Bootstrap CFTs with S_N symmetry

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Mar 20th, 2018 OIST

CFTs with global symmetry

OPE has a further decomposition according to irreps of the global symmetry

$$\phi_i \times \phi_j \sim \sum_I P^{(I)}_{ijkl} \left(\sum_{O^{(I)}} O^{(I)}_{kl} \right)$$

$O^{(I)}_{kl}$ transforms in the I-th irrep of the symmetry group.

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Similarly for four point function,

$$\langle \overbrace{\phi_i(x_1)\phi_j(x_2)}\overbrace{\phi_k(x_3)\phi_l(x_4)} \rangle = \frac{1}{x_{12}^{2\Delta_\phi} x_{34}^{2\Delta_\phi}} \sum_I P^{(I)}_{ijkl} \left(\sum_{\mathcal{O}\in I} \lambda^2_{\mathcal{O}} g_{\Delta_{\mathcal{O}}, l_{\mathcal{O}}}(u,v) \right)$$

$P^{(I)}_{ijkl}$ are projectors, written in terms of invariant tensors (CG- coefficients).
Projectors satisfy

$$P^{(I)}_{ijkl} P^{(I)}_{lkmn} = \delta_{IJ} P^{(I)}_{ijmn},$$

$$\sum_I P^{(I)}_{ijkl} P^{(J)}_{lkmn} = \delta_{il}\delta_{jk},$$

CFTs global symmetry

Suppose the CFT preserves O(N) symmetry, operators that appear in $\phi^i \times \phi^j$ OPE belong to three different channel

$$n \otimes n \rightarrow S \oplus A \oplus T$$

bounding the lowest operator in S-channel gives us a kink, corresponding to the O(N) vector. The projectors are written in terms of δ_{ij}.

$$P^{(S)}_{ijkl} = \frac{1}{n}\delta_{ij}\delta_{kl}$$
$$P^{(T)}_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{ik}\delta_{jl}) - \frac{1}{n}\delta_{ij}\delta_{kl}$$
$$P^{(A)}_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} - \delta ik\delta_{jl})$$

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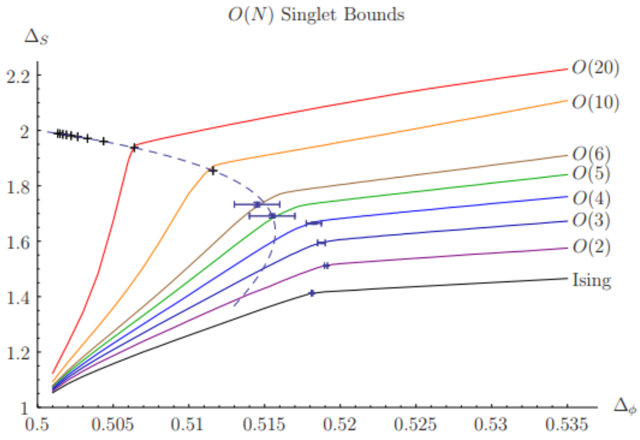

[Kos, Poland, Simmons-Duffin '13]

F_4 bootstrap

Suppose the symmetry has an extra invariant tensor d_{ijk},

$$\phi^i \times \phi^j \sim \lambda_{\phi\phi\phi} d_{ijk} \phi^k + \ldots .$$

We have an extra channel channel

$$n \otimes n \to S \oplus n \oplus A \oplus T' + \ldots .$$

F_4 bootstrap

F_4 group has such a rank-3 invariant tensor. Take ϕ^i to in the $n = 26$ dimensional representation of F_4 group.
Assuming

- First spin-0 operator in n-channel has scaling dimension Δ_ϕ,
- second spin-0 operator in n-channel has scaling dimension larger or equal to Δ_n,
- operators with other spins or in other channels satisfies the unitarity bound.

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Since F_4 is a subgroup of $O(26)$, bounding S-channel here would still give us the O(n) vector model kink.

F_4 bootstrap

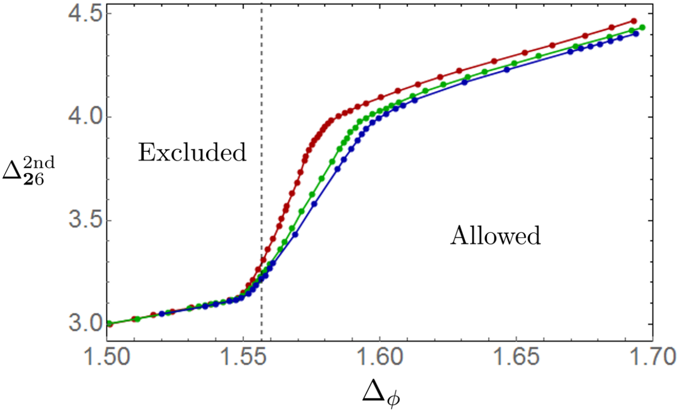

Figure: F_4 bootstrap in $D = 5$.[Pang, JR, Su '16]

F_4 bootstrap

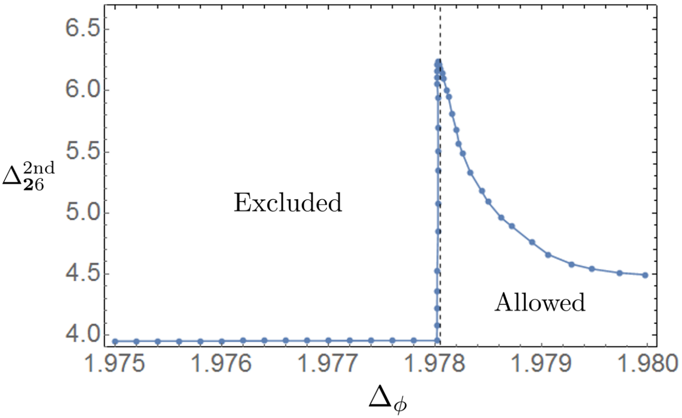

Figure: F_4 bootstrap in $D = 5.95$.[Pang, JR, Su '16]

F_4 bootstrap

Consider the ϕ^3 theory

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^i)(\partial_\mu \phi^i) + \frac{g}{6} d_{ijk} \phi^i \phi^j \phi^k,$$

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Beta function:

$$\beta(g) = -\frac{\epsilon}{2} g + \frac{19}{56} g^3 + \ldots .$$

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The operator $d_{ijk}\phi^j\phi^k$ becomes a descendant at this fixed point, as a result of the equation of motion

$$\Box \phi^i \sim d_{ijk} \phi^j \phi^k$$

S_N bootstrap

We consider the case when ϕ^i lives in the $n = N - 1$ representation of S_N (the reason to do this would become clear later).

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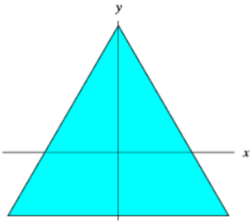
For S_3, it is simply

$$e^1 = (\frac{\sqrt{3}}{2}, -\frac{1}{2}), \quad e^2 = (-\frac{\sqrt{3}}{2}, -\frac{1}{2}), \quad e^3 = (0, 1).$$



S_N bootstrap

There exist an invariant tensor

$$d_{ijk} = \sum_{\alpha} e_i^{\alpha} e_j^{\alpha} e_k^{\alpha}$$

In this case

$$\mathbf{n} \otimes \mathbf{n} \to \mathbf{S} \oplus \mathbf{n} \oplus \mathbf{T}' \oplus \mathbf{A}$$

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$$\begin{aligned}
P^{(1)}_{ijkl} &= \frac{1}{n}\delta_{ij}\delta_{kl}, \\
P^{(n)}_{ijkl} &= \frac{n^3}{(n-1)(n+1)^2} d_{ijm} d_{klm}, \\
P^{(T')}_{ijkl} &= \frac{1}{2}\delta_{il}\delta_{jk} + \frac{1}{2}\delta_{ik}\delta_{jl} - \frac{1}{n}\delta_{ij}\delta_{kl} - \frac{n^3}{(n-1)(n+1)^2} d_{ijm} d_{klm}, \\
P^{(A)}_{ijkl} &= \frac{1}{2}\delta_{il}\delta_{jk} - \frac{1}{2}\delta_{ik}\delta_{jl}.
\end{aligned}$$

S_N bootstrap

Using crossing symmetry

$$\langle \phi_i(x_1)\phi_j(x_2)\phi_k(x_3)\phi_l(x_4)\rangle = \langle \phi_i(x_1)\phi_j(x_2)\phi_k(x_3)\phi_l(x_4)\rangle .$$

Each projector would give us one equation.

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Each projector would give us one equation.

$$\sum_I \sum_{O\in I} \lambda^2_{\phi\phi O} \vec{V}^{(I)}_{\Delta_O, l_{\mathcal{O}}}(u,v) = 0\,, \quad \text{with} \quad I \in \{1^+, n^+, T'^+, A^-\}\,,$$

where

$$\vec{V}^{(1^+)}_{\Delta_O,l_{\mathcal{O}}}(u,v) = \begin{pmatrix} 0 \\ 0 \\ \frac{F}{n} \\ -\frac{H}{n} \end{pmatrix}, \vec{V}^{(n^+)}_{\Delta_O,l_{\mathcal{O}}}(u,v) = \begin{pmatrix} F \\ 0 \\ \frac{F}{1-n} \\ \frac{H}{n-1} \end{pmatrix}, \vec{V}^{(T'^+)}_{\Delta_O,l_{\mathcal{O}}}(u,v) = \begin{pmatrix} -F \\ \frac{F}{2} \\ \frac{F\left(n^2-n+2\right)}{2(n-1)n} \\ \frac{H\left(n^2-n-2\right)}{2(n-1)n} \end{pmatrix}, \vec{V}^{(A^-)}_{\Delta_O,l_{\mathcal{O}}}(u,v) = \begin{pmatrix} 0 \\ -\frac{F}{2} \\ \frac{F}{2} \\ \frac{H}{2} \end{pmatrix}.$$

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Here F and H are short for $F_{\Delta,l}$ and $H_{\Delta,l}$, defined by

$$\begin{aligned}F_{\Delta,l}&=v^{\Delta_\phi}G_{\Delta,l}(u,v)-u^{\Delta_\phi}G_{\Delta,l}(v,u),\\H_{\Delta,l}&=v^{\Delta_\phi}G_{\Delta,l}(u,v)+u^{\Delta_\phi}G_{\Delta,l}(v,u).\end{aligned}$$

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Using these projectors, we could derived the following crossing equations

$$\sum_I \sum_{O \in I} \lambda^2_{\phi\phi O} \vec{V}^{(I)}_{\Delta_O, l_O}(u,v) = 0\,, \quad \text{with} \quad I \in \{1^+, n^+, A^-\}\,,$$

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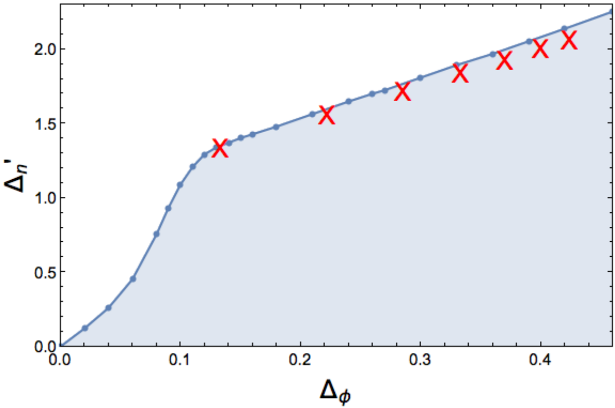
In two space time dimensions, we get the [JR, Su '17]



Minimal models of W3 algebra

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W_3 algebra contains a spin 3 current.

$$[L_n, W_n] = (2n - m) W_{n+m},$$

$$[W_n, W_m] = \frac{C}{3 \cdot 5!}(n^2 - 4)(n^2 - 1) n \delta_{n+m,0} + b^2 (n - m) \Lambda_{n+m}$$

$$+ (n - m)\left[\tfrac{1}{15}(n + m + 2)(n + m + 3) - \tfrac{1}{6}(n + 2)(m + 2)\right] L_{n+m},$$

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Roughly speaking, the minimal models are

$$L = \partial_\mu \phi^\dagger \partial_\mu \phi + V(\phi^\dagger, \phi)$$

with $V(\phi^\dagger, \phi)$ being a polynomial invariant under a Z_3 rotation, $\phi \to \exp\left(\frac{2}{3}\pi \mathrm{i}\right)\phi$, with highest degree term to be $(\phi^\dagger \phi)^{p-2}$.

Explain the spectrum

Their central charges are

$$C_p = 2\left(1 - \frac{12}{p(p-1)}\right).$$

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Scaling dimensions (of $\mathcal{W}_3$ primaries) are

$$\Delta\left[\Phi\begin{pmatrix} n & m \\ n' & m' \end{pmatrix}\right] = \frac{1}{12p(p+1)}\Big(3((p+1)(n+n') - p(m+m'))^2 + ((p+1)(n-n') - p(m-m'))^2 - 12\Big),$$

where m, n, m', n' and p are positive integers whose range are $n + n' \leq p - 1,\ m + m' \leq p$ and $p \geq 4$.

Minimal models of W3 algebra

In the bootstrap result

$$\Delta_\phi = 2 \times \Delta\left[\Phi\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}\right] = \frac{2(p-3)}{3(p+1)},$$

and

$$\Delta'_n = 2 \times \Delta\left[\Phi\begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}\right] = \frac{4(2p-3)}{3(p+1)},$$

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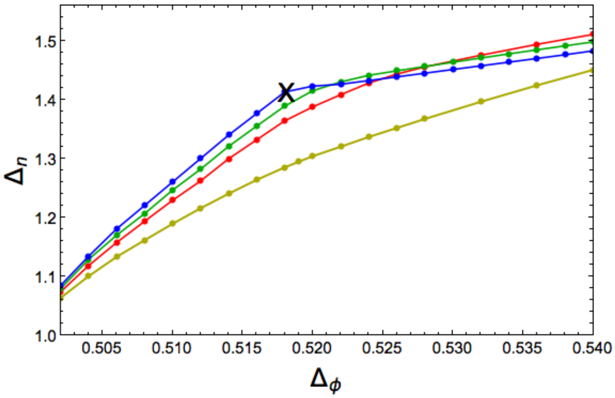
Minimal models of Virasoro algebra shows a similar behaviour when bootstrapping 2D CFT's with Z_2. [Rychkov, Vichi '09]

Result

The first operator in **n**-channel [JR, Su '17]



Check the actual symmetry

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For single correlator bootstrap, their crossing equations look the same.

$\lambda\phi^4$ theory

Write down the scalar Lagrangian

$$L = \frac{1}{2}\partial_\mu\phi^i\partial_\mu\phi^i + \frac{g_1}{8}d_{ijm}d_{klm}\phi^i\phi^j\phi^k\phi^l + \frac{g_2}{8}(\phi^i\phi^i)^2$$

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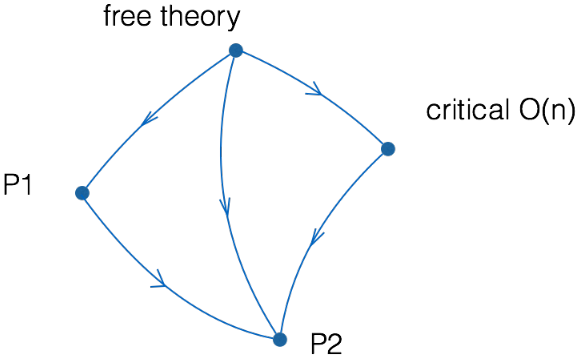
It has four fixed points



At the O(N) fixed point $g_1 = 0$.

The large N limit

One could compare the ϵ-expansion result with Ising model, for example:

$$\Delta_\phi^{P_1} = 1 - \frac{\epsilon}{2} + \frac{\left(n^2+8n+7\right)\epsilon^2}{108(n+3)^2} + \dots,$$

$$\Delta_\phi^{P_2} = 1 - \frac{\epsilon}{2} + \frac{\left(n^4-9n^3+31n^2-45n+22\right)\epsilon^2}{108\left(n^2-5n+8\right)^2} + \dots.$$

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The corresponding Ising series is

$$\Delta_\sigma^{\text{Ising}} = 1 - \frac{\epsilon}{2} + \frac{\epsilon^2}{108} + \dots.$$

The large N limit

Operator	$\Delta_{n\to\infty}$
$\phi \in \mathrm{n}_-$	$\Delta_\sigma^{\mathrm{Ising}}$
$\phi^2 \in \mathrm{S}$	$\Delta_\epsilon^{\mathrm{Ising}}$
$\phi^4 \in \mathrm{S}$, 1st	$2 \times \Delta_\epsilon^{\mathrm{Ising}}$
$\phi^4 \in \mathrm{S}$, 2st	$\Delta_{\epsilon'}^{\mathrm{Ising}}$
$\phi^2 \in \mathrm{n}_+$	$\Delta_\epsilon^{\mathrm{Ising}}$
$\phi^2 \in \mathrm{T}'$	$2 \times \Delta_\sigma^{\mathrm{Ising}}$

Table: large N spectrum for P_1.

The large N limit

Operator	$\Delta_{n\to\infty}$
$\phi \in \mathsf{n}_-$	$\Delta_\sigma^{\text{Ising}}$
$\phi^2 \in \mathsf{S}$	$D - \Delta_\epsilon^{\text{Ising}}$
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Table: large N spectrum for P_2.

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The operators

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transforms in the N-dimension representation.
In $\sigma \times \sigma$ OPE, we have

$$N \otimes N \to S \oplus n \oplus A \oplus T'$$

S-channel operators are

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Notice that $\sum_{\alpha} \epsilon_{\alpha}$, is a singlet, here $n = N - 1$.

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A-channel:

$$\sigma_{[\alpha}\partial_\mu \sigma_{\beta]} \dots\dots$$

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$\phi^4 \in$ S, 1st	$2 \times \Delta_\epsilon^{\text{Ising}}$	$\sum_{\alpha\neq\beta} \epsilon_\alpha \epsilon_\beta$
$\phi^4 \in$ S, 2st	$\Delta_{\epsilon'}^{\text{Ising}}$	$\sum_\alpha \epsilon'_\alpha$
$\phi^2 \in n_+$	$\Delta_\epsilon^{\text{Ising}}$	$\epsilon_\alpha e^i_\alpha$
$\phi^2 \in$ T$'$	$2 \times \Delta_\sigma^{\text{Ising}}$	$\sigma_{(\alpha}\sigma_{\beta)}\big\vert_{i\neq j}$

Table: large N behaviour from ϵ-expansion, for fixed point P_1.

The large N limit

At large N, P_2 is related to P_1 by a "double trace" flow

$$S_{P_2} = S_{P_1} + \lambda \int d^3x \sum_{i \neq j} \epsilon_i \epsilon_j$$
$$\approx S_{P_1} + \lambda \int d^3x \quad O \cdot (\sum \epsilon_i)$$

In IR, the Hubbard-Stratonovich auxiliary field O appears in the spectrum.

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$\phi^4 \in \mathrm{S}$, 1st	$2 \times (D - \Delta_\epsilon^{\text{Ising}})$	OO
$\phi^4 \in \mathrm{S}$, 2st	$\Delta_{\epsilon'}^{\text{Ising}}$	$\sum_\alpha \epsilon'_\alpha$
$\phi^2 \in \mathrm{n}_+$	$\Delta_\epsilon^{\text{Ising}}$	$\epsilon_\alpha e^i_\alpha$
$\phi^2 \in \mathrm{T}'$	$2 \times \Delta_\sigma^{\text{Ising}}$	$\sigma_{(\alpha}\sigma_{\beta)}\vert_{\alpha \neq \beta}$

Table: large N behaviour from ϵ-expansion, for fixed point P_2.

The large N limit

$S_N \otimes Z_2^N$ invariant operators of the decoupled Ising model
$O(x) = \frac{1}{\sqrt{N}} \sum_\alpha \epsilon_\alpha(x)$:

$$\langle O(x_1) O(x_2) \rangle \sim 1$$

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Three point function

$$\langle O(x_1) O(x_2) O(x_3) \rangle = \frac{1}{N^{3/2}} \sum_\alpha \langle \epsilon_\alpha \epsilon_\alpha \epsilon_\alpha \rangle \sim \frac{1}{\sqrt{N}} \lambda_{\epsilon\epsilon\epsilon}$$

The large N limit

Four point function point function

$$
\begin{aligned}
&\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle \\
&= \frac{1}{N^2}\sum_{i,j,k,l}\langle O^i(x_1)O^j(x_2)O^k(x_3)O^l(x_4)\rangle \\
&= \frac{1}{N^2}\sum_{i=j\neq k=l}\frac{1}{x_{12}^{2\Delta_O}x_{34}^{2\Delta_O}} + \frac{1}{N^2}\sum_{i=k\neq j=l}\frac{1}{x_{13}^{2\Delta_O}x_{24}^{2\Delta_O}} + \frac{1}{N^2}\sum_{i=l\neq j=l}\frac{1}{x_{14}^{2\Delta_O}x_{23}^{2\Delta_O}} \\
&\quad + \frac{1}{N^2}\sum_{i=j=k=l}\langle OOOO\rangle \\
&= (1-\frac{1}{N})\left(\frac{1}{x_{12}^{2\Delta_O}x_{34}^{2\Delta_O}} + \frac{1}{x_{13}^{2\Delta_O}x_{24}^{2\Delta_O}} + \frac{1}{x_{14}^{2\Delta_O}x_{23}^{2\Delta_O}}\right) + \frac{1}{N}\langle OOOO\rangle. \quad (2.11)
\end{aligned}
$$

The leading piece is disconnected, equivalent to a free scalar in AdS with mass $m^2L^2 = (3-\Delta_\epsilon)\Delta_\epsilon$.

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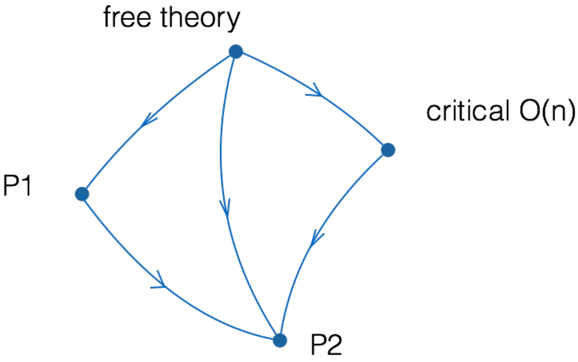

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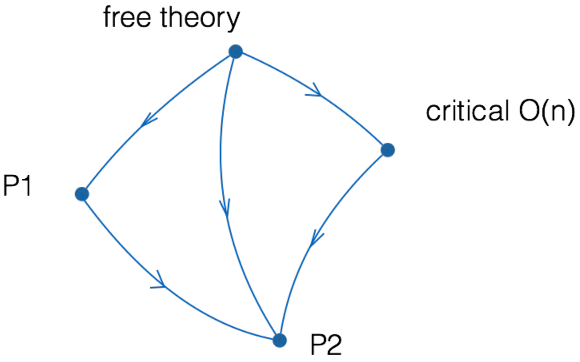

It is restored at the free theory point. The spectrum contains both higher spin current and other massive modes.

Cubic fixed point

The N=4 case deserves special attention.

$$S_4 \otimes Z_2 = S_3 \otimes (Z_2)^3$$

The order of the group is $4! \times 2 = 3! \times 2 \times 2 \times 2 = 48$.

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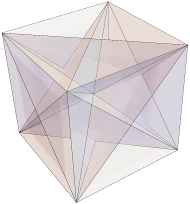

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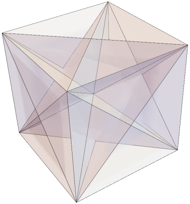
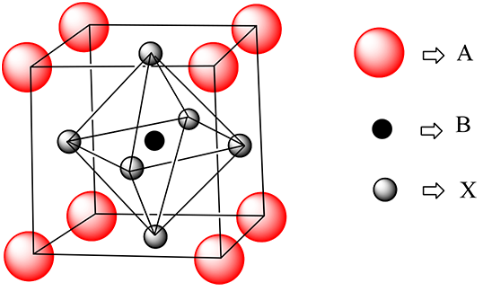


$S_3 \otimes (Z_2)^3$: permutation and reflection of the three axis.
$S_4 \otimes Z_2$: surface diagonals forms two tetrahedrons.

Cubic fixed point

Crystal of the type ABX_3 (perovskites)



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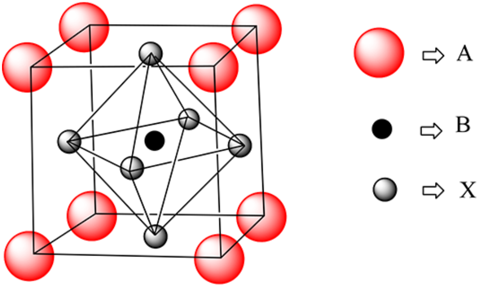


The Ginzburg–Landau theory has a term breaks O(3) to $S_3 \otimes (Z_2)^3$.

Extremal Functional

To study bootstrap equation, we apply a linear functional on the crossing equation. Numerical bootstrap use the basis

$$\alpha = \sum_{m+n \text{ is odd}, m+n<\Lambda} \alpha_{mn} (\partial_z)^m (\partial_{\bar{z}})^n$$

Remember for the crossing equation is

$$\sum_{O \in \phi} F_{\Delta,l}(u,v) = 0,$$
$$\lambda^2_{O_0} F_{\Delta_0,l_0}(u,v) = -F_{0,0}(u,v) - \sum_{O} \lambda^2_{O} F_{\Delta,l}(u,v)$$

we try to find a linear functional such that

$$\alpha(F_{\Delta_0,l_0}(u,v)) = 1,$$
$$\alpha(F_{\Delta,l}(u,v)) \geq 0 \text{ for others}$$

Cubic fixed point

If such an α exist, we have

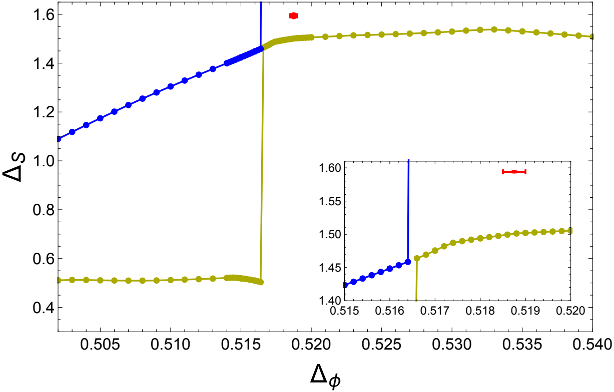
$$\lambda_{O_0}^2 = -\alpha(F_{0,0}(u,v)) - \sum_O \lambda_O^2 \alpha(F_{\Delta,l}(u,v)) \leq -\alpha(F_{0,0}(u,v))$$

We try to find the most restrictive bound, which minimize $-\alpha(F_{0,0}(u,v))$. Such an α should satiefy

$$\sum_O \lambda_O^2 \alpha(F_{\Delta,l}(u,v)) = 0.$$

Cubic fixed point

The in S-channel scalar from extremal functional [JR, Su '17]



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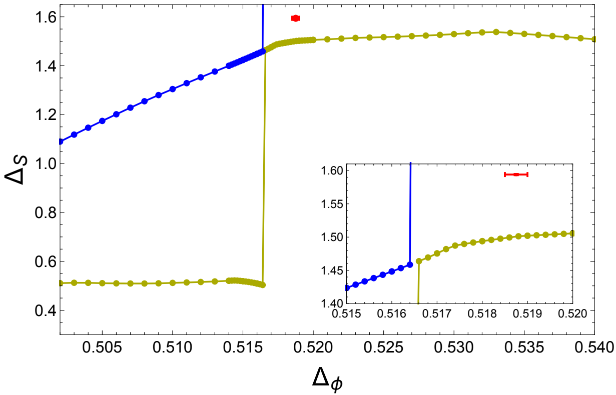

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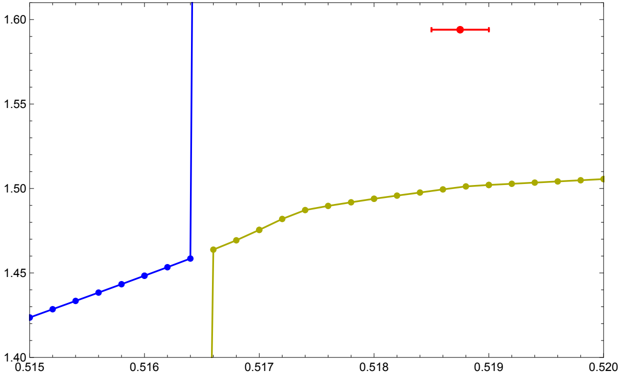

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A recent preprint [Stergiou '18] studies the S-channel operator, and also discovered this difference.

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This seems to favor the numerical bootstrap result!

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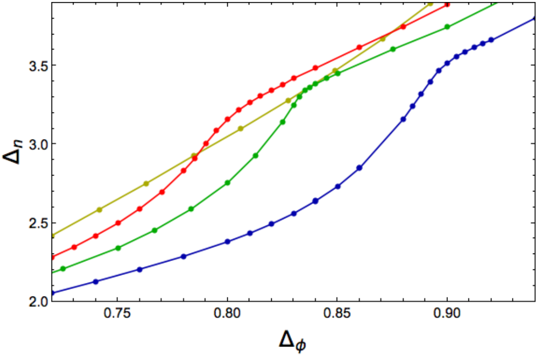
A lattice simulation would be interesting.

$$\mathcal{H}/k_{\mathrm{B}}T = -\sum_{\langle ij\rangle}[K\vec{s_i}\cdot\vec{s_j} + M(\vec{s_i}\cdot\vec{s_j})^2]\,,$$

Our Monte Carlo friends are working on it.

Another kink

Another kink could be found in the larger Δ_ϕ region [JR, Su '17]



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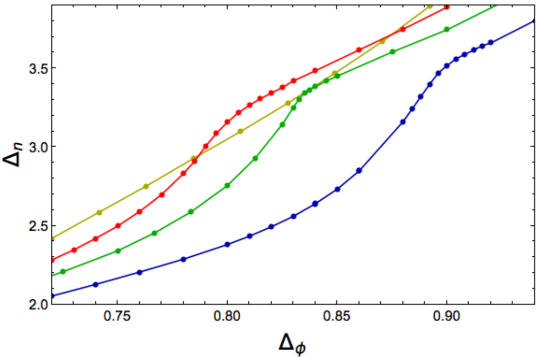

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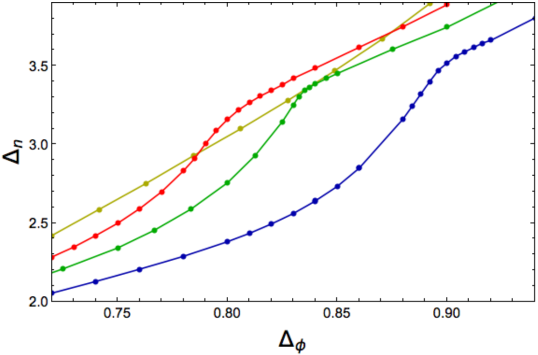

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Similar kink could be found in other on bootstrap curve with other symmetry. [Nakayama '17]

Consider the ϕ^3 theory

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At $6 > D > 3$, Potts models undergo a first order fixed point, related to the fact that the fixed point is non-unitary.

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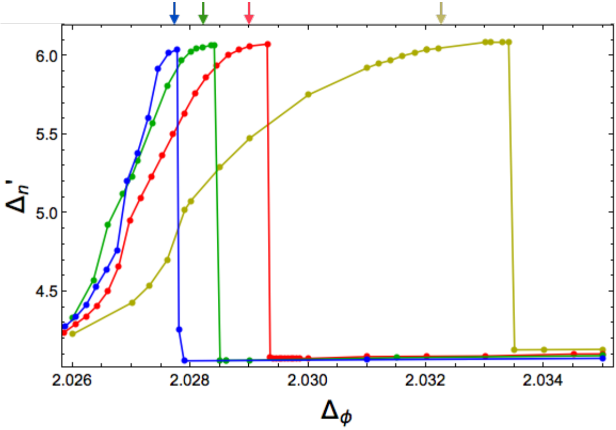

Figure: S_N bootstrap at $D = 6.05$.

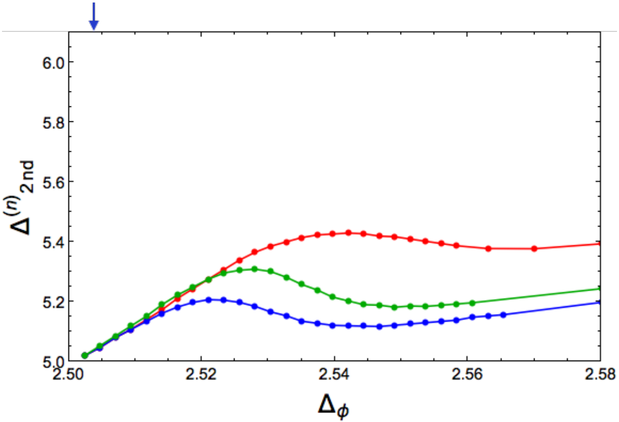

Figure: S_N bootstrap at $D = 7$.

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This fixed point describes Lee-Yang edge singularity.

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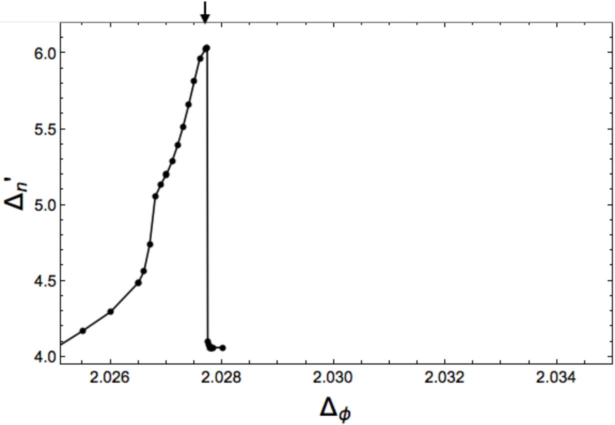

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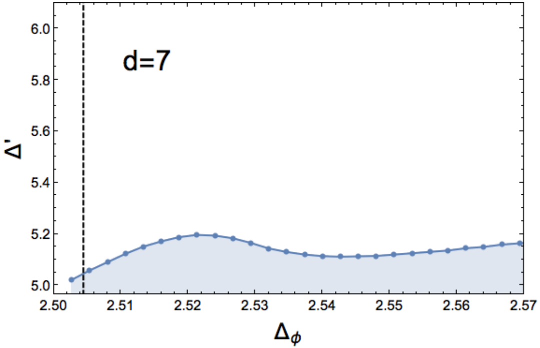

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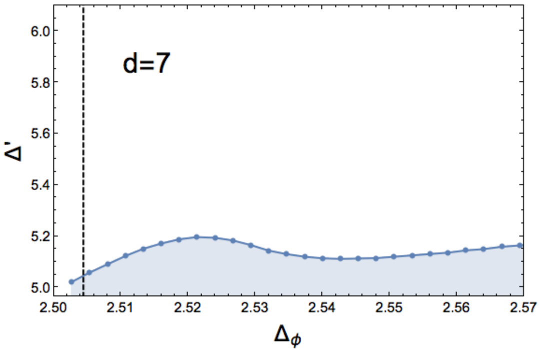

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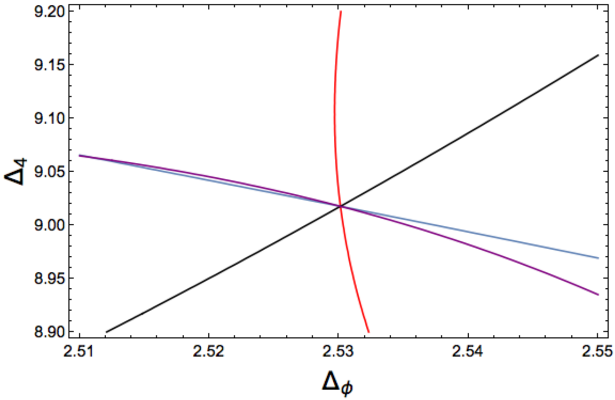
Fusion rule truncation



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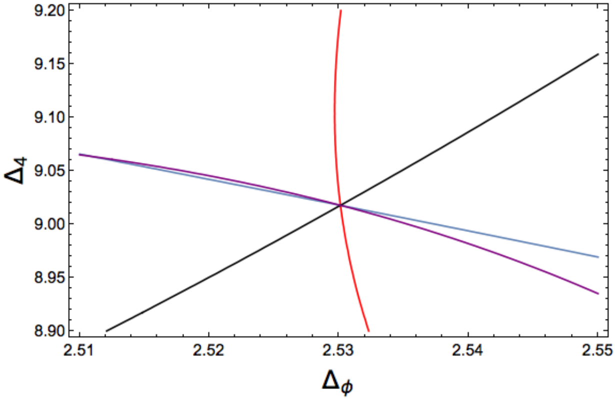

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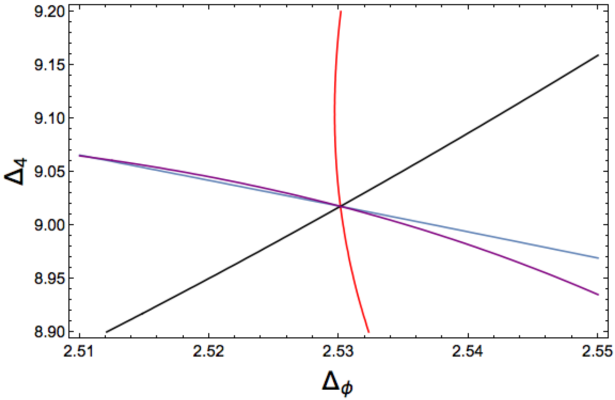

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