. \end{array}$$

Back again to the brain teaser

$((z_1 - z_2) \perp (z_1 + z_2)) \Longrightarrow \frac{z_1 - z_2}{z_1 + z_2} = (z_1 - z_2) \cdot \frac{1}{z_1 + z_2}$.

Multivariate decomposition theorem

Theorem (L. Guo, S.-P., B. Zhang/ N. Berline, M. Vergne 2015)

$\mathcal{M}(\mathbb{C}^k) = \mathcal{M}_-(\mathbb{C}^k) \oplus^\perp \mathcal{M}_+(\mathbb{C}^k)$, where $\mathcal{M}_-(\mathbb{C}^k) \ni \frac{h(\ell_1, \cdots, \ell_n)}{L_1^{s_1} \cdots L_n^{s_n}}$ with $\mathrm{Dep}(h) \perp \langle L_1, \cdots, L_n \rangle$ and $f_1 \perp f_2 \Longleftrightarrow \mathrm{Dep}(f_1) \perp \mathrm{Dep}(f_2)$.

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- Regularised evaluator $\mathrm{ev}_0{}^{\mathrm{reg}} : \mathcal{M}(\mathbb{C}^k) \underset{\pi_+}{\longrightarrow} \mathcal{M}_+(\mathbb{C}^k) \underset{ev_0}{\longrightarrow} \mathbb{C}$

$$f \longmapsto f^{\mathrm{reg}}(0) := \mathrm{ev}_0{}^{\mathrm{reg}}(f)$$

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Multiplicativity of the regularised evaluator

The regularised evaluator is multiplicative on mutually independent germs: $f_1 \perp f_2 \Longleftrightarrow (f_1 \cdot f_2)^{\mathrm{reg}}(0) = (f_1{}^{\mathrm{reg}}(0))\,(f_2{}^{\mathrm{reg}}(0))\,.$

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RENORMALISATION and LOCALITY
reconciled

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Our task

Build a locality character $\Phi^{\mathrm{reg}} : (\mathcal{A}, \top_A, m_A) \longrightarrow (\mathbb{C}, \cdot)$

$$a_1 \top_A a_2 \Longrightarrow \Phi^{\mathrm{reg}}\left(m_A(a_1, a_2)\right) = \Phi^{\mathrm{reg}}(a_1) \cdot \Phi^{\mathrm{reg}}(a_2). \tag{1}$$

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Back to our main protagonist

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Theorem

A locality morphism $\Phi : (\mathcal{A}, \top) \longrightarrow (\mathcal{M}(\mathbb{C}^k), \perp)$ *gives rise to a locality character*

$$\Phi^{\mathrm{reg}} := ev_0{}^{\mathrm{reg}} \circ \Phi : (\mathcal{A}, \top) \longrightarrow \mathbb{C}.$$

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Summary

A multivariate regularisation provides a renormalisation scheme which respects locality .

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1. exponential integrals/sums on a cone **C**:
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3. Feynman amplitudes $(z_e, e \in \mathcal{E}(\Gamma)) \longmapsto \prod_e G(z_e)$, with $G(z_e)$ the kernel of $(\Delta + m^2)^{-1+z_e}$, on each edge e of the graph Γ.

Conclusions

One can renormalise at poles while preserving locality

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Open questions

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Group actions

- Group G acting on $\mathcal{A}$ which induces an action on $\Phi(\mathcal{A}) \subset \mathcal{M}(\mathbb{C}^\infty)$

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- Group G acting on $\mathcal{A}$ which induces an action on $\Phi(\mathcal{A}) \subset \mathcal{M}(\mathbb{C}^\infty)$
- How does it act on $\Phi^{\mathrm{reg}}(\mathcal{A})$?

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THE END

THANK YOU !

EXTRA SLIDES

Further examples

Probability theory: independence of events

Given a probability space $\mathcal{P} := (\Omega, \Sigma, P)$ and two events $A, B \in \Sigma$:

$$A \top B \Longleftrightarrow P(A \cap B) = P(A)\, P(B).$$

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Geometry: transversal manifolds

Given two submanifolds L_1 and L_2 of a manifold M:

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Number theory: coprime numbers

Given two positive integers m, n in $\mathbb{N}$:

$$m \top n \Longleftrightarrow m \wedge n = 1.$$

Locality cont'd.

Local functionals in QFT

Functionals F on fields ϕ of the form $F(\phi) = \int_M f\left(j_x^k(\phi)\right)\, dx$, where $j_x^k(\phi)$ is the k-th jet of ϕ at x. Here, $\mathrm{Supp}\,(f\,(\psi)) \subset \mathrm{Supp}\,(\psi)\,.$

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Link between various concepts of locality

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