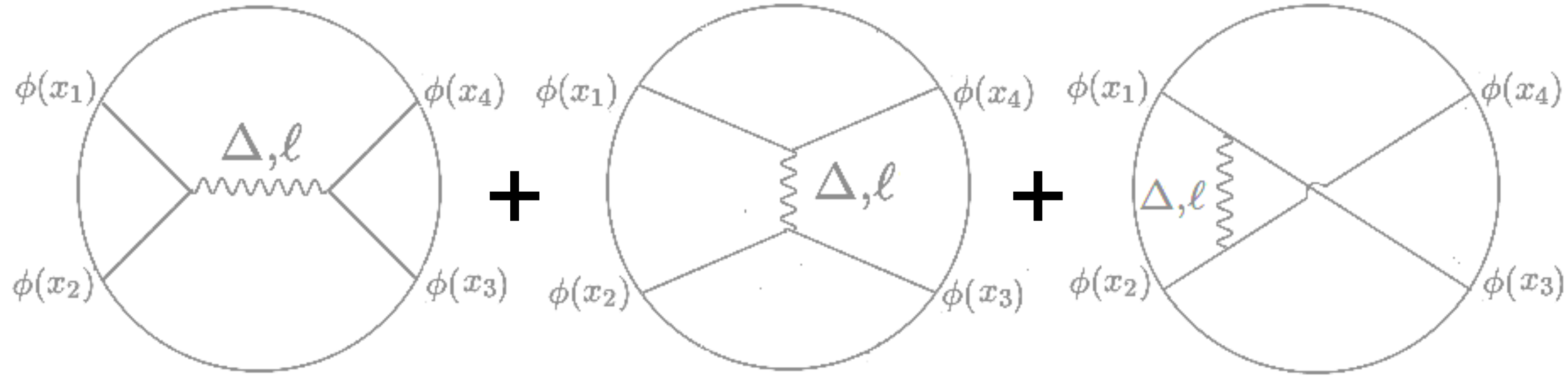
$$\mathcal{M}(s,t) = \sum_{\Delta,\ell} c_{\Delta,\ell}$$




# Demystifying Mellin bootstrap

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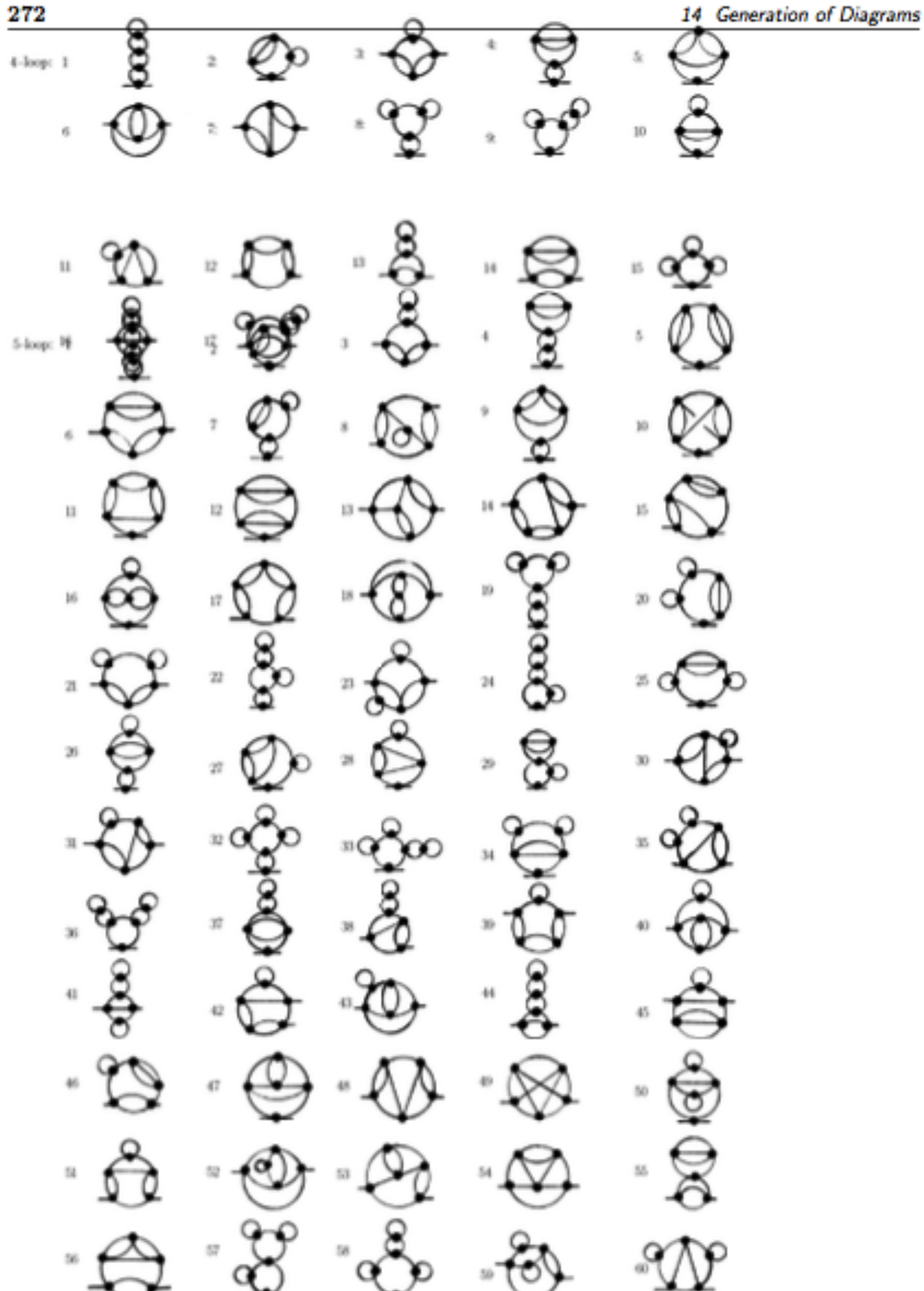
*OIST, March 2018.*

# In the beginning….
(almost)

# Conformal Scream




You were asked to verify this……

# With bootstrap (ok, not quite there but getting better)

## Anomalous dimension at 3 loops from Caron-huot or Mellin Bootstrap

```
Quit[]
```

- Expression either using Caron-huot or using Polyakov bootstrap.

```
$Assumptions := {ℓ ∈ Integers && ℓ ≥ 0}
```

$$\gamma2 := -\Big(\Big(2\,\text{Gamma}\Big[\frac{\text{ell}+\Delta}{2}\Big]^2\,\text{Gamma}\Big[1-h+qq+rr+\frac{\Delta0}{2}\Big]^2\,\text{Gamma}[\Delta0]\;\text{Gamma}[1-h+\Delta0]\;\text{Gamma}\Big[\frac{1}{2}(2+\text{ell}+\Delta-2\Delta\phi)\Big]\;\text{Gamma}\Big[1+rr+\frac{\Delta0}{2}-\Delta\phi\Big]^2\;\text{Gamma}[\Delta\phi]^2\;\text{Gamma}\Big[\frac{\text{ell}}{2}-h+qq+\frac{\Delta}{2}+\Delta\phi\Big]\;\text{Sin}\Big[\frac{1}{2}\pi(\Delta0-2\Delta\phi)\Big]^2\Big)\Big/\Big(\pi^2\,qq!\,rr!\;\text{Gamma}\Big[1-h+\frac{\Delta0}{2}\Big]^2\,\text{Gamma}\Big[\frac{\Delta0}{2}\Big]^2\,\text{Gamma}[1-h+rr+\Delta0]\;\text{Gamma}\Big[\frac{1}{2}(2+\text{ell}-2h+2qq+2rr+\Delta+\Delta0)\Big]^2\;\text{Gamma}\Big[\frac{1}{2}(\text{ell}+2h+\Delta-2\Delta\phi)\Big]\;\text{Gamma}\Big[-1+\frac{\text{ell}+\Delta}{2}+\Delta\phi\Big]\Big)\Big)$$

```
Δ0 := 2 - 2 / 3 ϵ + 19 / 162 ϵ^2
h := 2 - ϵ / 2
Δϕ := 1 - ϵ / 2 + ϵ^2 / 108 + aa ϵ^3
Δ := 2 Δϕ + ell
```

- $\frac{\text{Gamma}\left[1-h+qq+rr+\frac{\Delta0}{2}\right]^2}{\text{Gamma}\left[1-h+\frac{\Delta0}{2}\right]^2}$ means that we need to set qq,rr to zero.

In Mellin bootstrap we do not take any Feynman diagram input!!

## Anomalous dimension of $J_l$ to 3 – loops.

```
Series[(2 - 2 / 3 ϵ - 34 / 81 ϵ^2) γ2 /. {qq → 0, rr → 0, ell → ℓ}, {ϵ, 0, 3}] //
  FullSimplify // Normal
```

$$-\frac{\epsilon^2}{9\,\ell\,(1+\ell)}-\frac{\epsilon^3\,\big(63+68\,\ell-22\,\ell^2+36\,\ell\,(1+\ell)\,(\text{EulerGamma}+\text{PolyGamma}[0,\ell])\big)}{486\,\ell^2\,(1+\ell)^2}$$

In August 2016, Mathematica 11.0 was released. It was timed aptly since our Mellin bootstrap papers were to come out in a month.

```
In[60]:= InverseMellinTransform[Gamma[-t]^2 Gamma[s+t]^2, t, y]
Out[60]= y^s Gamma[s]^4 Hypergeometric2F1Regularized[s, s, 2 s, 1 - y]
```

*–Look it is your destiny.*

"There's more in calculus too. Like Green's functions for general equations in general domains. And, long awaited (at least by me): Mellin transforms. (They've been a favorite of mine ever since they were central to a 1977 particle physics paper of mine.)"

*–S. Wolfram*

# Sampling of results

$$O_{\Delta,\ell} \sim \phi \partial_{\mu_1} \cdots \partial_{\mu_\ell} \phi$$

$$\Delta_\ell = d - 2 + \ell + \left(1 - \frac{6}{\ell(\ell+1)}\right)\frac{\epsilon^2}{54} + \frac{373\ell^2 - 384\ell - 324 + 109\ell^3(\ell+2) - 432\ell(\ell+1)H_\ell}{5832\ell^2(\ell+1)^2}\epsilon^3$$

[agrees with Derkachov, Gracey, Manashov's 3-loop Feynman diagram calc!!]

$$\begin{aligned}\frac{C_\ell}{C_\ell^{MFT}} &= 1 + \frac{\epsilon^2\left(\ell(1+\ell)\left(H_{2\ell} - H_{\ell-1}\right) - 1\right)}{9\ell^2(1+\ell)^2} \\ &+ \frac{\epsilon^3}{486\ell^2(1+\ell)^3}\Big[27 + (59 - 22\ell)\ell - 36\ell(1+\ell)^2 H_\ell^2 + (1+\ell)H_\ell\left(22\ell^2 + 4\ell - 27 + 36\ell(1+\ell)H_{2\ell}\right) \\ &+ (1+\ell)\Big(\left(27 + 32\ell - 22\ell^2\right)H_{2\ell} + 18\ell(1+\ell)\big(3H_{2\ell}^{(2)} - 2H_\ell^{(2)}\big)\Big)\Big] + O(\epsilon^4)\,.\end{aligned}$$

[not available using FD; 1609.00572, 1611.08407 using Mellin bootstrap, see Alday et al 2017 for 4th order cT]

Hjalmar Mellin (Finnish mathematician)

$$f(x) = \int_{c-i\infty}^{c+i\infty} \tilde{f}(s) x^{-s}\, ds$$

# Outline

- Caron-huot/Alday in Mellin space $\ell > 1$
- Polyakov in Mellin space or Mellin bootstrap $\ell \geq 0$
- Difficulties in Mellin space

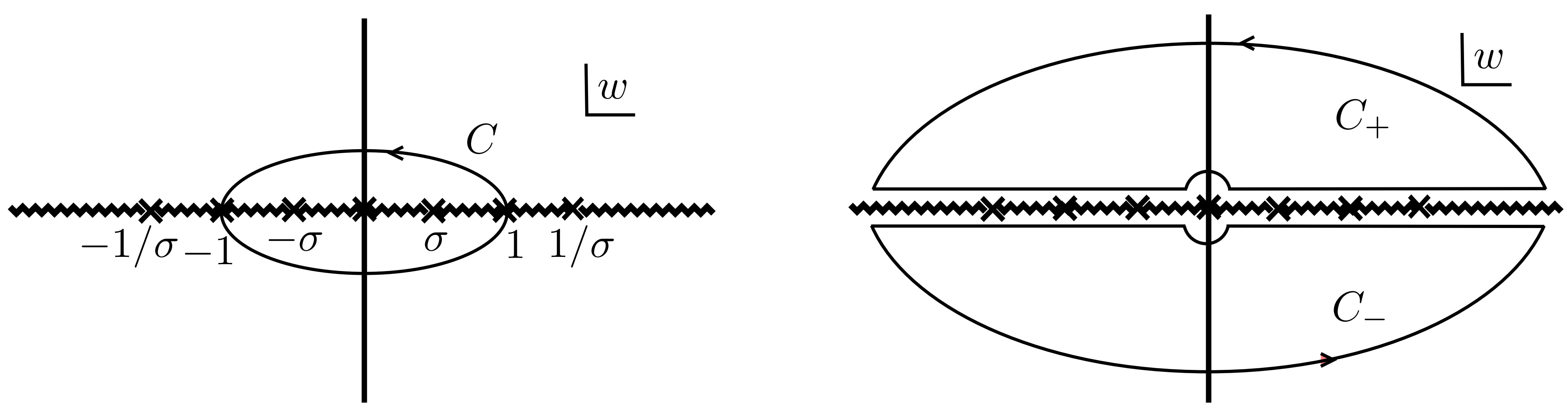
**Based partly insights drawn from 1609.00572, 1611.08407, 1612.05032, 1709.06110 but mainly on ongoing work with R.Gopakumar 1xxx.xxxxx**

- I will begin by reviewing an amazing formula derived by Caron-Huot in 1703.00278.

- Caron-huot derived an inversion formula. Given a 4 point function, using this formula, we can compute the OPE data directly. Similar to LSPT.
- Logic is to start with the Euclidean OPE inversion formula [using Harmonic function analysis] and analytically continue to Lorentzian signature.
- Eventually, only the discontinuity of the four point function is needed.

Usual block

$$c(\Delta,\ell) = N(\Delta,\ell)\int d^2z\,\mu(z,\bar z)F_{\Delta,J}(z,\bar z)\langle\phi\phi\phi\phi\rangle$$ "Trivial"




$$u = z\bar z, v = (1-z)(1-\bar z)$$

$$z = \frac{4\rho}{(1+\rho)^2}, \bar z = \frac{4\bar\rho}{(1+\bar\rho)^2}$$

$$\rho = \sigma w, \bar\rho = \sigma/w$$

Add "spurious pieces" to close contour to the centre. Get unusual block!

!!dimension, spin interchanged!!
Analyticity in spin

“non-trivial”

$$\int_0^1 dudv\mu(u,v) \times G_{\underbrace{J+d-1,\Delta-d+1}}(u,v) \times dDisc[4pt]$$

$$dDisc[v^t] = 2\sin^2(\pi t)v^t$$

# Lorentzian OPE integral in Mellin space

$$\gamma_\ell = \int_0^1 \frac{dy}{y^2} k_\beta(y) \mathrm{dDisc}\, \mathcal{G}(1-y)$$

$$\ell > 1$$

**Mellin bootstrap removes this restriction.**

$$k_\beta(y) = y^{\beta/2} {}_2F_1(\beta/2, \beta/2, \beta, y)$$

$$\beta = \Delta + \ell$$

$$y^{\ell}\,{}_2F_1(\Delta_\phi+\ell,\Delta_\phi+\ell,2\Delta_\phi+2\ell,y)=$$

$$\int_{-i\infty}^{i\infty}\frac{dt}{2\pi i}\,\Gamma^2(-t)\Gamma^2(\Delta_\phi+t)(1-y)^t\,\frac{\Gamma(2\Delta_\phi+2\ell)}{\Gamma^2(\Delta_\phi)\Gamma^2(\Delta_\phi+\ell)}\,{}_3F_2\left(\begin{matrix}-\ell,2\Delta_\phi+\ell-1,-t\\ \Delta_\phi,\Delta_\phi\end{matrix}\right)$$

**Truncating, polynomial of degree $\ell$ in t**

$$\gamma_\ell = \int_0^1 \frac{dy}{y^2} k_\beta(y) \mathrm{dDisc}\ \mathcal{G}(1-y)$$

$$\Delta = 2\Delta_\phi + \ell + \gamma_\ell$$

**“double trace” operators**

$$\int dy y^{2\Delta_\phi - 2} y^\ell {}_2F_1(\Delta_\phi + \ell, \Delta_\phi + \ell, 2\Delta_\phi + 2\ell, y)$$

$$\times \mathrm{dDisc}_{y=1} (1-y)^{\Delta'/2 - \Delta_\phi} {}_2F_1(\frac{\Delta'}{2}, \frac{\Delta'}{2}, \Delta' - h + 1, 1-y)$$

$$= \# \int dt \Gamma^2(-t) \Gamma^2(\Delta_\phi + t) Q_{\ell,0}^{2\Delta_\phi + \ell}(t) \ \times \boxed{\sin^2(\pi t)}$$

$$\times {}_3F_{2r}\left( \begin{matrix} \frac{\Delta'}{2} - 2\Delta_\phi - t + 1, \frac{\Delta'}{2}, \frac{\Delta'}{2} \\ \Delta' - h + 1, \frac{\Delta'}{2} - t \end{matrix} \right) \Gamma(\frac{\Delta'}{2} - 2\Delta_\phi - t + 1)$$

**3F2s appear again and again**

# Aside on continuous Hahn polynomials

$$G_{\Delta,\ell}(u,v) = \int_{-i\infty}^{i\infty} ds dt\, u^s v^t \Gamma^2(-t)\Gamma^2(s+t)\Gamma^2(\Delta_\phi - s)\frac{\Gamma(\frac{\Delta-\ell}{2} - s)\tilde{\Gamma}(\frac{(2h-\Delta-\ell)}{2} - s)}{\Gamma^2(\Delta_\phi - s)} P_{\Delta,\ell}(s,t)$$

$$Q^{\Delta}_{\ell,0}(t) \propto P_{\Delta,\ell}(\frac{\Delta-\ell}{2}, t)$$

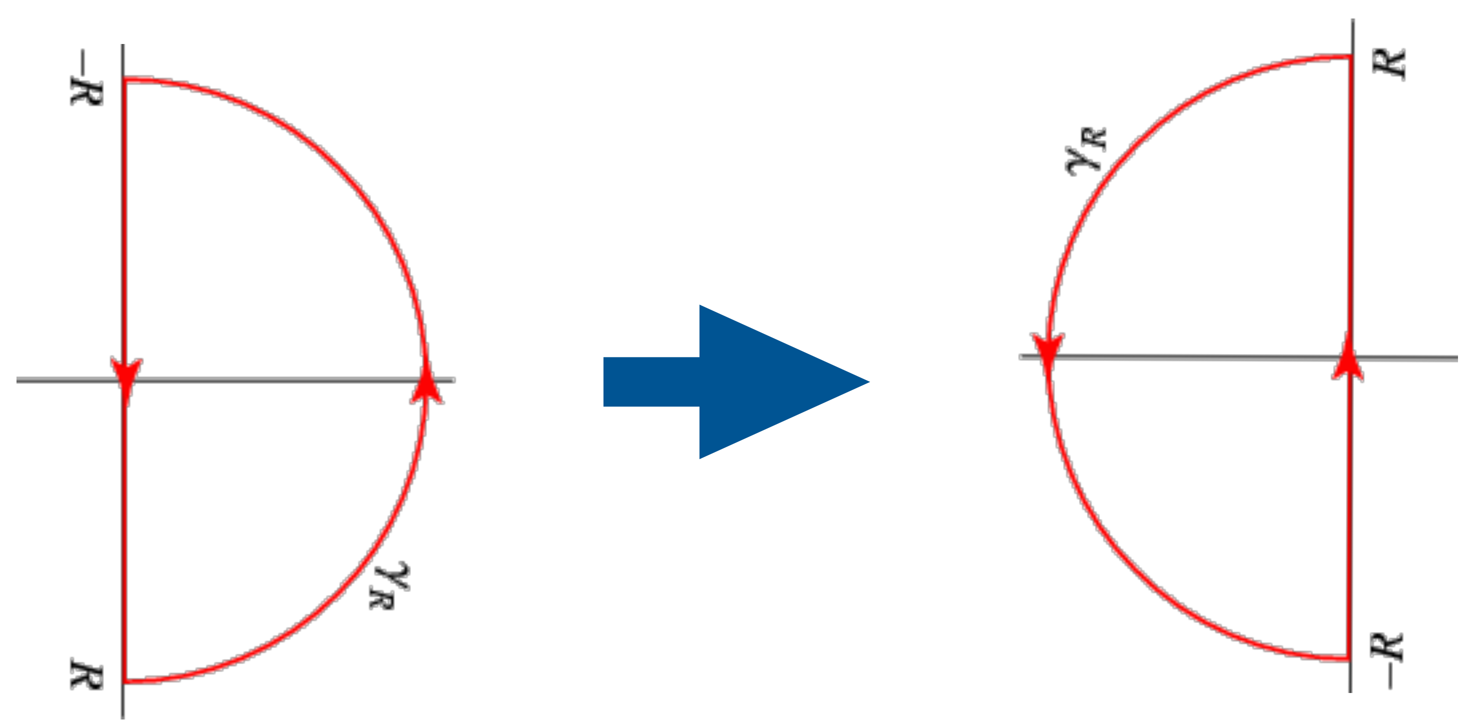
$$\int_{-i\infty}^{i\infty} dt\, \Gamma^2(\sigma + t)\Gamma^2(-t) Q^{2\sigma+\ell}_{\ell,0}(t) Q^{2\sigma+\ell'}_{\ell',0}(t) \propto \delta_{\ell,\ell'}$$

introduced by Pafnuty Chebyshev in 1875 (Chebyshev 1907) and rediscovered by Wolfgang Hahn (Hahn 1949)

**Mellin-Barnes representation**

$$ {}_3F_2\left[\begin{matrix}-\ell,\ 2s+\ell-1,\ s+t \\ s\ \ ,\ \ s\end{matrix};1\right] $$

$$ = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} d\xi\, \frac{\Gamma^2(s)\Gamma(2s+\ell-1+\xi)\Gamma(s+t+\xi)\Gamma(-\xi)}{\Gamma(s+t)\Gamma(\ell+1-\xi)\Gamma(2s+\ell-1)\Gamma^2(s+\xi)}\ell! $$



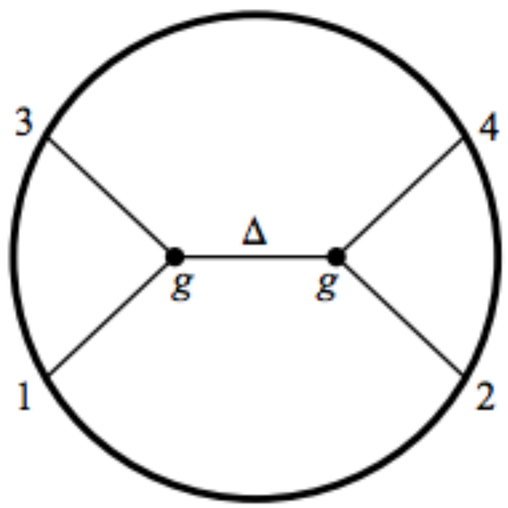

Usual

Useful for large spin asymptotic

Dey, Ghosh, AS

# Witten diagrams in Mellin space

Penedones, Paulos……




$$= \frac{1}{2s-\Delta}\frac{\Gamma^2(\Delta_\phi+\frac{\Delta-2h}{2})}{\Gamma(1+\Delta-h)}\,{}_3F_2\left[\begin{matrix}1-\Delta_\phi+\frac{\Delta}{2},1-\Delta_\phi+\frac{\Delta}{2},\frac{\Delta}{2}-s\\ 1+\frac{\Delta}{2}-s,1+\Delta-h\end{matrix};1\right]$$

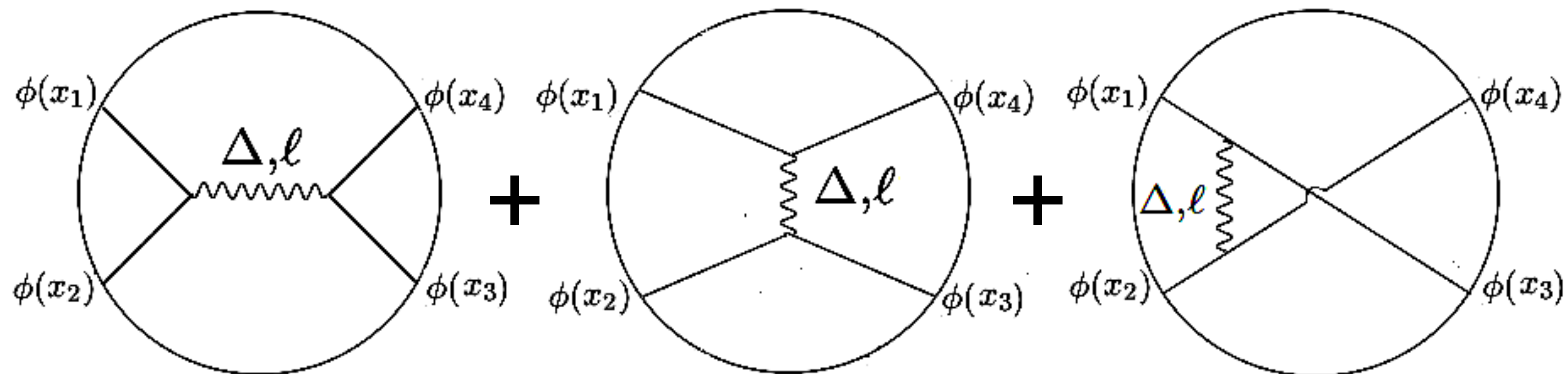
$$= \frac{1}{4\pi i\Gamma(\Delta_\phi-s)^2}\int_{-i\infty}^{i\infty} d\nu q[\nu]q[-\nu]$$

$$q[\nu]=\frac{\Gamma(\frac{h+\nu}{2}-s)\Gamma^2(\frac{2\Delta_\phi-h+\nu}{2})}{(\Delta-h)+\nu}$$

$$\mathcal{M}(\nu)=q[\nu]q[-\nu]$$

Spectral function. Polyakov (1974!!) gave a different physical argument for the double poles. Exactly the same form!! Momentum/position space are not ideal to see the simplification we saw in Mellin space.

$$\mathcal{M}(s,t) = \sum_{\Delta,\ell} c_{\Delta,\ell}$$




Residue fixed by conformal invariance; Mack polynomials

Mellin amplitudes should factorize on physical poles.
m: descendant label, $\ell$: spin
Q: polynomials in t

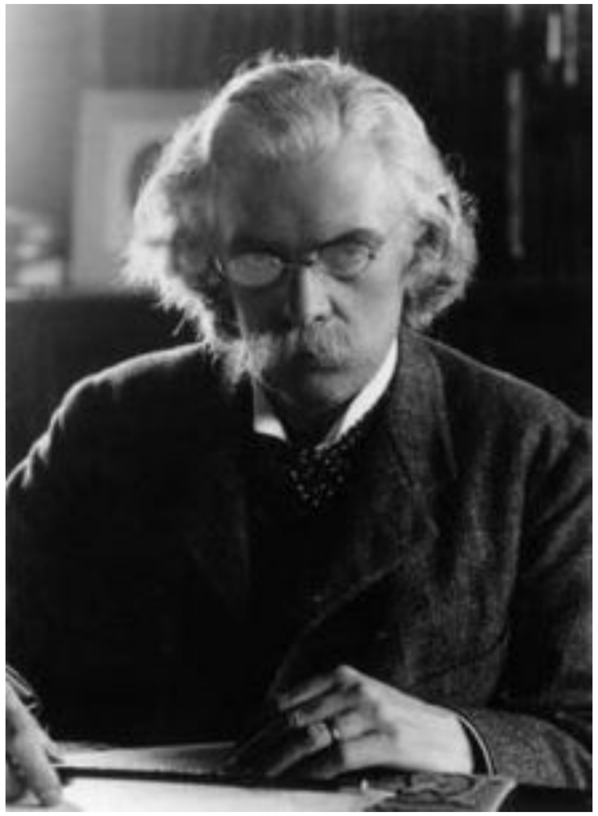
$$\sum_m \frac{Q^{\Delta}_{\ell,m}(t)}{s - \frac{\Delta-\ell}{2} - m} + \boxed{\cdots\cdots}$$

Modern Mellin amplitude literature: Mack; Penedones; Paulos;……….

.....

Difference between usual conformal block expansion and Witten diagram expansion lies in the …. pieces. The regular piece for usual conformal block is exponential at infinity while for the Witten block it is polynomial. These polynomial pieces will have an important role in my story.

**Strictly**



Mellin's advisor!

Gosta Mittag-Leffler
(Swedish mathematician)

Thm: Existence of meromorphic functions with prescribed poles

**Roughly**

$$\Gamma(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(x+n)} \qquad + \Gamma_1(x)$$

Entire function

$$\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(x+n)} \frac{\Gamma(y)}{\Gamma(y-n)}$$

famous formula responsible for the birth of string theory!

no extra entire function piece

# A quick derivation of the Witten diagram meromorphic piece

$$\frac{\Gamma(\tau/2-s)\Gamma(\tilde{\tau}/2-s)}{\Gamma^2(\Delta_\phi-s)} = \frac{B(\underbrace{\tau/2-s,}\Delta_\phi-\tau/2)\Gamma(\tilde{\tau}/2-s)}{\Gamma(\Delta_\phi-\tau/2)\Gamma(\Delta_\phi-s)}$$

$$= \sum_n \frac{(-1)^n(\Delta_\phi-\tau/2-n)_n}{(\tau/2+n-s)\Gamma(\Delta_\phi-\tau/2)} \frac{\Gamma(\tilde{\tau}/2-\tau/2-n)}{\Gamma(\Delta_\phi-\tau/2-n)} + \text{regular}$$

$$= \frac{\Gamma(h-\Delta)}{(\tau/2-s)\Gamma^2(\Delta_\phi-\tau/2)}\, {}_3F_2\left[\begin{matrix}\tau/2-s, \tau/2-\Delta_\phi+1, \tau/2-\Delta_\phi+1 \\ \tau/2-s+1, \Delta-h+1\end{matrix};1\right]$$

$$+\text{regular}$$

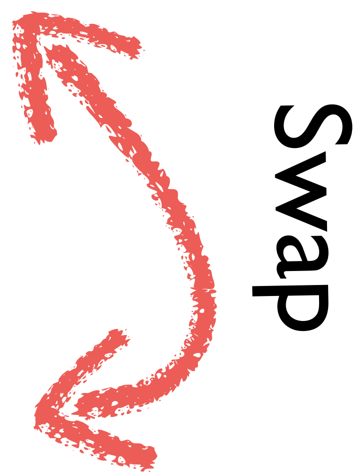
This gets multiplied by the Mack polynomial

$$G_{\Delta,\ell}(u,v) = \int_{-i\infty}^{i\infty} ds dt\, u^s v^t \Gamma^2(-t)\Gamma^2(s+t)\Gamma^2(\Delta_\phi - s)\ \left(W^{(s)}_{\Delta,\ell}(s,t) + \rho(s,t)\right) \sin^2(\Delta_\phi - s)$$

**Explains connection between large spin perturbation and MB for the WF fixed point.**

- As an aside, 3F2's need to be analytically continued for the scalar exchange at the WF point.
- This is a key step which enables us to get the anomalous dimension and OPE coefficients for the scalar exchange.

- In the traditional approach we expand in terms of partial waves which are consistent with OPE.



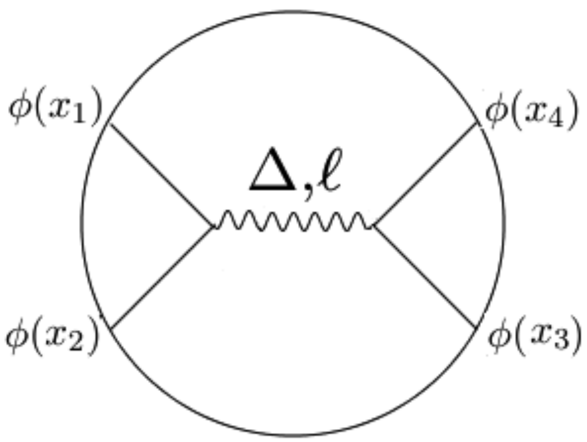

- Impose crossing symmetry as constraint.

# New approach: key point

- In the new approach we expand in terms of crossing symmetric partial waves.
- Impose OPE consistency as constraint.

$$G_{\Delta,\ell}(u,v) = \int_{-i\infty}^{i\infty} ds dt \, u^s v^t \Gamma^2(-t)\Gamma^2(s+t)\Gamma^2(\Delta_\phi - s) \, \left( W^{(s)}_{\Delta,\ell}(s,t) + \rho(s,t) \right) \sin^2(\Delta_\phi - s)$$

$$w_{\Delta,\ell}(u,v) = \int_{-i\infty}^{i\infty} ds dt u^s v^t \Gamma^2(-t)\Gamma^2(s+t)\Gamma^2(\Delta_\phi - s) \left( W^{(s)} + W^{(t)} + W^{(u)} + \rho_c(s,t) \right)$$



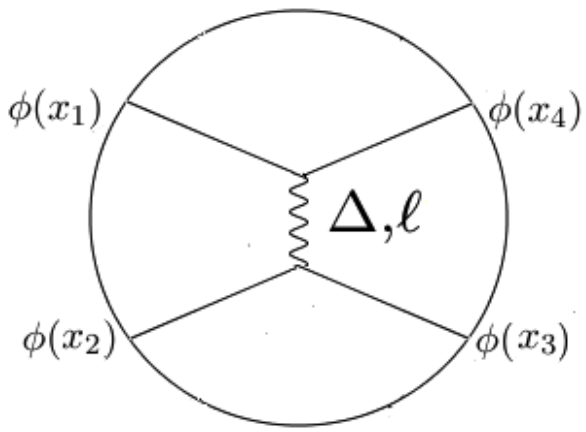

$$\sim \int \frac{ds dt}{(2\pi i)^2} \Gamma(-t)^2 \Gamma(s+t)^2 \Gamma(\Delta_\phi - s)^2 \frac{\sum_m Q^{\Delta}_{\ell,m}(t)}{s - \frac{\Delta-\ell}{2} - m} u^s v^t$$

$$\rightarrow u^{\Delta_\phi + r} \log u, u^{\Delta_\phi + r}$$

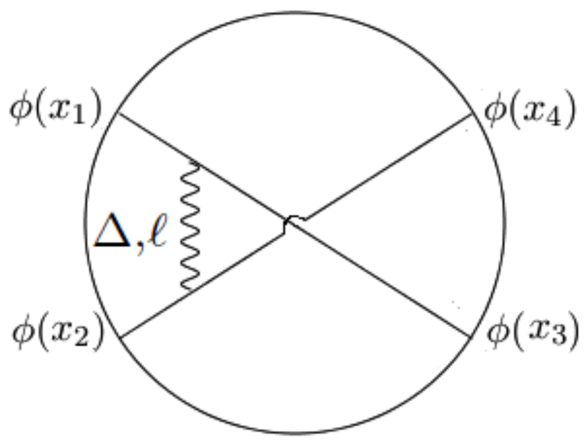
$$\rightarrow \text{incompatible with s} - \text{channel OPE}$$

$$\rightarrow \text{conditions needed to cancel}$$

NB: log u is genuinely spurious as it comes singly and not due to expanding some u-power




$$\sim \int \frac{dsdt}{(2\pi i)^2}\Gamma(-t)^2\Gamma(s+t)^2\Gamma(\Delta_\phi - s)^2 \frac{\sum_m Q^{\Delta}_{\ell,m}(s-\Delta_\phi)}{t+\Delta_\phi - \frac{\Delta-\ell}{2} - m} u^s v^t$$

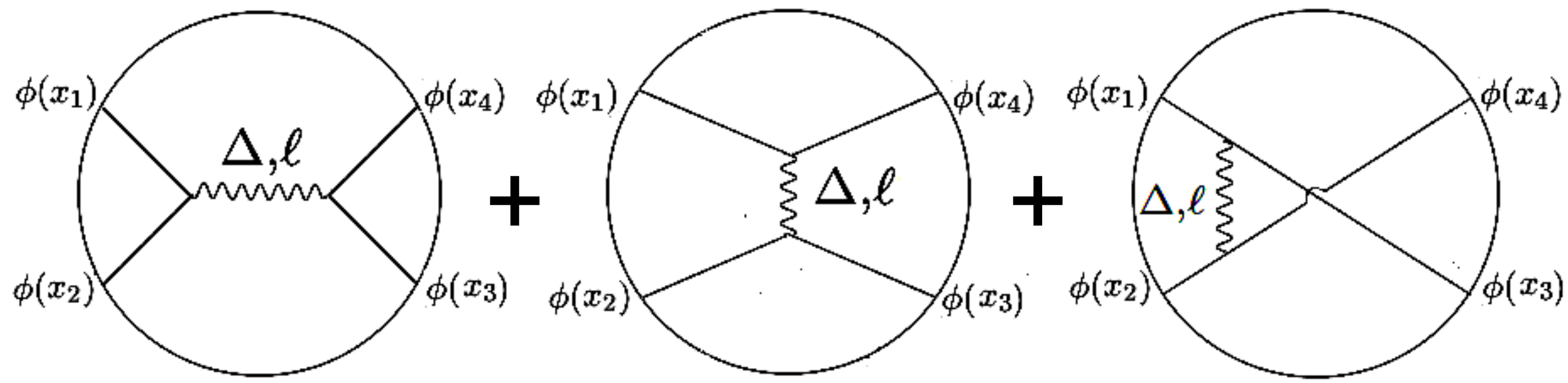



$$\sim \int \frac{dsdt}{(2\pi i)^2}\Gamma(-t)^2\Gamma(s+t)^2\Gamma(\Delta_\phi - s)^2 \frac{\sum_m Q^{\Delta}_{\ell,m}(t)}{\Delta_\phi - s - t - \frac{\Delta-\ell}{2} - m} u^s v^t$$

$$\to u^{\Delta_\phi + r}\log u,\, u^{\Delta_\phi + r}$$

Demand that the spurious poles in s+t+u cancel to have compatibility with OPE

$$\mathcal{M}(s,t) = \sum_{\Delta,\ell} c_{\Delta,\ell}$$




$$\text{should not have } (s-\Delta_\phi - r)^0, (s-\Delta_\phi - r)^1 \text{ terms}$$

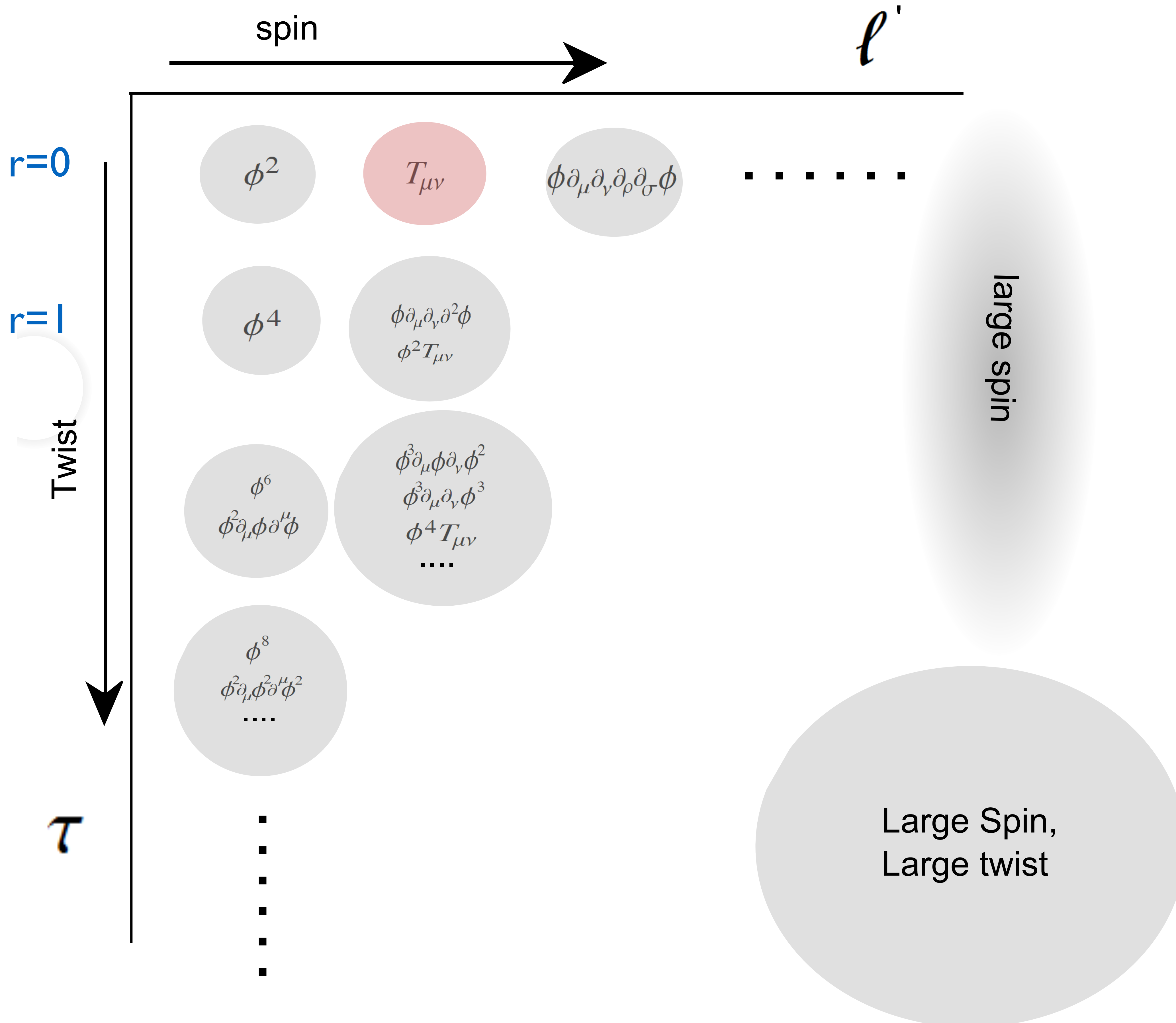
$$\forall r \in \mathbb{Z}_+$$

This should hold for all t.

Lots of conditions!!

Show scalar example on board [if time]

# Structure of equations



To get anomalous dimension of twist 4 scalar $Q_{\ell=0}\ basis$

$$s=\Delta_\phi+1$$

Equation in s channel has contributions from all twist scalars

$$s=\Delta_\phi$$

Equation in s channel has contributions from all twist scalars

BUT difference has contributions only from twist 2 and twist 4 scalars and spin 2 stress tensor only

$$\implies \Delta_{\phi^4}=4\Delta_\phi+2\epsilon, \quad C_{\phi\phi\phi^4}=\frac{\epsilon^2}{54}$$

Similarly for spin-2 twist 4 which is non-degenerate

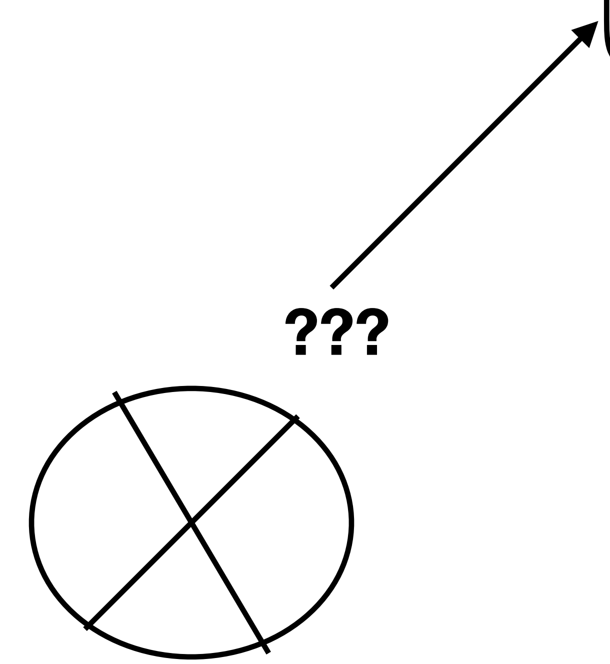
- Turns out that the Caron-huot formula and the Mellin space formula for anomalous dimension for the double twist operators is exactly the same upto cubic order but is different at fourth order.

- The difference is due to contact terms [Rajesh’s talk] that we can/must add to the Witten diagram basis which we need a clever argument to fix.

- The contact term ambiguity persists even at spin-0.

# Way forward in Mellin bootstrap

$$w_{\Delta,\ell}(u,v) = \int_{-i\infty}^{i\infty} ds dt u^s v^t \Gamma^2(-t)\Gamma^2(s+t)\Gamma^2(\Delta_\phi - s)\left(W^{(s)} + W^{(t)} + W^{(u)} + \rho_c(s,t)\right)$$



- Need an equation that fixes the contact term. May be a differential equation? Will help address issues such as completeness.

- For a while we thought that the “split-function” representation would help resolve this issue but it does not.

- May be even without fixing this issue one can hope to address general questions like transcendentality that Fernando was using—prove it.

- Can we shed some light on the AdS dual to the WF point?