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Introduction to the I. F. P. toward resolution of singularities in positive characteristic

Idealistic Filtration Program

OIST / RIMS workshop

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by

Kenji Matsuki

Day I.

Part I.

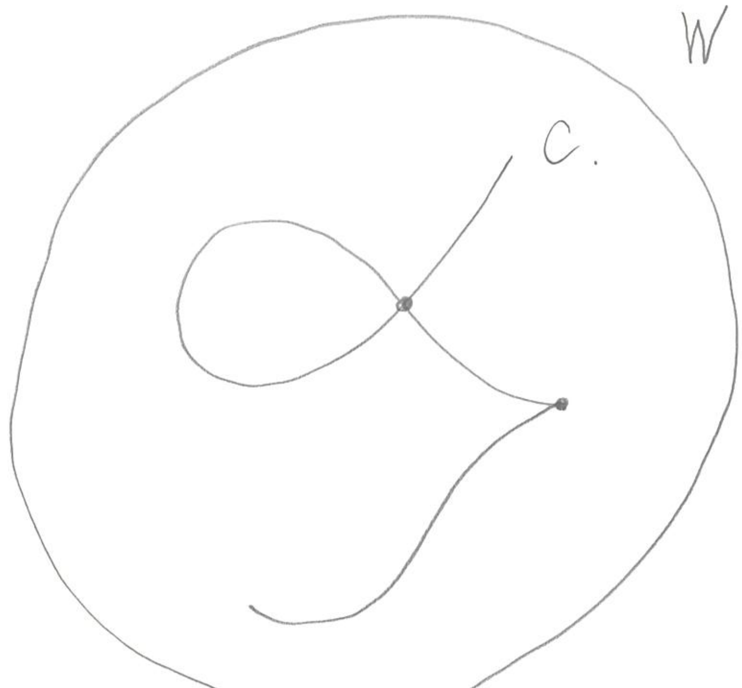
Theme : Resolution of singularities of a curve embedded in a nonsingular surface

Situation / $k = \bar{k}$, $\mathrm{char}(k) = 0$

$C \subset W$

a nonsingular surface.

a curve embedded as a closed subvariety



Theorem

$\exists$ a seq. of blowups
with centers being closed points

$$
\begin{array}{ccccccccc}
W & = & W_0 & \longleftarrow & W_1 & \longleftarrow & & & \\
\cup & & \cup & & \cup & & & & \\
C & = & C_0 & \longleftarrow & C_1 & \longleftarrow & & &
\end{array}
$$

$$
\begin{array}{cccccccc}
 & W_i & \xleftarrow{\pi_i} & W_{i+1} & & \longleftarrow & W_l \\
\cdots & \cup & & \cup & \cdots & & \cup \\
 & C_i & \longleftarrow & C_{i+1} & & \longleftarrow & C_l
\end{array}
$$

C_{i+1} : the strict transform of C_i

s.t.

C_l : nonsingular

Choice of the center

(4)

- We do NOT blow up nonsingular pts.
  (if we do, then superfluous!)
- We ONLY blow up the singular pts, and HAVE to.

→

Aside from which singular pt. to blow up 1st, 2nd ...

there is a UNIQUE way to blowup.

Issue : Does the seq. terminate?

Goal : Develop the invariants which measure the improvement of each blow up.

1st invariant μ.

$$p \in C = \{f = 0\} \subset U \subset W$$

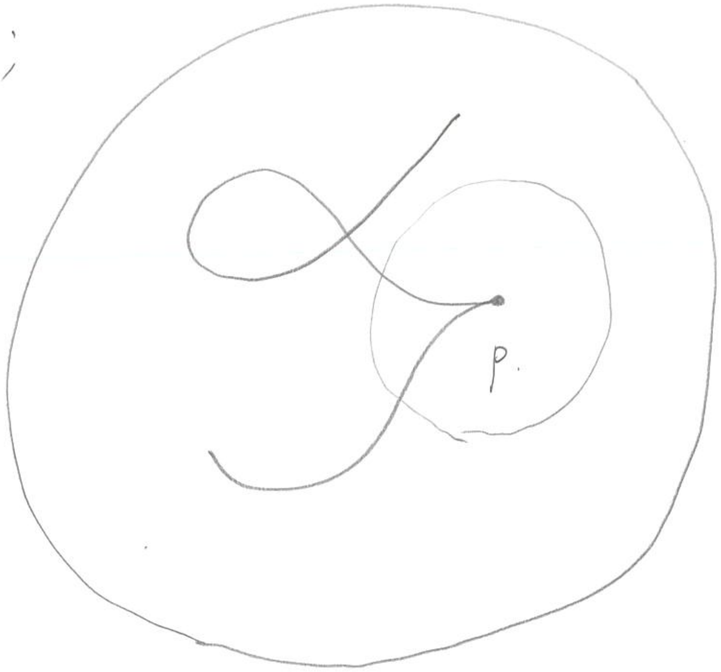
affine open

$$\mu(P) = \mathrm{ord}_P(f)$$

$$= \max\{n \geq 0 ;\ f \in \mathfrak{m}_P^{n}\}.$$

$$\left(f = \sum \alpha_{ij} x^i y^j \in \mathcal{O}_{W,p} \subset \widehat{\mathcal{O}_{W,p}} \right)$$

$$= \min\{i + j ;\ \alpha_{ij} \neq 0\}.$$



Note ① $\mu(P) = \mathrm{ord}_P(C)$.

independent of the choice of f.

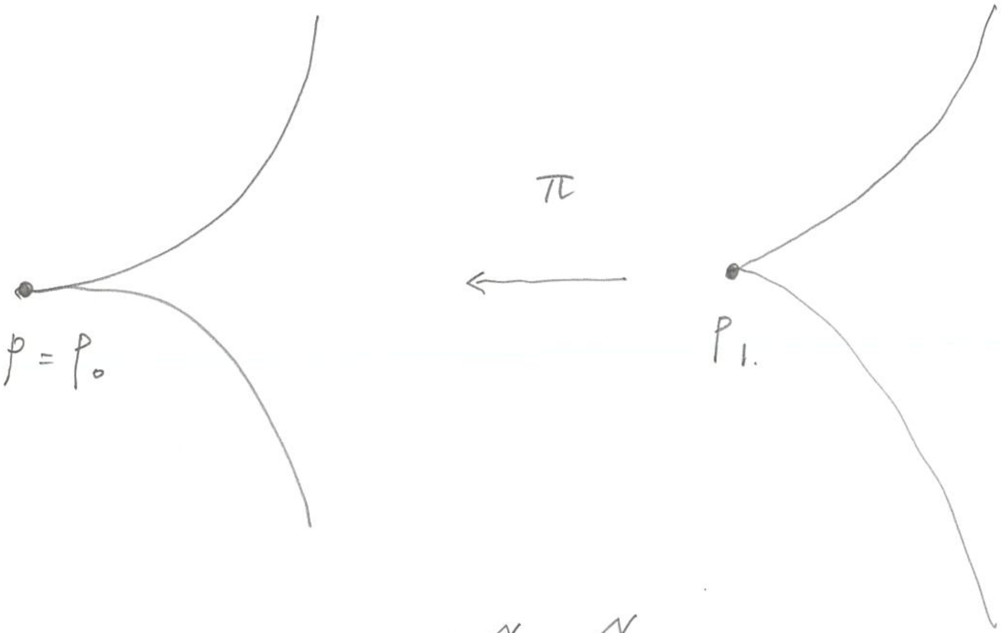
(or U)

Remark : The invariant "μ" by itself is NOT good enough.

② C nonsingular at P $\iff$ $\mu(P) = 1$. | singular at P $\iff$ $\mu(P) > 1$.

Example

$$f = f_0 = y^2 - x^5 \qquad \pi^* f = (x_1 y_1)^2 - x_1^5$$
$$= x_1^2 (y_1^2 - x_1^3)$$



$$\begin{cases} x = x_1 \\ y = x_1 y_1 . \end{cases}$$

i.e.

$$\begin{cases} x_1 = x \\ y_1 = x/y \end{cases}$$

$$f_1 = y_1^2 - x_1^3$$

$$2 = \mu(P_0) = \mu(P_1) = 2 .$$

$\boxed{\text{2nd invariant } \nu}$

Weierstrass Preparation Theorem

$$f = \underset{\substack{\uparrow \\ \text{unit}}}{u} \left(y^{\mu} + c_1 y^{\mu-1} + \cdots + c_{\mu} \right) \qquad c_i \in k[[X]]$$

Tschirnhausen Transformation

$$y' = y + \frac{1}{\mu} c_1 \quad (\leftarrow \mathrm{char}(k) = 0)$$

May assume

$$f = y^{\mu} + \underbrace{c_1}_{\substack{\shortparallel \\ 0}} y^{\mu-1} + c_2 y^{\mu-2} + \cdots + c_{\mu}.$$

Question : Why do we want to make $c_1 = 0$?

Answer : Later ☺

(actually because
$c_1 = 0 \rightarrow y \in \Delta^{\mu-1}(\mathrm{dc})_p$)

$$\nu := \min \left\{ \frac{\mathrm{ord}_p(C_i)}{i} \;;\; i = 2, \dots, \mu \right\}$$

<u>Motivation</u>

Inductive scheme!

(for simplicity, pretend $\begin{cases} U = \operatorname{Spec} k[X, Y] \\ \& \; C_i \in k[X] \end{cases}$)

$p \in W$

$\mathrm{ord}_p(f) = \mu \iff p \in H = \{Y = 0\}$.
& $\mathrm{ord}_p(C_i) \geq i$.
for $i = 2, \dots, \mu$.

$\iff p \in H = \{Y = 0\}$.
& $\mathrm{ord}_p\Big(\underbrace{\sum_{i=2}^{\mu} (C_i)^{\frac{\mu!}{i}}}_{J_H}\Big) \geq \mu!$

$$\nu = \frac{\mathrm{ord}_p(J_H)}{\mu!}$$

Big Question : Is ν independent of the choices ?

Answer : Yes !

We will prove this later
in the spirit of

- Wlodarczyk's homogenization
- Kawanoue's differential saturation

Note :

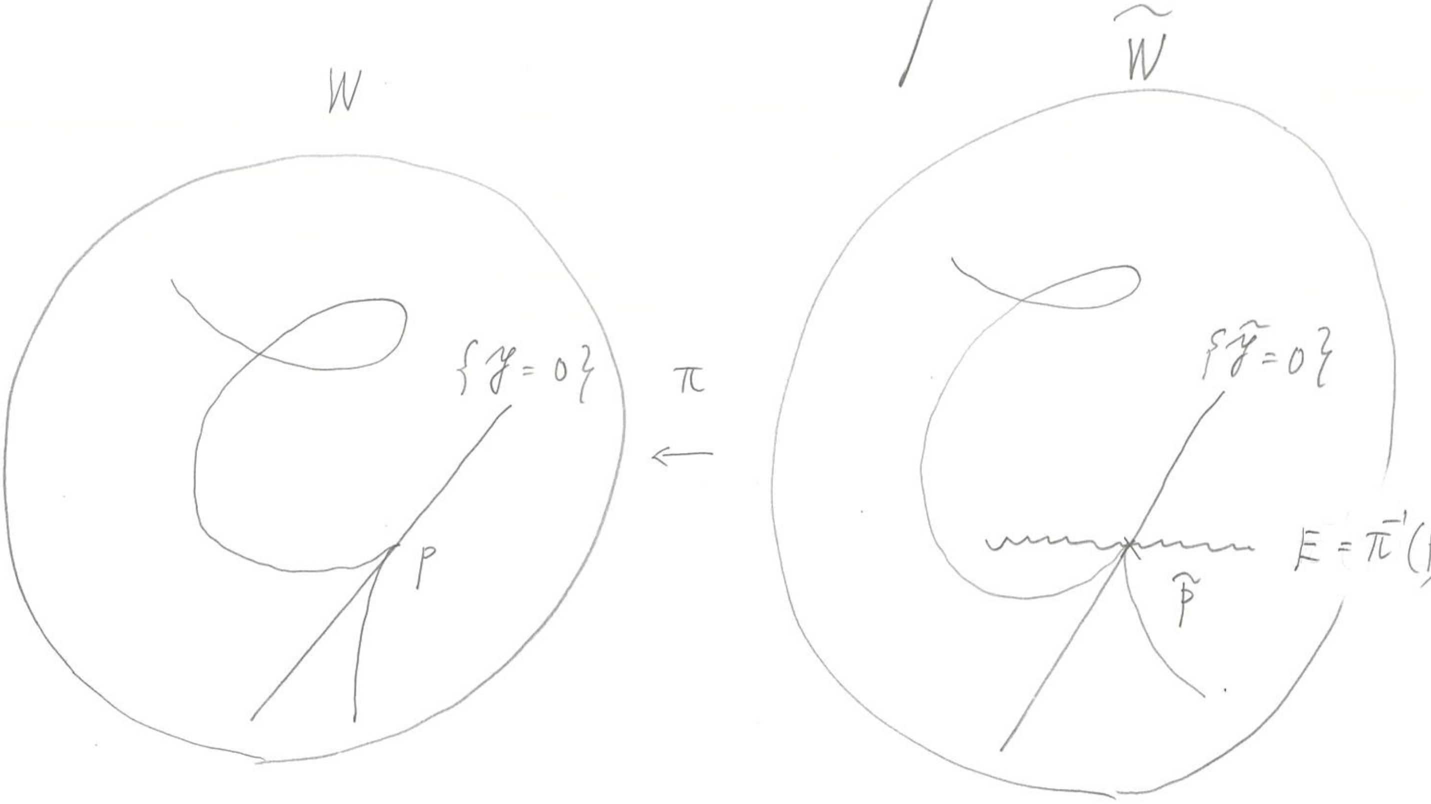
- Hironaka proves this independence
by Hironaka's Trick.
- Mumford cheats (avoids this question)

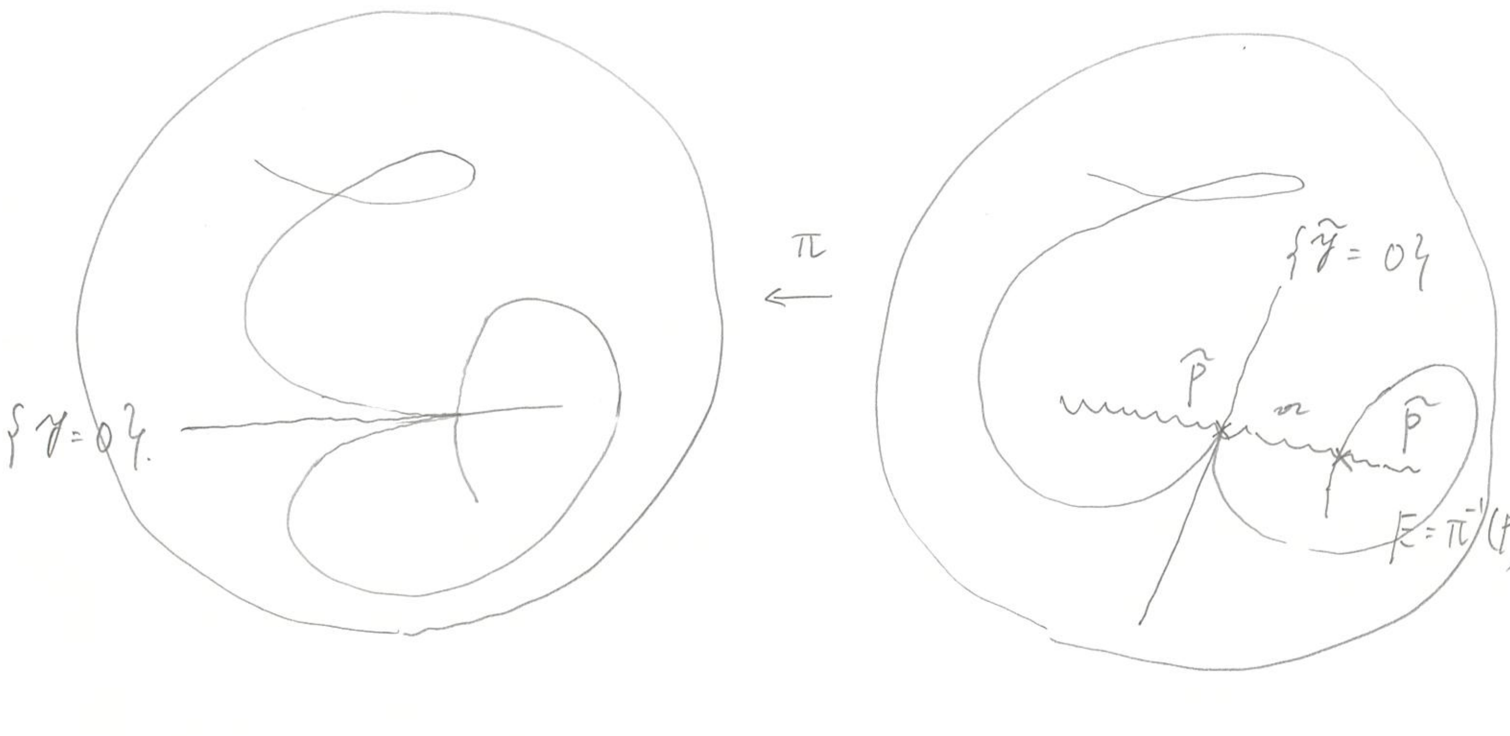
by defining

$$\text{Mumford's } \nu = \min_{\left[\substack{\text{among all}\\ \text{such choices}}\right]} \{\text{our } \nu\}$$

$\longrightarrow$ enough to prove termination !

Behavior of μ & ν under blow up.





Conclusion

① $\hat{\tilde{P}} \in \{\hat{\tilde{y}} = 0\}$.

: the strict transform of $\{y = 0\}$

a hypersurface of maximal contact

$\longrightarrow$

$$\mu(P) \geqq \mu(\hat{\tilde{P}})$$

& if $\mu(P) = \mu(\hat{\tilde{P}})$

then $\nu(P) > \nu(\hat{\tilde{P}})$

② $\hat{P} \notin \{\hat{\tilde{y}} = 0\}$.

$\longrightarrow$

$$\mu(P) > \mu(\hat{\tilde{P}})$$

Proof.

①

$$f = y^{\mu} + \overset{0}{\overset{\shortparallel}{c_1}}\, y^{\mu-1} + c_2 y^{\mu-2} + \cdots + c_{\mu-1} y + c_{\mu}.$$

$$\left(\begin{array}{l} \mathrm{ord}_p(c_i) \geq i \quad i = 1, 2, \ldots, \mu. \\ \text{since} \\ \mathrm{ord}_p(f) = \mu. \end{array}\right)$$

at p

$$\begin{cases} x = \hat{x} \\ y = \hat{x}\hat{y} \end{cases}$$

at $\hat{p}$

$$\begin{cases} \hat{x} = x \\ \hat{y} = \dfrac{y}{x} \end{cases}$$

$$\left(\begin{array}{l} \text{usually } = \dfrac{y}{x} - \beta \\ \text{but } \beta = 0 \text{ since} \\ \hat{p} \in \{\hat{y} = 0\}. \end{array}\right)$$

$$\hat{f} = \frac{\pi^* f}{\hat{x}^{\mu}} \qquad = 0$$

$$= \hat{y}^{\mu} + \frac{c_1(\hat{x})}{\hat{x}} \cdot \hat{y}^{\mu-1} + \frac{c_2(\hat{x})}{\hat{x}^2} \cdot \hat{y}^{\mu-2} + \cdots + \frac{c_{\mu}(\hat{x})}{\hat{x}^{\mu}}$$

$$= \tilde{y}^{\mu} + \underset{0}{\overset{\shortparallel}{\hat{c}_1}} \cdot \tilde{y}^{\mu-1} + \hat{c}_2\, y^{\mu-2}$$

$$\cdots + \widehat{C_{\mu-1}}\, \hat{y} + \widehat{C_\mu}$$

Conclusion

$$\mu(P) = \mu \geqq \mu(\hat{P})$$

Moreover

if $\mu(P) = \mu = \mu(\hat{P})$

then

$$\frac{\mathrm{ord}_{\hat{p}}(\hat{C}_i)}{i} = \frac{\mathrm{ord}_{\hat{p}}(C_i/x^i)}{i}$$

$$= \frac{\mathrm{ord}_p(C_i)}{i} - 1$$

$$\therefore \quad \nu(\hat{P}) = \min\left\{\frac{\mathrm{ord}_{\hat{p}}(\hat{C}_i)}{i}\ ;\ i=2, \ldots, \mu\right\}$$

$$= \min\left\{\frac{\mathrm{ord}_p(C_i)}{i} - 1\ ;\ i=2, \ldots, \mu\right\}$$

$$= \min\left\{\frac{\mathrm{ord}_p(C_i)}{i}\ ;\ i=2, \ldots, \mu\right\} - 1$$

$$= \nu(P) - 1.$$

Review

$p \in C \subset W$

$\{f = 0\}$

W. P. Th.

$$f = y^{\mu} + c_1 y^{\mu-1} + c_2 y^{\mu-2} + \cdots c_{\mu-1} y + c_{\mu}$$

$$c_i \in k[[X]]$$

1st. min $\mu = \mathrm{ord}_p(f)$

$$\mathrm{ord}_p(c_i) \geq i$$

via Tschirnhausen transformation

$$y' = y + \frac{1}{\mu} c_1$$

may assume $c_1 = 0$.

2nd. min

$$\nu = \min \left\{ \frac{\mathrm{ord}_p(c_i)}{i} \; ; \; i = 2, \ldots, \mu \right\}$$

Behavior of μ & ν

under blowup.

① on HMC

i.e.

$\hat{P} \in \{\tilde{y} = 0\}$

str. trans. of $\{y = 0\}$.

$$\mu(P) \geq \mu(\hat{P})$$

and

if $\mu(P) = \mu(\hat{P})$

then $\nu(P) > \nu(\hat{P}) \geq \nu(P) - 1$

② away from HMC.

Giraud's Lemma.

Giraud's Lemma.

Begining of Day 2.

(16)

$$\mathcal{I} \subset \mathcal{O}_W.$$

$$\cap$$

$$\Delta(\mathcal{I})$$

$$\Delta(\mathcal{I})_p = \langle f \in \mathcal{I}_p,\ \delta f\ ;\ f \in \mathcal{I}_p,\ \delta \text{ diff. op. of deg} \leq 1. \rangle$$

Specify the level.

$$\begin{array}{ccc}
W & \xleftarrow{\pi} & \widehat{W} \\
(\mathcal{I}, b) & & (\widehat{\mathcal{I}}, b) \\
\mathrm{ord}_p(\mathcal{I}) \geq b. & \xrightarrow[\text{"}\sim\text{"}]{} & \pi^{-1}(\mathcal{I})\mathcal{O}_{\widehat{W}} \cdot \mathcal{I}(E)^{-b} \\
\text{"}\Delta\text{"}\downarrow & & \text{"}\Delta\text{"}\downarrow \\
 & & (\Delta(\widehat{\mathcal{I}}),\ b-1) \\
 & & \cup \\
(\Delta(\mathcal{I}),\ b-1) & \xrightarrow[\text{"}\sim\text{"}]{} & (\widehat{\Delta(\mathcal{I})},\ b-1) \\
 & & \| \\
 & & \pi^{-1}(\Delta(\mathcal{I}))\mathcal{O}_{\widehat{W}} \cdot \mathcal{I}(E)^{b-1}.
\end{array}$$

Proof

② $\mathcal{d} = \mathcal{d}_c$

$$y \in \Delta^{\mu-1}(\mathcal{d}) \qquad \left((\Delta^{\mu-1}(\mathcal{d}), 1) \right.$$

$$\hat{y} \in \widehat{\Delta^{\mu-1}(\mathcal{d})} \qquad \left. (\widehat{\Delta^{\mu-1}(\mathcal{d})}, 1) \right)$$

$$\cap \quad \leftarrow \text{Giraud's Lemma.}$$

$$\Delta^{\mu-1}(\hat{\mathcal{d}})$$

Suppose

$$\mathrm{ord}_{\hat{p}}(\hat{\mathcal{d}}) \geqq \mu.$$

$$\rightarrow \mathrm{ord}_{\hat{p}}(\Delta^{\mu-1}(\hat{\mathcal{d}})) \geqq \mu - (\mu - 1) = 1.$$

$$\Downarrow$$

$$\hat{y}$$

i.e. $\hat{y} = 0$ at $\hat{p}$

i.e. $\hat{p} \in \{\hat{y} = 0\}$

All the ord $= \mu$ pts are on HMC $\rightarrow$ induction

$\therefore \hat{p} \notin \{\hat{y} = 0\}$

$\Rightarrow \mu(\hat{p}) = \mathrm{ord}_{\hat{p}}(\hat{\mathcal{d}}) < \mu.$ $\square$

Remark

What Prof. Encinas (& Prof. Hauser) was trying to tell me yesterday (which I ignored ☺)

$$f = y^n + c_1 y^{n-1} + c_2 y^{n-2} + \cdots + c_{n-1} y + c_n.$$

What happens if we do NOT apply Tschirnhausen trans. yet still define

$$\nu_{\text{primitive}} = \min \left\{ \frac{\operatorname{ord}_p(c_i)}{i} ; i = 1, \cdots, n \right\}.$$

highly dependent of the choices

Exercise

(1) Show

$$\max_{\text{all the choices}} \nu_{\text{primitive}} = \nu.$$

2 $\min_{\text{all the choices}} \nu_{\text{primitive}} = 1$.

Proposition

Invariant ν is independent of the choice

(i) $\nu_y \overset{\text{def}}{=} \min \left\{ \frac{\mathrm{ord}_p(c_i)}{i} \; ; \; i = 2, \dots, \mu \right\}$

$= \min \left\{ \frac{\mathrm{ord}_p \left(\Delta^{\mu - i} (\widehat{\mathcal{I}_p}) \big|_{\{y=0\}} \right)}{i} \; ; \right.$

$\left. i = 1, 2, \dots, \mu \right\}$

(ii) $\nu_y = \nu_{y'}$

In fact

$$\left[\begin{array}{l} \forall y \in \Delta^{\mu-1}(\widehat{\mathcal{I}_p}) \\ \qquad \text{with } \mathrm{ord}_p(y) = 1 \\ \min \left\{ \frac{\mathrm{ord}_p \left(\Delta^{\mu-i}(\widehat{\mathcal{I}_p}) \big|_{\{y=0\}} \right)}{i} \; ; \; i = 1, 2, \dots, \mu \right\} \end{array} \right]$$

is independent of the choice of y.

In particular,

take $y \in \Delta^{\mu-1}(\mathcal{O})$

with $\mathrm{ord}_p(y) = 1$.

$$\left(\begin{array}{l} \leftarrow \quad \mathrm{ord}_p(\mathcal{O}) = \mu \\ \text{i.e.} \\ \quad \exists\, f \in \mathcal{O}_p \\ \text{s.t.} \\ \quad \mathrm{ord}_p(f) = \mu. \\ \therefore\ \exists\, \delta \ \text{diff. op. of deg} \leq \mu - 1 \\ \text{s.t.} \\ \quad \mathrm{ord}_p(\delta f) = 1. \\ \quad \delta f \in \Delta^{\mu-1}(\mathcal{O}_p) \end{array} \right)$$

$$\nu = \left\{ \frac{\mathrm{ord}_p\left(\Delta^{\mu-i}(\widehat{\mathcal{O}_p})\big|_{\{y=0\}}\right)}{i} ; \ i = 1, 2, \ldots, \mu \right\}$$

$$= \left\{ \frac{\mathrm{ord}_p\left(\Delta^{\mu-i}(\mathcal{O}_p)\big|_{\{y=0\}}\right)}{i} ; \ i = 1, 2, \ldots, \mu \right\}$$

$\boxed{\text{algebraic}}$.

Proof.

(i) $\widehat{\mathcal{O}_p} = (f = y^\mu + c_1 y^{\mu-1} + \cdots + c_{\mu-1} y + c_\mu)$

$$\Delta^{\mu-i}(\widehat{\mathcal{O}_p})\Big|_{\{y=0\}}$$

$i = \mu.$ $\quad \Delta^0(\widehat{\mathcal{O}_p}) \ni c_\mu, \quad \mathrm{ord}_p(\quad)/\mu$

$i = \mu - 1$ $\quad \Delta^1(\widehat{\mathcal{O}_p}) \ni c_{\mu-1} \quad \mathrm{ord}_p(\quad)/\mu-1$

$$\frac{\partial}{\partial y}(f)\Big|_{\{y=0\}}$$

$\vdots$

$i = 2$ $\quad \Delta^{\mu-2}(\widehat{\mathcal{O}_p}) \ni c_2 \quad \mathrm{ord}_p(\quad)/2.$

$$\frac{\partial^{\mu-2}}{\partial y^{\mu-2}} f\Big|_{\{y=0\}}$$

$i = 1$ $\quad \Delta^{\mu-1}(\widehat{\mathcal{O}_p}) \ni c_1 \quad \mathrm{ord}_p(\quad)/1.$

Conclusion

$$\nu_y \overset{\mathrm{def}}{=} \min\left\{ \frac{\mathrm{ord}_p(c_i)}{i} \,;\, i = 1, 2, \ldots, \mu \right\}.$$

$$\geqq \min\left\{ \frac{\mathrm{ord}_p\left(\Delta^{\mu-i}(\widehat{\mathcal{O}_p})\big|_{\{y=0\}}\right)}{i} \,;\, i = 1, 2, \ldots, \mu \right\}.$$

On the other hand

generated by

$i = \mu$ $\quad \Delta^0(\widehat{d_p})\big|_{\{y=0\}}$ $\quad C_\mu$.

$i = \mu - 1$ $\quad \Delta^1(\widehat{d_p})\big|_{\{y=0\}}$ $\quad$ plus $\frac{\partial}{\partial x} C_\mu$, $C_{\mu-1}$

$i = \mu - 2$ $\quad \Delta^2(\widehat{d_p})\big|_{\{y=0\}}$ $\quad$ plus $\frac{\partial^2}{\partial x^2} C_\mu$, $\frac{\partial}{\partial x} C_{\mu-1}$, $C_{\mu-2}$

$\vdots$ $\quad \vdots$ $\quad \vdots$

$i = 2$ $\quad \Delta^{\mu-2}(\widehat{d_p})\big|_{\{y=0\}}$ $\quad$ plus

$i = 1$ $\quad \Delta^{\mu-1}(\widehat{d_p})\big|_{\{y=0\}}$ $\quad \frac{\partial^{\mu-1}}{\partial x^{\mu-1}} C_\mu$, $\frac{\partial^{\mu-2}}{\partial x^{\mu-2}} C_{\mu-1}$, $\dots$, $\frac{\partial}{\partial x} C_2$, C_1.

$i = \mu$

$$\boxed{\frac{\operatorname{ord}_p(C_\mu)}{\mu}.}$$

$i = \mu - 2$

$$\frac{\operatorname{ord}_p(C_\mu)}{\mu} \leq \left\{ \begin{array}{l} \dfrac{\operatorname{ord}_p(C_\mu)}{\mu - 1} \\[2ex] \dfrac{\operatorname{ord}_p\left(\frac{\partial}{\partial x} C_\mu\right)}{\mu - 1} \end{array} \right. \qquad \boxed{\frac{\operatorname{ord}_p(C_{\mu-1})}{\mu - 1}.}$$

since $\operatorname{ord}_p(C_\mu) \geq \mu$.

$i = \mu - 2$

$$\frac{\operatorname{ord}_p(C_\mu)}{\mu} \leq \left\{ \begin{array}{l} \dfrac{\operatorname{ord}_p(C_\mu)}{\mu - 2} \\[2ex] \dfrac{\operatorname{ord}_p\left(\frac{\partial}{\partial x} C_\mu\right)}{\mu - 1.} \\[2ex] \dfrac{\operatorname{ord}_p\left(\frac{\partial^2}{\partial x^2} C_\mu\right)}{\mu - 2.} \end{array} \right. , \quad \frac{\operatorname{ord}_p(C_{\mu-1})}{\mu - 1} \leq \left\{ \begin{array}{l} \dfrac{\operatorname{ord}_p(C_{\mu-1})}{\mu - 2}, \\[2ex] \dfrac{\operatorname{ord}_p\left(\frac{\partial}{\partial x} C_{\mu-1}\right)}{\mu - 2} \end{array} \right. \qquad \boxed{\frac{\operatorname{ord}_p(C_{\mu-2})}{\mu - 2}}$$

Conclusión

$$\nu_y \overset{\text{def}}{=} \min \left\{ \frac{\operatorname{ord}_p(C_i)}{i} \;;\; i = 1, 2, \ldots, \mu \right\}$$

$$\leq \min \left\{ \frac{\operatorname{ord}_p\left(\Delta^{\mu - i}(\widehat{d_p})\big|_{\{y=0\}}\right)}{i} \;; \right.$$

$$\left. i = 1, 2, \ldots, \mu \right\}.$$

Grand Conclusión

$$\nu_y = \min \left\{ \frac{\operatorname{ord}_p\left(\Delta^{\mu - i}(\widehat{d_p})\big|_{\{y=0\}}\right)}{i} \;; \right.$$

$$\left. i = 1, 2, \ldots, \mu \right\}.$$

(ii) Take another

$$y' \in \Delta^{\mu-1}(\widehat{d_p})$$

with

$$\mathrm{ord}_p(y') = 1.$$

Want to show

$$\min \left\{ \frac{\mathrm{ord}_p\left(\Delta^{\mu-i}(\widehat{d_p})\big|_{\{y'=0\}}\right)}{i} \; ; \; i = 1, 2, \ldots, \mu \right\}$$

$$= \min \left\{ \frac{\mathrm{ord}_p\left(\Delta^{\mu-i}(\widehat{d_p})\big|_{\{y=0\}}\right)}{i} \; ; \; i = 1, 2, \ldots, \mu \right\}.$$

Set

$$y' = y + \underset{\Delta^{n-1}(\widehat{\mathcal{O}_p})}{\underset{\cap}{h}}.$$

Choose

$$z \in \widehat{\mathfrak{m}_{W,p}}$$

s.t.

$$\begin{matrix} (y, z) \\ (y', z) \end{matrix} \quad \text{r.s.p. for } \widehat{\mathcal{O}_{W,p}}$$

Consider

$$\begin{array}{ccccc} \phi^* : & \widehat{\mathcal{O}_{W,p}} & \xrightarrow{\sim} & \widehat{\mathcal{O}_{W,p}} \\ & \text{SII} & & \text{SII} \\ & k[[y, z]] & \xrightarrow{\sim} & k[[y', z]] \\ & \text{Ш} & & \text{Ш} \\ & y & \longmapsto & y' \\ & z & \longmapsto & z \end{array}$$

k-alg. autom.
continuous

Take $g(y, z) \in \Delta^{\mu - i}(\widehat{d_p})$ (27)

We compute

$$\mathrm{ord}_p\left(g(y, z)|_{\{y=0\}} \right)$$

$$= \mathrm{ord}_p\left(\phi^* g(y, z) |_{\{\phi^* y = 0\}} \right)$$

$$= \mathrm{ord}_p\left(g(\phi^* y, \phi^* z) |_{\{\phi^* y = 0\}} \right)$$

$$= \mathrm{ord}_p\left(g(y', z) |_{\{y' = 0\}} \right)$$

$$= \mathrm{ord}_p\left(g(y + h, z) |_{\{y' = 0\}} \right)$$

$$= \mathrm{ord}_p\Bigg(\underbrace{\sum_{l=0}^{i-1} \underbrace{\frac{1}{l!}\left\{\frac{\partial^l}{\partial y^l} g(y, z)\right\}}_{\in\, \Delta^{\mu - i + l}(\widehat{d_p}),\ \mu - i + l = \mu - (i - l)} \underbrace{h^l}_{\in\, \Delta^{\mu - 1}(\widehat{d_p})} \Big|_{\{y'=0\}}}_{\downarrow}$$

$$\mathrm{ord}_p\left(\qquad \right)$$

$$\geqq (i - l) \cdot \nu_{y'} + l \cdot \nu_{y'} = i \cdot \nu_{y'}$$

$$+ \underbrace{\sum_{l=i}^{\infty} \frac{1}{l!}\left\{\frac{\partial^l}{\partial y^l} g(y, z)\right\} h^l \Big|_{\{y'=0\}}}_{\downarrow}$$

$$\mathrm{ord}_p\left(\qquad \right)$$

$$\geqq \mathrm{ord}_p\left(\underbrace{h^i}_{\in\, \Delta^{\mu - 1}(\widehat{d_p})} |_{\{y'=0\}} \right) \geqq i \cdot \nu_{y'}$$

<u>Conclusion</u>

$$g(y, z) \in \Delta^{\mu-i}(\widehat{\mathfrak{d}_p})$$

$$\longrightarrow \quad \mathrm{ord}_p\left(g(y,z)\big|_{\{y=0\}} \right) \geq i \cdot \nu_{y'}$$

$$\longrightarrow \quad \frac{\mathrm{ord}_p\left(\Delta^{\mu-i}(\widehat{\mathfrak{d}_p})\big|_{\{y=0\}} \right)}{i} \geq \nu_{y'}$$

$$\longrightarrow$$

$$\nu_y = \min\left\{ \frac{\mathrm{ord}_p\left(\Delta^{\mu-i}(\widehat{\mathfrak{d}_p})\big|_{\{y=0\}} \right)}{i} ; \; i = 1, 2, \cdots, \mu \right\}$$

$$\geq \nu_{y'}$$

i.e.

$$\nu_y \geq \nu_{y'}$$

Changing the roles of y & y', we have

$$\nu_{y'} \geq \nu_y$$

<u>Grand Conclusion</u>

$$\nu_{y'} = \nu_y$$

Remark.

A different proof
is given
in terms of

- $\mathcal{D}$-saturated idealistic filtration
- * formal coefficient lemma

by Kawanoue - ~~M~~atsuki

Part 2:

Theme: Resolution of singularities of a basic object

Why?

Implications

Resolution of singularities
of a basic object

(functorial) ⇓

Embedded resolution of singularities
in arbitrary dimension

(functorial) ⇓

Abstract resolution of singularities

Problem of embedded resolution of singularities

Situation

$$X \subset W$$

a variety embedded as a closed subvariety

a nonsingular variety called an ambient space

Formulation

Construct a sequence of blowups

$$\begin{array}{ccccccccccc}
W = W_0 & \leftarrow & W_1 & \leftarrow & \cdots & \leftarrow & W_i & \leftarrow & W_{i+1} & \cdots \leftarrow & W_e \\
\cup & & \cup & & & & \cup & & \cup & & \cup \\
X = X_0 & \leftarrow & X_1 & \leftarrow & \cdots & \leftarrow & X_i & \leftarrow & X_{i+1} & \cdots \leftarrow & X_e
\end{array}$$

where

(i) π_i blow up with center C_i

requirement on C_i

① C_i nonsingular (maybe reducible)

② w $C_i \not\ni X_i$ weak form

or

② s $C_i \subset \mathrm{Sing}(X_i)$ strong form.

(→ ② w)

optional

③ $C_i \pitchfork F_i$

the union of the exceptional divisors

(ii) X_{i+1} the strict transform of X_i.

s.t.

X_ℓ nonsingular

Problem of resolution of singularities of a basic object

Def. a basic object

$$(W, (\mathcal{J}, b), E)$$

- W: a nonsingular variety
- $\mathcal{J}$: a coherent sheaf of ideals
- $b \in \mathbb{Z}_{>0}$
- E: an SNC divisor

$$\mathrm{Sing}(\mathcal{J}, b) := \{p \in W \,;\, \mathrm{ord}_p(\mathcal{J}) \geq b\}.$$

Formulation

Construct a seq. of transformations

$$\underset{(W, (\mathcal{I}, b), E)}{\overset{\substack{\mathcal{O} \\ \Uparrow}}{}} \quad \longrightarrow \quad \text{Choice of the center in the inductive scheme}$$

$$(W, (\mathcal{I}, b), E) = (W_0, (\mathcal{I}_0, b), E_0) \xleftarrow{\pi_0} (W_1, (\mathcal{I}_1, b), E_1)$$

$$\longleftarrow \cdots$$

$$(W_i, (\mathcal{I}_i, b), E_i) \xleftarrow{\pi_i} (W_{i+1}, (\mathcal{I}_{i+1}, b), E_{i+1})$$

$$\longleftarrow \cdots$$

$$\xleftarrow{\pi_{\ell-1}} (W_\ell, (\mathcal{I}_\ell, b), E_\ell)$$

s.t.

$$\mathrm{Sing}(\mathcal{I}_\ell, b) = \emptyset.$$

where

(i) π_i blowup with center C_i

requirement on C_i

① C_i nonsingular (maybe reducible)

② $C_i \subset \mathrm{Sing}(\mathcal{I}_i, b)$

③ $C_i \pitchfork E_i$

(ii) $\mathcal{J}_{i+1} = \pi_i^{-1}(\mathcal{J}_i)\mathcal{O}_{W_{i+1}} \cdot \mathcal{J}(\pi_i^{-1}(C_i))^{-b}$.

$(\leftarrow$ (i) ② $)$

(iii) $E_{i+1} = E_i \cup \pi_i^{-1}(C_i)$.

the strict transform of

Proposition

Res. of sing. of a basic object

$\Downarrow$

Embedded res. of sing. (weak form)

Proof.

Given $X \subset W$

consider a basic object

$$(W, (\mathcal{I}, b), E)$$

$$= (W, (\mathcal{I}_X, 1), \phi)$$

Res. of sing. of the basic object

$$(W, (\mathcal{I}, b), E)$$

$$\parallel$$

$$(W_0, (\mathcal{I}_0, b), E_0) \leftarrow \cdots \leftarrow (W_\ell, (\mathcal{I}_\ell, b), E_\ell)$$

s.t.

$$\mathrm{Sing}(\mathcal{I}_\ell, b) = \phi$$

Observe

Year 0 : $\mathrm{Sing}(d_0, b) = \mathrm{Sing}(d_X, 1) = X = X_0$

- X_0 is an irred. comp. of $\mathrm{Sing}(d_0, b)$

Assume

Year i :

- for $j < i$
  - X_j is an irred. comp. of $\mathrm{Sing}(d_j, b)$
  - $C_j \not\supset X_j$

    and hence X_{j+1} makes sense

  ($\rightarrow$ X_{j+1} is an irred. comp. of $\mathrm{Sing}(d_{j+1}, b)$)

- X_i is an irred. comp. of $\mathrm{Sing}(d_i, b)$

  ($\rightarrow \neq \emptyset$)

Case A. $C_i \not\supset X_i$

and hence X_{i+1} makes sense.

$\longrightarrow$

X_{i+1} is an irred. comp. of $\mathrm{Sing}(\mathcal{I}_{i+1}, b)$

$(\rightarrow \neq \emptyset)$

Case B $\quad C_i \supset X_i$: an irred. comp. of

$$\begin{array}{cc} C_i \supset X_i & \\ \cap & \mathrm{Sing}(\mathcal{I}_i, b) \\ \mathrm{Sing}(\mathcal{I}_i, b) & \end{array}$$

Conclusion : X_i an irred. comp. of C_i.

$\longrightarrow$

X_i nonsingular

since

C_i nonsingular ☺

Grand Conclusion

For some year $i_0 < l$, Case B has to happen, since $\mathrm{Sing}(\mathcal{I}_l, b) = \emptyset$.

$$\begin{array}{ccc} W_0 \leftarrow & \cdots & \leftarrow W_{i_0} \\ \cup & & \cup \\ X_0 \leftarrow & & X_{i_0} \end{array}$$

provides embedded res.

Remark.

① Setting

$(W, (\mathcal{I}, b), E) = (W, (\mathcal{I}_X, 2), \phi)$

only works

when X ; a hypersurface

② Actually

Res. of sing. of a basic object

$\Downarrow$

Embedded res. of sing

(strong form)

involves Hilbert-Samuel function

but
· much more difficult
· characteristic free.

Naive inductive scheme.

Given $(W, (\mathcal{I}, b), E)$,

find

$$\begin{array}{c} (W, (\mathcal{I}, b), E) \\ \cup \\ (H, (\mathcal{J}, c), F = E|_H) \end{array}$$

s.t.

res. of sing. for

$$\begin{array}{l} (H, (\mathcal{J}, c), F) \\ \quad \| \\ (H_0, (\mathcal{J}_0, c), F_0) \leftarrow \cdots \leftarrow (H_\ell, (\mathcal{J}_\ell, c), F_\ell) \end{array}$$

s.t.

$$\mathrm{Sing}(\mathcal{J}_\ell, c) = \emptyset$$

induces res. of sing. for

$$\boxed{\begin{array}{c} (H_i, (\mathcal{J}_i, c), F_i) \\ \cap \quad \boxed{\begin{array}{c} \mathrm{Sing}(\mathcal{J}_i, c) \\ \| \\ \mathrm{Sing}(\mathcal{I}_i, b) \end{array}} \\ (W_i, (\mathcal{I}_i, b), E_i) \end{array}}$$

$$\begin{array}{l} (W, (\mathcal{I}, b), E) \\ \quad \| \\ (W_0, (\mathcal{I}_0, b), E_0) \leftarrow \cdots \leftarrow (W_\ell, (\mathcal{I}_\ell, b), E_\ell) \end{array}$$

s.t.

$$\mathrm{Sing}(\mathcal{I}_\ell, b) = \emptyset.$$

$\longrightarrow$ Induction on dimension.

Prototype $\quad \mathrm{char}(k) = 0.$

$$\left\{\begin{array}{l}
W = \mathbb{A}_k^d = \mathrm{Spec}\, k[x_1, \ldots, x_{d-1}, \underset{\underset{\displaystyle y}{\shortparallel}}{x_d}] \\[2em]
\mathcal{I} = (f) \\
\qquad f = y^{\mu} + \underbrace{c_1}_{\underset{\displaystyle 0}{\shortparallel}} y^{\mu-1} + c_2 y^{\mu-2} + \cdots + c_{\mu-1} y + c_{\mu} \\[1em]
\qquad\qquad c_i \in k[x_1, \ldots, x_{d-1}] \\[1em]
b = \mu. \\[1em]
E = \emptyset.
\end{array}\right.$$

Consider $(W, (\mathcal{I}, b), E)$

$$\shortparallel$$

$$(\mathbb{A}_k^d, ((f), \mu), \emptyset)$$

Then

$$\underline{(H, \overline{(\mathcal{J}, c)}, F)}$$

$$H = \{y = 0\}, \quad (\mathcal{J}, c) = \left(\sum_{i=2}^{\mu} (c_i)^{\frac{\mu!}{i}},\ \mu!\right), \quad F = E|_H = \emptyset$$

will do.

$p \in \mathrm{Sing}(\mathcal{d}, b)$

$\Longleftrightarrow \quad \mathrm{ord}_p(\mathcal{d}) \geqq b$

$\Longleftrightarrow \quad \mathrm{ord}_p(f) \geqq \mu.$

$\Longleftrightarrow$

$p \in \{y = 0\} = H \quad (\leftarrow c_1 = 0)$

& $\mathrm{ord}_p(c_i) \geqq i$

$\Longleftrightarrow$

$p \in H$

& $\sum_{i=1}^{\mu} (c_i)^{\frac{\mu!}{i}} \geqq \mu!$

$\Longleftrightarrow$

$p \in H.$

$p \in \mathrm{Sing}(\mathcal{J}, c)$

This relation continues after blow up.

Remark.

- One extra condition for the prototype

$$\boxed{f \neq y^{\mu}.}$$

Question ① What happens if $f = y^{\mu}$?

Answer : Then

$$\left(\underset{\{y=0\}}{\overset{\|}{H}}, \left(\underset{\sum_{i=2}^{\mu} (C_i)^{\frac{\mu!}{i}}}{\overset{\|}{J}}, \underset{\mu!}{\overset{\|}{c}} \right), \underset{E|_H}{\overset{\|}{F}} \right)$$

$\sum_{i=2}^{\mu} (C_i)^{\frac{\mu!}{i}} = 0$ ideal, since $C_i = 0$ for $i = 2, \dots, \mu$

Remember

we do NOT consider resolution problem for $(W, (d, b), E)$ when $d = 0$.

Question ② : What do we do then when $f = y^n$?

Answer : We blow up with center

$$C = \{y = 0\} \subset \mathrm{Sing}(d, b)$$

$$\left(\overset{\text{actually}}{=}\right)$$

Then

$$
\begin{array}{ccc}
(W, (d, b), E) & & \\
\| & & \\
(\mathbb{A}^d_k, ((y^n), n), \phi) & \xleftarrow{\pi} & (\mathbb{A}^d_k, ((1), n), \{y = 0\}) \\
\| & & \| \\
(W_0, (d_0, b), E_0) & \xleftarrow{\pi} & (W_1, (d_1, b), E_1)
\end{array}
$$

$$\mathrm{Sing}(d_1, b) = \phi$$

☺

Main Inductive Lemma. (MIL)

$$(W, (\mathcal{I}, b), E)$$

$$p \in \operatorname{Sing}(\mathcal{I}, b)$$

Assume

1. $\operatorname{ord}_p(\mathcal{I}) = b = \mu$

$$\left(\longrightarrow \quad \begin{array}{c} \exists\, y \in \Delta^{\mu-1}(\mathcal{I})_p \\ \operatorname{ord}_p(y) = 1 \\ \text{Set } \; H = \{y = 0\}. \end{array} \right)$$

2. $H \pitchfork E$ at p. $H \not\subset E$

Set

$$\mathcal{J} = \operatorname{Coeff}(\mathcal{I}) = \left. \sum_{i=1}^{\mu} \{\Delta^{\mu-i}(\mathcal{I})\}^{\frac{\mu!}{i}} \right|_H .$$

Case $\mathcal{J} = 0$

$\longrightarrow \mathcal{I} = (\mathcal{Y}^{\cdot b})$

$\longrightarrow H = \mathrm{Sing}(\mathcal{I}, b)$

in a neighborhood of P

$$(W, (\mathcal{I}, b), E) \xleftarrow[\pi]{} (\widehat{W}, (\widehat{\mathcal{I}}, b), \widehat{E})$$

blow up with center H.

$\mathrm{Sing}(\widehat{\mathcal{I}}, b) = \emptyset$

☺

Case $\mathcal{J} \neq 0$.

res. of sing. for $(W, (\mathcal{I}, b), E)$

$\Uparrow$

res. of sing. for $(H, (\mathcal{J}, c), F)$

$\mathcal{J} = \mu!$, $\quad F = E|_H$

(locally around P)

i.e. naive inductive scheme is realized!

Proof.

Skip

Case $j = 0$.

$$\Delta^{\mu-1}(\mathcal{I})\big|_H = 0$$

$$\to \operatorname{ord}_{\eta_H}(\Delta^{\mu-1}(\mathcal{I})) \geq 1$$

$$\to \operatorname{ord}_{\eta_H}(\mathcal{I}) \geq \mu.$$

$\to$

$$y^\mu \mid \mathcal{I} \quad \text{plus} \quad \operatorname{ord}_P(\mathcal{I}) = \mu.$$

$\to$

$$\mathcal{I} = (y^\mu) \quad (\text{in a neighborhood of } P)$$

the rest is easy.

Case $\mathcal{J} \neq 0$

Want to show

$$\text{res. of sing. for } (W, (\mathcal{d}, b), E)$$

$$\Uparrow$$

$$\text{res. of sing. for } (H, (\mathcal{J}, c), F)$$

Assume inductively

$$(H, (\mathcal{J}, c), F)$$

$$\shortparallel$$

$$(H_0, (\mathcal{J}_0, c), F_0) \leftarrow \cdots \leftarrow (H_i, (\mathcal{J}_i, c), F_i)$$

while

$$\begin{array}{ccc} H_t & \supset & \mathrm{Sing}(\mathcal{J}_t, c) \\ \cap & & \shortparallel \\ W_t & \supset & \mathrm{Sing}(\mathcal{d}_t, b) \end{array} \quad \text{for } t = 1, \ldots, i.$$

$$(W_0, (\mathcal{d}_0, b), F_0) \leftarrow \cdots \leftarrow (W_i, (\mathcal{d}_i, b), F_i)$$

$$\shortparallel$$

$$(W, (\mathcal{d}, b), E)$$

Take

$$\begin{array}{ccccc} C_i & \subset & \mathrm{Sing}(J_i, c) & \subset & H_i \\ \| & & \| & & \cap \\ C_i & \subset & \mathrm{Sing}(d_i, b) & \subset & W_i \end{array}$$

Enough to show.

$$\begin{array}{ccc} \mathrm{Sing}(J_{i+1}, c) & \subset & H_{i+1} \\ \| & & \cap \\ \mathrm{Sing}(d_{i+1}, b) & \subset & W_{i+1}. \end{array}$$

Claim $\quad \mathrm{Sing}(d_{i+1}, b) \overset{②}{\subset} \mathrm{Sing}(g_{i+1}, c)$

$$\begin{array}{ccc} ① \cap & & \cap \\ H_{i+1} & = & H_{i+1} \end{array}$$

$\because)$ ① $Q_{i+1} \in \mathrm{Sing}(d_{i+1}, b)$

$\Longleftrightarrow \mathrm{ord}_{Q_{i+1}}(d_{i+1}) \geqq b = \mu.$

$\Longleftrightarrow \mathrm{ord}_{Q_{i+1}}(\Delta^{\mu-1}(d_{i+1})) \geqq 1.$

$$\cup \qquad \text{Giraud's Lemma.}$$

$\Rightarrow \mathrm{ord}_{Q_{i+1}}([\Delta^{\mu-1}(d)]_{i+1}) \geqq 1$

$$\begin{array}{c} \cup \\ y_{i+1} \end{array}$$

$$H_{i+1} = \{y_{i+1} = 0\}$$

$\Rightarrow \quad Q_{i+1} \in H_{i+1}$

$$\therefore \quad \begin{array}{c} \mathrm{Sing}(d_{i+1}, b) \\ \cap \\ H_{i+1} \end{array}$$

② $Q_{i+1} \in \mathrm{Sing}(d_{i+1}, b)$

$$\iff \mathrm{ord}_{Q_{i+1}}(d_{i+1}) \geq b = \mu.$$

$$\iff \mathrm{ord}_{Q_{i+1}}(\Delta^{\mu-j'}(d_{i+1})) \geq \mu-(\mu-j') = j'$$

for $0 \leq j' \leq \mu-1$.

$\Downarrow$ Giraud's Lemma.

$$\Rightarrow \mathrm{ord}_{Q_{i+1}}([\Delta^{\mu-j'}(d)]_{i+1}) \geq j'$$

for $0 \leq j' \leq \mu-1$.

$$\Rightarrow \mathrm{ord}_{\underset{H_{i+1}}{\overset{\cap}{Q_{i+1}}}}(J_{i+1}) \geq \mu!$$

$$\Rightarrow Q_i \in \mathrm{Sing}(J_{i+1})$$

$$\therefore \underset{H_i}{\underset{\cap}{\mathrm{Sing}(d_{i+1}, b)}} \subset \underset{H_i}{\underset{\cap}{\mathrm{Sing}(J_{i+1}, b)}}$$

$$H_i = H_i$$

Claim $\mathrm{Sing}(\mathcal{J}_{i+1}, c) \subset \mathrm{Sing}(\mathcal{I}_{i+1}, b)$

$\because$) Take $f_i \in \mathcal{I}_i$.

Write

$$f_i = \sum \gamma_\alpha y^\alpha$$

$$\gamma_\alpha \in k[[x_1, \dots, x_{d-1}]]$$

$$y_i = y.$$

$$= \sum_{\alpha=0}^{b-1} \gamma_\alpha y^\alpha + \sum_{\alpha=b}^{\infty} \gamma_\alpha y^\alpha.$$

Assume by induction

$$\gamma_\alpha \in \widehat{\Delta^\alpha(\mathcal{I})_i}. \qquad \alpha = 0, \dots, b-1$$

Now $C_i = \{\{x_{\underline{0}} = 0; \underline{0}\} \cap \{y = 0\}$

one of them $= x$

Compute over x-chart.

$$f_{i+1} = \frac{f_i}{x^b}$$

$$= \sum_{\alpha=0}^{b-1} \underbrace{\frac{\gamma_\alpha}{x^{b-\alpha}}}_{\cap} \cdot \left(\frac{y}{x}\right)^\alpha + \sum_{\alpha=b}^{\infty} \gamma_\alpha x^{\alpha-b} \left(\frac{y}{x}\right)^b$$

$$\Delta^\alpha(\mathcal{d})_{i+1}\Big|_{H_{i+1}} \qquad H_{i+1} = \left\{\frac{y}{x} = 0\right\}$$

$$\mathrm{ord}_{Q_{i+1}} \left(\quad \right) \left(\quad \right) \left(\quad \right)$$

$$\begin{array}{ccc} \text{VII.} & \text{VII} & \text{VII} \\ b-\alpha & +\ \alpha & b \end{array}$$

if $Q_{i+1} \in \mathrm{Sing}(\mathcal{J}_{i+1}, c)$

$$\underbrace{\hphantom{b-\alpha + \alpha \qquad\qquad b}}$$

$$\text{VII}$$

$$b$$

$\therefore Q_{i+1} \in \mathrm{Sing}(\mathcal{J}_{i+1}, c)$

$\Rightarrow Q_{i+1} \in \mathrm{Sing}(\mathcal{d}_{i+1}, b)$

$\square$

Shortcomings of MIL toward achieving the Genuine Inductive Scheme

① "ord" is NOT a good invariant:

When $\mathrm{ord}_P(\mathcal{I}) > b$, after blow up, "ord" may strictly increase!

Ex.

$$(\mathbb{A}^2_k,\ \mathcal{I} = (f),\ b = 1), \quad f = y^2 - x^5$$

$$\mathrm{ord}_P(\mathcal{I}) = 2$$

blow up the origin

Compute over the x-chart

$$\pi^* f = (x_1 y_1)^2 - x_1^5 = x_1^2 (y_1^2 - x_1^3)$$

$$f_1 = \frac{\pi^* f}{x_1^b} = x_1 (y_1^2 - x_1^3)$$

$$\mathrm{ord}_{P_1}(\mathcal{I}_1) = 3$$

② When $\mathrm{ord}_p(\mathcal{I}) > b$,

a hypersurface of maximal contact does NOT exist

i.e.

$$\nexists\, y \in \Delta^{b-1}(\mathcal{I})_p$$

with

$$\mathrm{ord}_p(y) = 1.$$

since

$$\mathrm{ord}_p(\Delta^{b-1}(\mathcal{I})) \geq \underset{\displaystyle b}{\underset{\vee}{\mathrm{ord}_p(\mathcal{I})}} - (b-1) > 1$$

Objection : You might say

We only have to deal ? with the case

where

$$\mathrm{ord}_p(d) = b$$

e.g.

choose $b = \max \{\mathrm{ord}_p(d)\}$.

and then drop max

one by one.

Answer to Objection : Not quite!

$$(W, (\mathcal{I}), \overset{=\mu}{b}), \overset{=\emptyset}{E})$$

$$\sim (H, ((\begin{smallmatrix} \mathrm{ord}_p(c_i) \geq i \\ i = 2, \dots, \mu \end{smallmatrix}), \mu!), \emptyset)$$

$$H = \{y = 0\}$$

$$\sum_{i=2}^{\mu} (c_i)^{\frac{\mu!}{i}}$$

We may have $'' > ''$ for all $i = 2, \dots, \mu$.

Induction will NOT work

unless we consider the case

where

$$\mathrm{ord}_p(d) > b.$$

$\square$

③ Even when

$$\exists\, y \in \Delta^{b-1}(\mathcal{I})_p \text{ with } \operatorname{ord}_p(y) = 1,$$

$H = \{y = 0\}$ may NOT be $\pitchfork$ to E

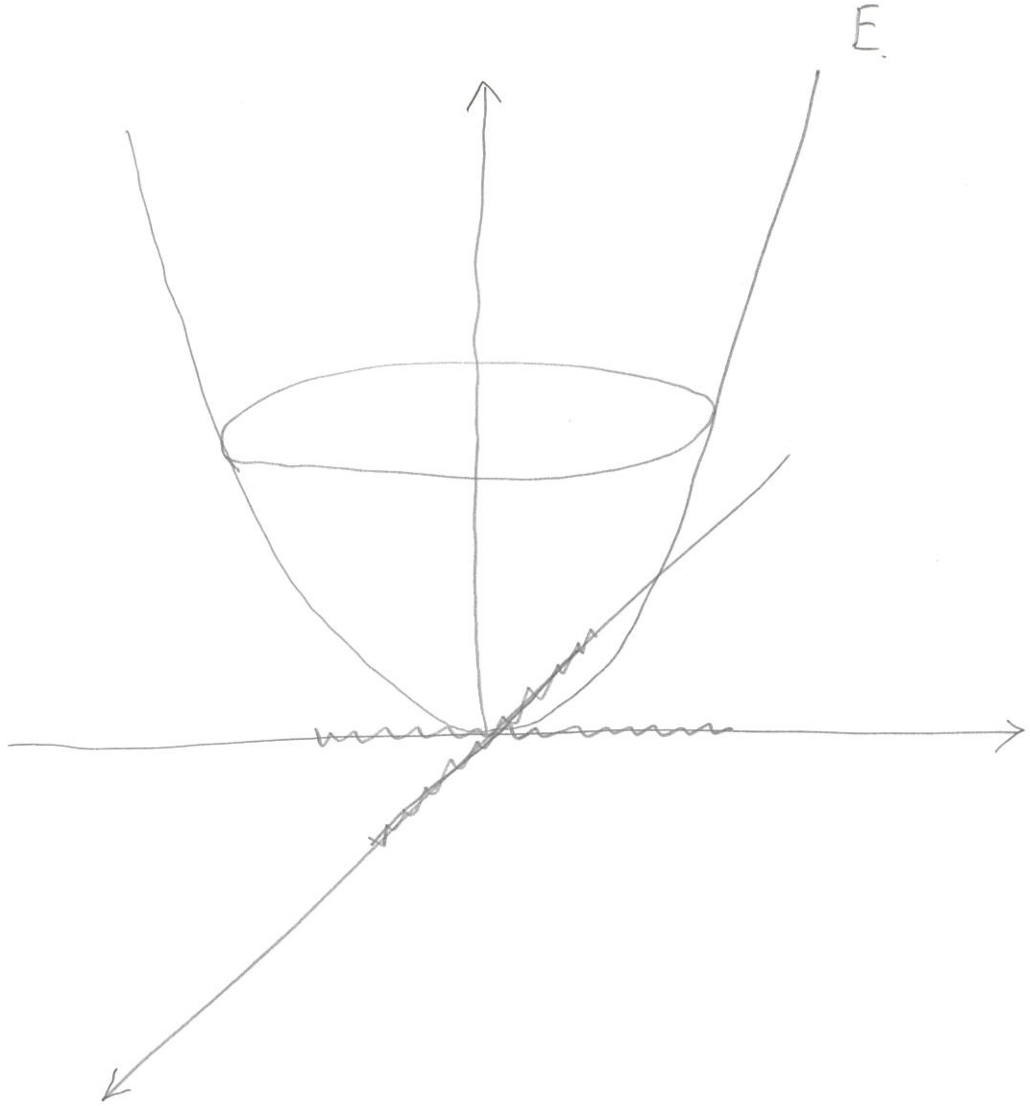
$\left(\begin{array}{c} \# \\ \phi \\ \text{in general} \end{array} \right)$

Ex.

$$\begin{cases} W = \mathbb{A}_k^3 = \operatorname{Spec} k[x, y, z] \\ \mathcal{I} = (z^2, (z - xy)^2) \\ b = 2 \\ E = \{z - x^2 - y^2 = 0\} \end{cases}$$

$$\begin{aligned} \operatorname{Sing}(\mathcal{I}, b) &= \{z = z - xy = 0\} \\ &= \{z = xy = 0\} \end{aligned}$$

$\nexists\, H \supset \operatorname{Sing}(\mathcal{I}, b)$ smooth hyper & $\pitchfork E$.


E

(4) A hypersurface of maximal contact exists only locally (even when it exists).

How can we globalize the inductive scheme ?

Globalization Problem.

Hironaka's BRILLIANT solution to overcome these shortcomings

Note: I learned this from Villamayor. So maybe Villamayor's ☺

(a) Introduction of (w-ord, s)

$$\operatorname{ord}_P(\bar{I}) \qquad \# \text{ of "bad" (old-) components}$$

$$\| $$

$$I / \prod I(F_\alpha)^{a_\alpha}$$

as much as possible

\# of the strict transforms of the components, going back to the year when the current value of "w-ord" started

(b) Construction of the modification

f2

$$m\,(W,\ (d,b),\ E)$$

$$= (W,\ \underbrace{m(d,b)}_{=},\ \underbrace{m(E)}_{=})$$

$$m(d,b) = (d,\ b) \subset (\bar{d},\ w-vd) \subset \bigcap_{F_\alpha : \text{bad}} (d(F_\alpha),\ 1)$$

$$m(E) = E \setminus F_{\cancel{bad}\ old}$$

Arithmetic among basic objects

$$(d,b) \cap (f,c)$$
$$= (d^c + f^b,\ bc)$$

s.t.

① $\mathrm{Sing}(m(d, b)) = \mathrm{Sing}(d, b) \cap \mathrm{MaxLocus}(w\text{-ord}, s)$

② $m(W, (d, b), E)$ i.e. res. of sing. for is reduced to res. of sing. for

$\cancel{\sim (H, (J, c), F)}$

satisfies the assumption of MIL.

Case A res. of sing. by blowing up H.

Case B

$\longrightarrow$

(c) ~~Induction on dimension~~

res. of sing. for $(H, (J, c), F)$

$\longrightarrow$ res. of sing for $m(W, (d, b), E)$ by induction

$\longrightarrow$

(d) decrease of $\max(w\text{-ord}, s)$

$\longrightarrow$ repeat the procedure $w\text{-ord} = 0$

i.e. $d = \prod d(F_\alpha)^{a_\alpha}$

$\underbrace{\text{monomial case.}}$

$\longrightarrow$

(e) res. of sing. in $\downarrow$ is EASY! ☺

Shortcoming (4)

Globalization Problem

Classical Answer : Hironaka's trick

Modern Answer :

- Wlodarczyk's homogenization
- Kawamone's differential saturation

What I learned from ~~Comment on~~ Villamayor's lecture yesterday

Difficulty his method faces in positive characteristi

$e = n$ pts $\neq$ pts with n-ple roots

(But if $p \nmid n$, then $\underline{\underline{=}}$)

Example

$\operatorname{char}(k) = 2$

$$\underbrace{Z^2 + \underset{\underset{0}{\shortparallel}}{a_1} Z + \underset{\underset{-x^3}{\shortparallel}}{a_2}}_{f} = 0$$

$$X = \{ Z^2 - x^3 = 0 \}$$

$$\begin{array}{ccc} \cap & & \\ \mathbb{A}^2 & \longrightarrow & \mathbb{A}^1 \\ & & \in \\ (z, x) & \longmapsto & x \end{array}$$

Observation

① It has double roots over all pts. on $\mathbb{A}^1$. values x

② $t = 2$ pts are singular pts.

$$\frac{\partial}{\partial z} f = 0. \quad \text{actually } \frac{\partial}{\partial z} f \equiv 0.$$

$$\frac{\partial}{\partial x} f = 0.$$

$$\parallel$$

$$3x^2. \qquad \text{i.e. } x = 0.$$

cannot express as a polynomial in a_1, a_2 $\qquad \rightarrow \quad z = 0.$

the only singular pt is $(0, 0)$

$$\mathrm{Sing}(z^2 - x^3, 2) = F_2(X) = \{(0,0)\}.$$

☺ $\qquad S \downarrow$ projection

$$\mathrm{Sing}(3x^2, 1) = \beta(F_2(X)) = \{x = 0\}.$$

What happens after blow up
(the origin)

$(f, 2)$ $\qquad$ $(\hat{f} = \hat{z}^2 - \hat{x}, 2)$

$\mathrm{Sing}(\hat{f}, 2) = \emptyset$

on the other hand.

$(3x^2, 1)$ $\qquad$ $(3\tilde{x}, 1)$

g $\qquad$ $\tilde{g}$

$\mathrm{Sing}(\tilde{g}, 1) = \{(0)\}$

What is causing this?

Because we can only have

$$\begin{array}{ll|l} & & \text{example} \\ & \Delta(\hat{\mathfrak{d}}) & \neq \quad k[x] \\ \text{in Giraud's} & \cup & \quad \cup \\ \text{Lemma.} & \widehat{\Delta(\mathfrak{d})} & \quad (x) \end{array}$$

Day 2 — Joint work with H. Kawanoue

Architect : Kawanoue

Carpenter : —

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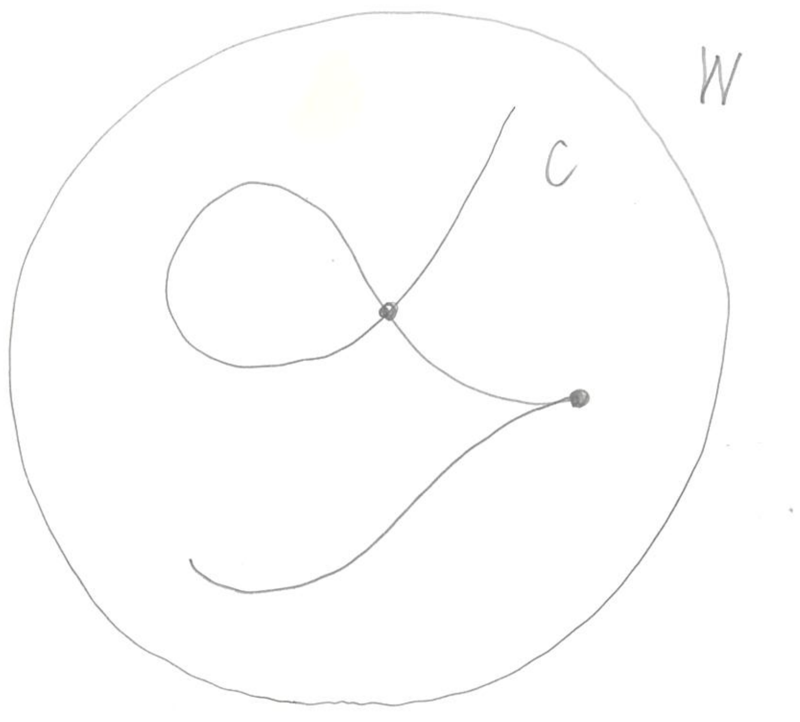
Part I.

<u>Theme</u> : Resolution of singularities of a curve embedded in a nonsingular surface. : a baby, baby version of IFP program.

<u>Situation</u> / $k = \overline{k}$, $\mathrm{char}(k) = p > 0$

$C \subset W$ a nonsingular surface

C : a curve embedded as a closed subvariety



Theorem

$\exists$ a seq. of blowups
with centers being closed pts.

$$
\begin{array}{ccccccc}
W & = W_0 & \leftarrow & W_1 & \leftarrow & & \\
\cup & \cup & & \cup & & & \\
C & C_0 & \leftarrow & C_1 & \leftarrow & &
\end{array}
$$

$$
\begin{array}{cccccc}
 & W_i & \leftarrow & W_{i+1} & & \leftarrow W_\ell \\
\cdots & \cup & & \cup & \cdots & \cup \\
 & C_i & \leftarrow & C_{i+1} & & \leftarrow C_\ell
\end{array}
$$

C_{i+1} : the strict transform of C_i.

s.t.

C_ℓ : nonsingular

Choice of the center

- We do NOT blow up nonsingular pts.
  (if we do, then superfluous!)
- We ONLY blow up the singular pts,

and HAVE to.

→

aside from which singular pt.
to blow up 1st.
2nd ...

there is a UNIQUE way to blow up.

Issue: Does the seq. terminate?

Goal: Develop the invariants
which measure the improvement
of each blow up.

$\boxed{\text{1 st invariant } \mu}$

$$P \in C \subset W$$

$$\mu(P) = \mathrm{ord}_P(dc)$$

Note: C nonsingular at P | singular at P

$\Longleftrightarrow$ | $\Longleftrightarrow$

$\mu(P) = 1.$ | $\mu(P) > 1.$

$\boxed{\text{2nd invariant } (\sigma, \nu)}$

Weierstrass Preparation Theorem

$$f = \underset{\substack{\uparrow \\ \text{unit}}}{u}\,(y^{\mu} + c_1 y^{\mu-1} + \cdots + c_{\mu})$$

$$c_i \in k[[X]]$$

May assume

$$f = y^{\mu} + c_1 y^{\mu-1} + c_2 y^{\mu-2} + \cdots + c_{\mu}$$

$$c_i \in k[[X]]$$

- Look for a substitute for a (smooth) HMC

$\boxed{\mathrm{char}(k) = 0}$

$\delta f = y + \text{higher}$ (after Tschirnhausen)

δ a diff. op. of $\deg = \mu - 1$.

or more generally a linear form in x & y. (without Tschirnhausen)

$\{\delta f = 0\}$ a smooth HMC.

$S_\ell = \{\delta f \bmod \mathfrak{M}_p^{\ell-1}$; δ a diff. op. of $\deg = \mu - \ell\}$.

$$\begin{array}{ccccccccccc} \mathbb{S} & = & S_1 & \oplus & S_2 & \oplus & S_3 & \oplus & \cdots & \oplus & S_\mu \\ & & \parallel & & \cap & & \cap & & & & \cap \\ \mathbb{L} & = & L_1 & \oplus & L_2 & \oplus & L_3 & \oplus & \cdots & \oplus & L_\mu \ \oplus \cdots \end{array}$$

$\mathbb{L}$ graded k-alg. generated by $\{S_\ell\}_{\ell=1}^{\mu}$.

$\mathbb{L} = G(\underset{\sim}{L_1}) \longrightarrow$ smooth HMC.

$\boxed{\mathrm{char}(k) = p > 0}$

$$S_\ell = \{ \delta f \bmod \mathfrak{m}_p^{\ell-1} \mid \delta \text{ a diff. op. of } \deg = \mu - \ell \}.$$

$$\left.\begin{array}{ccccccccc} \mathbb{S} & = & S_1 & \oplus & S_2 & \oplus & \cdots & \oplus & S_{p^{e_1}-1} \\ & & \cap & & \cap & & & & \cap \\ \mathbb{L} & = & L_1 & \oplus & L_2 & \oplus & \cdots & \oplus & L_{p^{e_1}-1} \end{array}\right| = 0$$

$$\begin{array}{ccc} S_{pe_1} \xrightarrow{p^{e_1}} & \oplus & S_{2pe_1} \xrightarrow{p^{e_1}} \\ \| & & \cap \\ L_{pe_1} & \oplus & L_{2pe_1} \\ \| & & \\ [L_{pe_1}]_{\text{pure}} & & \end{array} \qquad \cancel{\begin{array}{c} S_{pe_2} \\ \cap \\ L_{pe_2} \end{array}} \qquad \cdots$$

$\mathbb{L}$ the graded k-alg. generated by $\{S_\ell\}_{\ell\geq 1}^{\mu}$

$\tau = 1.$ { Case $\dim_k L_{pe_1} = 1.$ $\quad \mathbb{L} = G(L_{pe_1}) = G([L_{pe_1}]_{\text{pure}})$

$\tau = 2$ { Case $\dim_k L_{pe_1} = 1$ $\quad \mathbb{L} \neq G(L_{pe_1})$

$$= G(L_{pe_1} \sqcup L_{pe_2})$$

$$(S_{pe_2} \not\subset G(L_{pe_1}))$$

$$= G([L_{pe_1}]_{\text{pure}} \sqcup [L_{pe_2}]_{\text{pure}})$$

Case $\dim_k L_{pe_1} = 2$ $\quad \mathbb{L} = G(L_{pe_1}) = G([L_{pe_1}]_{\text{pure}})$

$\quad\quad\quad\quad \| $

$\quad\quad\quad\quad L_{p^{e_2}}$

Key Observation

(After a possible linear transformation of x & y)

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$\exists\ \delta_1$ a diff. op. of deg $= \mu - p^{e_1}$

s.t.

$h_1 = \delta_1 f = x_1^{p^{e_1}} + \text{higher}$

$\equiv x_1^{p^{e_1}} \mod \mathfrak{M}_p^{p^{e_1}+1}$

minus

~~some part generated by the lower degree terms of $\mathrm{Diff}((f, \mu))$~~

$\exists\ \delta_2$ a diff. op. of deg $= \mu - p^{e_2}$

s.t.

$h_2 = \delta_2 f = x^{p^{e_2}} + \text{higher}$

$\equiv x^{p^{e_2}} \mod \mathfrak{M}_p^{p^{e_2}+1}$

minus

some part generated by the lower degree terms of $\mathrm{Diff}((f, \mu))$

~~Skip!~~

~~$\{e_y, e_x\}$~~ $= \{e_1, e_2\}$

~~$\mathbb{L} = G(y^{p^{e_y}}, x^{p^{e_x}})$~~

~~Maybe ($e_1 = e_x$, $e_2 = e_y$; $e_1 = e_x = e_y$)~~

$\{(h_1, p^{e_1}), (h_2, p^{e_2})\} = \mathbb{H} \subset \mathrm{Diff}((f, \mu))$

$h_1 = x_1^{p^{e_1}} \in L_{p^{e_1}}$ the Leading Generator System

$h_2 = x_2^{p^{e_2}} \in L_{p^{e_2}}$

(the LGS)

LGS : a collective substitute
in positive characteristic
for the notion of
a smooth HMC
in characteristic zero.

Invariant σ

$$\sigma(P) = (h_n)_{n \in \mathbb{Z}_{\geq 0}}$$

$$h_n = 2 - \#\{\alpha \; ; \; e_\alpha \leqq n\}$$

$$\cap$$

$$\{1, 2\}$$

with Lex order.

Morale : The less the level of "e" is,
the better.
The "more" we have the "e",
the better.

Behavior of the invariants under blow up

$$
\begin{array}{ccc}
W & \xleftarrow{\ \pi\ } & \widetilde{W} \\
\cup & & \cup \\
C & \longleftarrow & \widehat{C} \quad \text{the strict transform} \\
\Uparrow & & \Uparrow \\
P & \longmapsfrom & \widehat{P} \in \pi^{-1}(P)
\end{array}
$$

Case $\tau = 2$

$$\mathrm{ord}_P(d_C) > \mathrm{ord}_{\widehat{P}}(d_{\widehat{C}})$$

Case $\tau = 1$

$$h_y = y^{p^{e_y}} + \text{higher}$$

- after replacing the original y with $y + \alpha x$, $\alpha \in k$ (since $y^{\mu} \in \mathbb{L}_{\mu}$.)

- ~~NO need to take "modulo the part generated by the lower degree terms.~~

May assume $\tilde{p}$ is in x-chart 177

with $\begin{cases} \hat{x} = x \\ \hat{y} = \dfrac{y}{x} \end{cases}$ r.s.p.

Since over y-chart

$$\mathrm{ord}_p(d_c) > \mathrm{ord}_{\tilde{p}}(d_{\tilde{c}})$$

Weierstrass Preparation Theorem.

$$h_y = u.(\underbrace{y^{p^{e_y}} + d_1 y^{p^{e_y}-1} + \cdots + d_{p^{e_y}-1}\, y + d_{p^{e_y}}})$$

$d_i \in k[[x]]$
$\mathrm{ord}_p(d_i) > i$.

May assume.

$h_y =$

Tempted to define

$$\nu = \min\left\{ \frac{\mathrm{ord}_p(d_j)}{j} \, ; \, j = 1, \ldots, p^{e_y}-1, p^{e_y} \right\}.$$

Trouble : As it is, ν may strictly increase (even when σ stays the same)

Example

$$h_y = y^{p^{e_y}} + \underbrace{d_1 y^{p^{e_y}-1} + \cdots + d_{p^{e_y}-1} \cdot y}_{\substack{\| \\ 0}} + \underbrace{d_{p^{e_y}}}_{\substack{\| \\ x^{2p^{e_y}} + x^{3p^{e_y}+1}}}$$

(h_y, p^{e_y})

$$\nu = \frac{\mathrm{ord}_p(d_{p^{e_y}})}{p^{e_y}} = 2.$$

$(\hat{h}_y, p^{e_y})$

$$\hat{h}_y = \frac{\pi^* h_y}{x^{p^{e_y}}} \qquad \left\{ \begin{array}{l} x = \hat{x} \\ y = \hat{x}\hat{y} \end{array} \right.$$

$$= \hat{y}^{p^{e_y}} + \hat{x}^{p^{e_y}} + \hat{x}^{2p^{e_y}+1}.$$

$$= (\hat{y} + \hat{x})^{p^{e_y}} + \hat{x}^{2p^{e_y}+1}.$$

$$\hat{\nu} = \frac{\mathrm{ord}_p(\hat{x}^{2p^{e_y}+1})}{p^{e_y}} = 2 + \frac{1}{p^{e_y}}.$$

"Cleaning" : We have to go through the process of "cleaning" before defining ν.

Case A : $min = min \left\{ \frac{ord_p(d_j)}{j} ; j = 1, \ldots, p^{e_g} - 1 \right\}$.

$\longrightarrow$ ☺

Case B : $min < \frac{ord_p(d_j)}{j}$ for $j = 1, \ldots, p^{e_g} - 1$.

& (and hence) $min = \frac{ord_p(d_{p^{e_g}})}{p^{e_g}}$

& $[d_{p^{e_g}}]_{lowest}$ is NOT a p^{e_g}-th power.

$\longrightarrow$ ☺

Case C : Otherwise

i.e. $min < \frac{ord_p(d_j)}{j}$ for $j = 1, \ldots, p^{e_g} - 1$

& (and hence) $min = \frac{ord_p(d_{p^{e_g}})}{p^{e_g}}$

& $[d_{p^{e_g}}]_{lowest}$ is a p^{e_g}-th power.

C $\longrightarrow$ Replace y

with $y' = y + \left([d_{p^{e_y}}]_{lowest} \right)^{1/p^{e_y}}$

$\longrightarrow$

$$\frac{\mathrm{ord}_p(d_{p^{e_y}})}{p^{e_y}} < \frac{\mathrm{ord}_p((d_{p^{e_y}})')}{p^{e_y}}$$

Case : $\exists\ j' = 1, \ldots, p^{e_y}-1$

s.t. $d_{j'} \neq 0$.

Set j'_0 = min of such

Then

$$\frac{\mathrm{ord}_p(d_{j'_0})}{j'_0} = \frac{\mathrm{ord}_p((d_{j'_0})')}{j'_0}$$

Therefore, after finitely many repetitions

either A or B.

Case : $d_{j'} = 0$ for $j' = 1, \ldots, p^{e_y}-1$.

Subcase : $d_{p^{e_y}}$ NOT a p^{e_y}-th power.

After finitely many repetitions

we reach B.

Subcase : $d_{p^{e_y}}$ is a p^{e_y}-th power.

Replace y.

with $y_{new} = y + (d_{p^{e_y}})^{1/p^{e_y}}$.

$$\boxed{\begin{array}{l} h_y = (y_{new})^{p^{e_y}} \\ \& \\ \{y_{new} = 0\} \\ \text{is a smooth HMC.} \end{array}}$$

In this subcase, we go back to f.
(after applying W. P. Th. dividing by a unit)

$$f = y_{new}^{\mu} + c_{1,new}\, y_{new}^{\mu-1} + \cdots + c_{\mu-1,new}\, y + c_{\mu,new}$$

$$\nu = \min\left\{ \frac{\mathrm{ord}_p(c_{i,new})}{i} \,;\, i = 1, \cdots, \mu \right\}.$$

(no cleaning needed.)

Grand Conclusion in Case $\tau = 1$

$$(\sigma, \nu)(P) > (\sigma, \nu)(P)$$

Note: Invariant ν is
highly dependent up
the choice of the coordinate system
(which we choose as specified
in the process
→ enough to show the termination)

only the last case
independent of the choice

Part 2.

Theme : Basic object

"=" Idealistic filtration

"=" Rees Algebra

Situation

$\mathrm{Spec}\, A = U \subset W$ a nonsingular variety $/ k = \bar{k}$

$\mathrm{char}(k) = 0$ or $p > 0$

$f \in A$

$b \in \mathbb{Z}_{\geq 0}$

$\mathrm{Sing}(f, b) = \{ p \in W ; \mathrm{ord}_p(f) \geq b \}$

$\mathrm{Sing}\ [\mathrm{U}(f_s, b_s)] = \bigcap \mathrm{Sing}(f_s, b_s)$

Observation

①

$$\mathrm{Sing}\begin{bmatrix}(f_1, b)\\(f_2, b)\end{bmatrix} = \mathrm{Sing}\left[\{(\alpha f_1 + \beta f_2, b);\ \alpha, \beta \in A\}\right]$$

②

$$\mathrm{Sing}\begin{bmatrix}(f, b)\\(g, c)\end{bmatrix} = \mathrm{Sing}\begin{bmatrix}(f, b)\\(g, c)\\(fg, bc)\end{bmatrix}$$

③

$$\mathrm{Sing}\left[(f, b)\right] = \mathrm{Sing}\begin{bmatrix}(f, b)\\(f, b-1)\\ \vdots \\(f, 1)\\(f, 0)\end{bmatrix}$$

Note : $\mathrm{Sing}\,(f, 0) = W \quad \forall f \in A.$

- Basic Object

$$(I, b) \quad \leftarrow \text{①}$$

- Idealistic Filtration

$$R = \oplus_{\mu \in \mathbb{Z}_{\geq 0}} (I_\mu, \mu)$$

(usually plus f.g. over A) graded A-algebra ← ①, ②

satisfying

$$A = I_0 \supset I_1 \supset I_2 \supset \cdots \quad \leftarrow \text{③}$$

- Rees Algebra

$$R = \oplus_{\mu \in \mathbb{Z}_{\geq 0}} (I_\mu, \mu)$$

(usually plus f.g. over A) graded A-algebra ← ①, ②

Global version   sheafification

- Basic Object

$$(\mathcal{I}, b)$$

$$\mathrm{Sing}(\mathcal{I}, b) = \{ p \in W ; \mathrm{ord}_p(\mathcal{I}) \geq b \}.$$

- Idealistic Filtration

$$\mathcal{R} = \oplus_{\mu \in \mathbb{Z}_{\geq 0}} (\mathcal{I}_\mu, \mu)$$

(f.g.) graded $\mathcal{O}_W$-algebra

satisfying

$$\mathcal{O}_W = \mathcal{I}_0 \supset \mathcal{I}_1 \supset \mathcal{I}_2 \supset \cdots$$

$$\mathrm{Sing}\,\mathcal{R} = \{ p \in W ; \mathrm{ord}_p(\mathcal{I}_\mu) \geq \mu \ \forall \mu \in \mathbb{Z}_{\geq 0} \}$$

- Rees Algebra

$$\mathcal{R} = \oplus_{\mu \in \mathbb{Z}_{\geq 0}} (\mathcal{I}_\mu, \mu)$$

(f.g.) graded $\mathcal{O}_W$-algebra

$$\mathrm{Sing}\,\mathcal{R} = \{ p \in W ; \mathrm{ord}_p(\mathcal{I}_\mu) \geq \mu \ \forall \mu \in \mathbb{Z}_{\geq 0} \}.$$

Resolution of singularities of an idealistic filtration

Formulation

Construct a seq. of transformations

$$\mathcal{O}_W \oplus \mathcal{O} \oplus \mathcal{O} \oplus \cdots$$

$$(W, \overset{*}{R}, E)$$

$$\parallel$$

$$(W_0, R_0, E_0) \xleftarrow{\pi_0} (W_1, R_1, E_1)$$

$$\leftarrow \cdots$$

$$(W_i, R_i, E_i) \xleftarrow{\pi_i} (W_i, R_i, E_i)$$

$$\leftarrow \cdots$$

$$\xleftarrow{\pi_{e-1}} (W_e, R_e, E_e)$$

s.t.

$$\mathrm{Sing}(R_e) = \phi$$

where

(i) π_i blow up with center C_i

requirement on C_i

① C_i nonsingular (maybe reducible)

② $C_i \subset \mathrm{Sing}(R_i)$

③ $C_i \pitchfork E_i$

(ii) $\mathcal{R}_i = \oplus_{\mu \in \mathbb{Z}_{\geq 0}} (\mathcal{I}_{i,\mu}, \mu)$

$$\mathcal{J}_{i+1,\mu} := \pi_i^{-1}(\mathcal{I}_{i,\mu})\mathcal{O}_{W_{i+1}} \cdot \mathcal{I}(\pi_i^{-1}(C_i))^{-\mu}$$

$$\mathcal{R}_{i+1} = G.\left(\oplus_{\mu \in \mathbb{Z}_{\geq 0}} (\mathcal{J}_{i+1,\mu}, \mu)\right)$$

generated as an idealistic filtration

i.e. the smallest idealistic filtration containing

Note:

(ii)$_{\text{Rees Algebra}}$

$$\mathcal{I}_{i+1,\mu} := \pi_i^{-1}(\mathcal{I}_{i,\mu})\mathcal{O}_{W_{i+1}} \cdot \mathcal{I}(\pi_i^{-1}(C_i))^{-\mu}$$

(iii) $E_{i+1} = E_i \sqcup \pi_i^{-1}(C_i)$

the strict transform of

Proposition

Res. of sing. for a basic object

$\Longleftrightarrow$ Res. of sing. for an f.g. idealistic filtration

$\Longleftrightarrow$ Res. of sing. for a f.g. Rees algebra.

Proof.

$(B \Rightarrow I)$

R generated by $\{(d_{\mu_s}, \mu_s)\}_{s \in S}$ $\#S < \infty$ as a graded $\mathcal{O}_W$-algebra.

Set
$$\begin{cases} d = \sum_{r \in S} (d_{\mu_r})^{\prod_{s \in S} \mu_s / \mu_r} \\ b = \prod_{s \in S} \mu_s \end{cases}$$

Res. of sing. for (d, b)

$\Rightarrow$ Res. of sing. for R

$(I \Rightarrow R)$

A a Rees algebra generated by $\{(J_{\mu_s}, \mu_s)\}_{s \in S}$ $\# S < \infty$ as a graded $\mathcal{O}_W$-algebra.

Set
R a graded $\mathcal{O}_W$-algebra generated by
$\{(J_{\mu_s}, \mu_s), (J_{\mu_s}, \mu_s - 1), \cdots, (J_{\mu_s}, 1)\}_{s \in S}$.

Then R (f.g.) idealistic filtration

Res. of sing. for R
$\Rightarrow$ Res. of sing. for A

$(R \Rightarrow B)$
(J, b)
Set A : a graded $\mathcal{O}_W$-algebra generated by $\{(J, b)\}$

Res. of sing for A
$\Rightarrow$ Res. of sing. for (J, b)

Def. (Leading algebra of an idealistic filtration)

$\mathcal{R}$ an idealistic filtration

$P \in \mathrm{Sing}(\mathcal{R})$

$$\mathbb{L}_P(\mathcal{R}) = \bigoplus_{\mu \in \mathbb{Z}_{\geq 0}} \{ f \bmod \mathfrak{m}_P^{\mu+1} ; (f, \mu) \in \mathcal{R} \}$$

$$\subset \bigoplus_{\mu \in \mathbb{Z}_{\geq 0}} \mathfrak{m}_P^{\mu} / \mathfrak{m}_P^{\mu+1}$$

$$\cong k[X_1, \cdots, X_d]$$

where $x_1, \cdots, x_d$ r.s.p. for $\mathcal{O}_{W,P}$

Def (Differential Saturation)

$$R = \bigoplus_{\mu \in \mathbb{Z}_{\geq 0}} (\mathcal{I}_\mu, \mu)$$

an idealistic filtration

def R diff. saturated

$\overset{\text{def}}{\Longleftrightarrow}$

$$\left[\begin{array}{l} (f, \mu) \in ((\mathcal{I}_\mu)_p, \mu) \in R_p. \\ \to \\ (\delta f, \max\{0, \mu - \deg\delta\}) \in R_p \\ \qquad \text{for any } \delta \text{ diff. op. of } \deg\delta. \end{array}\right]$$

$\mathcal{D}(R)$ the diff. saturation of R.

:= the smallest idealistic filtration which is diff. sat.

Construction $\mathcal{D}(R)$

R f.g. idealistic filtration generated by $\{(\mathcal{I}_{\mu_s}, \mu_s)\}_{s \in S}$ $\#S < \infty$ as a graded $\mathcal{O}_W$-algebra.

$\mathcal{D}(R)$ is the idealistic filtration generated by

$$\{ (\mathrm{Diff}^0(\mathcal{L}_{n_s}), n_s)$$
$$(\mathrm{Diff}^1(\mathcal{L}_{n_s}), n_s - 1)$$
$$\vdots$$
$$(\mathrm{Diff}^{n_s - 1}(\mathcal{L}_{n_s}), 1). \}_{s \in S}$$

as a graded $\mathbb{k}$-algebra

where

Diff^i diff. operators of deg $\leq i$.

Theorem (Kawanoue, Hironaka-Oda)

$\mathcal{R}$ an idealistic filtration which is diff. sat.

i.e. $\mathcal{R} = \mathcal{D}(\mathcal{R})$

$p \in \mathrm{Sing}(\mathcal{R}) \subset W$ a closed point

$\Rightarrow$

$\exists\ x_1, \cdots, x_d$ r.s.p. for $\mathcal{O}_{W,p}$

s.t.

$$\begin{array}{ccc} \mathbb{L}_p(\mathcal{R}) & \cong & k[x_1^{p^{e_1}}, \cdots, x_t^{p^{e_t}}] \\ \cap & & \cap \\ \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathfrak{m}_p^n / \mathfrak{m}_p^{n+1} & \cong & k[x_1, \cdots, x_t, x_{t+1}, \cdots, x_d] \end{array}$$

with

$$0 \leq e_1 \leq \cdots \leq e_t$$

Def. (Leading Generator System)

$$\{(h_\alpha, p^{e_\alpha})\}_{\alpha=1}^{t} = \mathbb{H} \subset R.$$

s.t.

$$h_\alpha \equiv x_\alpha^{p^{e_\alpha}} \mod \mathfrak{M}_p^{p^{e_\alpha}+1}.$$

the Leading Generator System

(the L.G.S.)

Invariant σ

$$\sigma(P) = (a_n)_{n \in \mathbb{Z}_{\geq 0}}$$

$$a_n = d - \#\{\alpha \,;\, e_\alpha \leq n\}$$

$$d = \dim W$$

Note : σ is independent of the choice of r.s.p.

Invariant $\tilde{\mu}$ ($\longleftrightarrow$ w-order)

- order modulo L.G.S.

$\mathbb{H}$ an LGS

(or more generally
the pull back of the LGS
in the previous year when the value
of σ stays the same.)

$f \in \mathcal{O}_{W,P}$ (or $\widehat{\mathcal{O}_{W,P}}$)

$\exists !$ the power series expansion w.r.t. $\mathbb{H}$
of the form

$$f = \sum C_{f,B} \underset{\displaystyle \prod_{\alpha=1}^{t} h_\alpha^{b_\alpha}}{\overset{\shortparallel}{H^B}} \qquad B = (b_1, \cdots, b_t)$$

where

$$C_{f,B} = \sum \gamma_{f,B,K} \, x_1^{k_1} \cdots x_t^{k_t}$$

$$K = (k_1, \cdots, k_t)$$

s.t.

$$0 \leq k_\alpha \leq p^{e_\alpha} - 1$$

$$\text{for } \alpha = 1, \cdots, t$$

&

$$\gamma_{f,B,K} \in k[[x_{t+1}, \cdots, x_d]]$$

$$\operatorname{ord}_p(f|_{\mathbb{H}}) := \operatorname{ord}_p(C_{f,0}).$$

Note: Independent of the choice of $\mathbb{H}$

$$\mu_p(\mathcal{R}) = \min\left\{ \frac{\operatorname{ord}_p(C_{f,0})}{n} \,;\right.$$

$$\left.(f, n) \in \mathcal{R}_p,\ n > 0 \right\}.$$

if $\mathcal{R}$ is $\mathcal{D}$-saturated

• $E = E_{young} \cup E_{aged}$

E_{young}: the union of the exceptional divisors created by the blowups after (including) the year when the current value of σ started.

$D \subset E_{young}$ an irred. comp.

$$\mu_{p,D}(\mathcal{R}) = \left\{ \frac{\mathrm{ord}_{\eta(D)}(C_f, 0)}{n} ; (f, n) \in \mathcal{R}_p, \ n > 0 \right\}.$$

$$\boxed{\hat{\mu}_p(\mathcal{R}) = \mu_p(\mathcal{R}) - \sum_{D \in E_{young}} \mu_{p,D}(\mathcal{R})}$$

Day 3 ... In positive characteristic

Inductive Scheme

$\boxed{\mathrm{char}(k) = 0.}$

classical strategy

: Induction on dimension

via a smooth HMC.

$\boxed{\mathrm{char}(k) \geq 0}$ $\mathrm{char}(k) = p > 0$ & $\mathrm{char}(k) = 0$

Idealistic Filtration Program (IFP)

: Induction on invariant σ

via $\underbrace{\text{LGS}}_{\text{a collection of singular HMC.}}$

Part I.

<u>Theme</u> Resolution of singularities of an idealistic filtration

(a) Introduction of $(\tilde{\mu}, \rho)$

$$\tilde{\mu}(P) \quad \# \text{ of "aged" components}$$

$$\tilde{\mu}_P(R)$$

(b) Introduction of the modification

$$m(W, R, E) = (W, m(R), m(E))$$

$$E_{young} = E \setminus E_{aged}$$

To be precise, it also depends on E, and on the history of the resolution seq.

s.t.

① $\mathrm{Sing}(m(R))$ "$=$" $\mathrm{Sing}(R) \cap \mathrm{MaxLocus}(\tilde{\mu}, \rho)$

actually we only deal with the "local" algorithm where we only have "$\supset$"

② $\sigma(\mathcal{R}) > \sigma(m(\mathcal{R}))$

(slightly oversimplified)

actually

$\longrightarrow$ decrease of $(\sigma, \# E)$

$\downarrow$ decrease of inv $= (\sigma, \hat{\mu}, s) \cdots (\sigma, \hat{\mu}, s)$

$\longrightarrow$

(c) Induction on invariant σ

res. of sing. for $m(W, \mathcal{R}, E)$

$\longrightarrow$

(d) Decrease of $\max(\hat{\mu}, s)$

$\longrightarrow$

repeat the procedure

$$(\hat{\mu}, s) = (0, 0)$$

: the monomial case

$\longrightarrow$

(e) res. of sing. in the monomial case

$\boxed{\mathrm{char}(k) = 0}$ EASY!

$\boxed{\mathrm{char}(k) = p > 0}$ Super Difficult!

Theme Resolution of singularities of an idealistic filtration in the Monomial Case

i.e.

$$(\sigma, \hat{\mu}, s) = (\sigma, 0, 0)$$

Situation

- Interpretation of $\hat{\mu} = 0$.

$\exists\ x_1, \ldots, x_d$ r.s.p. for $\mathcal{O}_{W,P}$

s.t.

(1) $\mathbb{H} = \{(h_\alpha, p^{e_\alpha})\}_{\alpha=1}^{t}$ (the pull back of an LGS)

with $0 \leqq e_1 \leqq \cdots \leqq e_t$

s.t.

$$h_\alpha \equiv x_\alpha^{p^{e_\alpha}} + \text{higher terms}$$

(2) $\exists\ M = \prod_{D \subset E_{young}} x_D^{a_D}$ a monomial

s.t.

$$(M, a) \in R_P.$$

for some $a \in \mathbb{Z}_{>0}$

where

$$\{x_D ; D \subset E_{young}\} \subset \{x_{t+1}, \ldots, x_d\}.$$

(3) $\mathcal{R}_p$ (and hence $\widehat{\mathcal{R}_p}$)

is saturated for

$$\left\{ \frac{\partial^\ell}{\partial x_\alpha^\ell} \ ; \ \ell \in \mathbb{Z}_{\geq 0}, \ \alpha = 1, \dots, t \right\}.$$

i.e.

$$(f, \lambda) \in \mathcal{R}_p \quad (\text{resp. } \widehat{\mathcal{R}_p})$$

$$\longrightarrow \left(\frac{\partial^\ell}{\partial x_\alpha^\ell} f, \ \max\{0, \lambda - \ell\} \right) \in \mathcal{R}_p$$

$$(\text{resp. } \widehat{\mathcal{R}_p})$$

(4) $(f, \lambda) \in \mathcal{R}_p$ (or $\widehat{\mathcal{R}_p}$)

$$f = \sum c_{f,\beta} H^\beta$$

the power series expansion

with respect to $\mathbb{H}$

(and $(x_1, \dots, x_d)$)

$$\longrightarrow \left(M^{\frac{1}{a}} \right)^\lambda \ \Big| \ c_{f,\emptyset}$$

divides

$M^{\frac{1}{a}} = M_{\text{usual}}$

· Interpretation of $S = 0$

No component of E_{aged} passing through P.

(All the components of E are in E_{young},

i.e. $E = E_{young}$ in a neighborhood of P)

Inductive scheme on invariant $\tau = t$

$\tau = 0$ As in the classical case Easy!

$\tau = 1$ $\boxed{\text{Most Difficult!}}$

$1 < \tau = j' < d.$ reduced to the case
$\tau = j' - 1$ in dimension $d-1$.
induction on dimension

$\tau = d$ does NOT happen!

Note: This confirms the folklore that
the most essential case in
the problem of res. of sing.
is the hypersurface case.

Analysis of the case where $\tau = 1$.

$$\mathbb{H} = \{(h, p^e)\} \quad \left(\text{the pull back of an LGS}\right)$$

May assume

via Weierstrass Preparation Theorem

$$h = x_1^{p^e} + a_1 x_1^{p^e-1} + a_2 x_1^{p^e-2} +$$

$$\cdots + a_{p^e-1} x_1 + a_{p^e}$$

with

$$a_i \in k[[x_2, \ldots, x_d]]$$

&

$$\mathrm{ord}_p(a_i) > i \text{ for } i = 1, \ldots, p^e-1, p^e.$$

<u>Observation</u>

(☆) Coefficients a_i for $i = 1, \ldots, p^e-1$

are "well-controlled"

in the sense

$$(M_{usual})^i \mid a_i$$

divides

$\leftarrow$ Situation (3) & (4)

(☆)

$$g = \frac{\partial^{p^e - i}}{\partial x_1^{p^e - i}}(h) = a_i + y(\qquad)$$

one think $(\partial x_1)^{p^e-i}$
NOT $(\partial x_1)^{p^e-i}$

$$i = 1, \dots, p^e - 1$$

$$(g, i) \in R_p$$

$$\| $$

$C_{g,0}.$

$\therefore \quad (\text{Mnomal})^i \mid g$

$\rightarrow$

$(\text{Mnomal})^i \mid a_i.$

In the following analysis,
we can "pretend" as if
$a_i = 0$ for $i = 1, \dots, p^e - 1$

i.e.

$$h = x_1^{p^e} + a_{p^e}$$

Note: This confirms the folklore that
the most essential case
in positive characteristic
is the purely inseparable case.

Analysis of G_{p^e}

We need to carry out "cleaning" in order to have well-defined invariants.

For example, $\mathrm{ord}_p(G_{p^e})$ is not well-defined without cleaning.

Ex.

$$h = X_1^{p^e} + G_{p^e}$$

$[G_{p^e}]_{\text{lowest}}$: the lowest degree term of G_{p^e} is a p^e-th power.

Replace X_1 with $X_1' = X_1^{p^e} + ([G_{p^e}]_{\text{lowest}})^{1/p^e}$

Then

$$h = (X_1')^{p^e} + G_{p^e}'$$

$$\mathrm{ord}_p(G_{p^e}) < \mathrm{ord}_p(G_{p^e}')$$

Real Invariant & Real Cleaning

- Real Invariant

$$\text{inv}_{\text{MON}, \diamond, p}(a_{p^e})$$

$$= \min \left\{ \frac{\text{ord}_p(a_j)}{p^e} \ ; \ j = 1, \dots, p^e - 1, p^e, \quad \& \quad \text{ord}_p(M_{\text{usual}}) \right\}$$

(☆)

$\downarrow$

$$= \min \left\{ \frac{\text{ord}_p(a_{p^e})}{p^e}, \ \text{ord}_p(M_{\text{usual}}) \right\}.$$

- Real Cleaning

A : $\min = \text{ord}_p(M_{\text{usual}})$

$\longrightarrow$ ☺

B : $\min < \text{ord}_p(M_{\text{usual}})$

& (and hence)

$\min = \dfrac{\text{ord}_p(a_{p^e})}{p^e}$

&

$[a_{p^e}]_{\text{lowest}}$ is NOT a p^e-th power.

$\longrightarrow$ ☺

C : Otherwise

i.e.

$\min < \mathrm{ord}_p(M_{\text{second}})$

&

(and hence)

$$\min = \frac{\mathrm{ord}_p(\mathfrak{a}_{pe})}{pe}$$

&

$[\mathfrak{a}_{pe}]_{\text{lowest}}$ is a pe-th power.

$\longrightarrow$

Replace y
with $y' = y + \left([\mathfrak{a}_{pe}]_{\text{lowest}} \right)^{1/pe}$

$$\frac{\mathrm{ord}_p(\mathfrak{a}_{pe})}{pe} < \frac{\mathrm{ord}_p(\mathfrak{a}'_{pe})}{pe}.$$

Therefore, after finitely many repetitions
either A or B.

After cleaning,
$\min_{\mathrm{MON}, \lozenge, p}(\mathfrak{a}_{pe})$ is independent
of the choices.

- M_{usual} vs M_{tight}

usual monomial — tight monomial

(after Benito - Villamayor)

$D \subset E_{young}$

$$\mathrm{inv}_{MON,\Diamond,\eta(D)}(h_{pe})$$
$$= \min\left\{\frac{\mathrm{ord}_{\eta(D)}(h_{pe})}{pe}, \mathrm{ord}_{\eta(D)}(M_{usual})\right\}$$

after cleaning

Def.
$$M_{tight} = \prod_{D \subset E_{young}} x_D^{\mathrm{inv}_{MON,\Diamond,\eta(D)}(h_{pe})}$$

w.r.t. $\eta(D)$

i.e.
$$h_{pe} = x_D^{r_D}\{g_{x_D}(\{x_2,\dots,x_d\}\setminus\{x_D\}) + x_D \cdot \varphi_{x_D}\}.$$

$$[h_{pe}]_{lowest} = x_D^{r_D} \cdot g_{x_D}.$$

Refined invariants

$$\boxed{\mathrm{inv}_{\mathrm{MON},\heartsuit,p}(G_{p^e})}$$

$$= \min\left\{\frac{\mathrm{ord}_p(G_{p^e})}{p^e} - \mathrm{ord}_p(M_{tight}),\right.$$

$$\left.\mathrm{ord}_p(M_{usual}) - \mathrm{ord}_p(M_{tight})\right\}.$$

$\geqq 0$

$\uparrow$ by definition

Note: Why subtract $\mathrm{ord}_p(M_{tight})$?
Otherwise, it increase just like "ord".
$\longleftrightarrow$ going from "ord" to "w-ord".

Def. (Benito - Villamayor)
Tight Monomial Case

$\overset{\Longleftrightarrow}{\text{def}}$ $\mathrm{inv}_{\mathrm{MON},\heartsuit,p}(G_{p^e}) = 0$

Note: Why?

Because
res. of sing. for an idealistic filtration
in the Tight Monomial Case
(Monomial Case & $\tau = 1$ & tight)
is Easy!

(Seemingly) Final Task 115

Bring $\underbrace{\text{inv}_{\mathrm{MON},\heartsuit,p}(G_{pe})}_{\text{hard to see the change under blow up.}}$ to 0.

one more refinement

$$\text{inv}_{\mathrm{MON},\spadesuit,p}(G_{pe})$$
$$= \min\{\rho_D - \mathrm{ord}_p(M_{tight})\ ;\ D \text{ bad},$$
$$\mathrm{ord}_p(M_{normal}) - \mathrm{ord}_p(M_{tight})\}.$$

where
$$\rho_D = \mathrm{ord}_p(x_D \cdot g_{x_D}) / p^e.$$

$$G_{pe} = x_D^{r_D}\{g_{x_D} + x_D \cdot \varphi_{x_D}\}.$$

with
$$r_D/p^e = \mathrm{ord}_{\mathrm{MON},\diamondsuit,\eta(D)}(G_{pe})$$

and where

$$D : \text{bad} \iff$$
$$\mathrm{ord}_{\eta(D)}(M_{tight}) < \mathrm{ord}_{\eta(D)}(M_{usual})$$

(Actual) Final Task (116)

lower dimension $\dim W \leq 3$
Not much choice ☺

Bring $\mathrm{inv}_{\mathrm{MON}, ♠, p}(\mathrm{Ape})$ to 0

Big Problem in higher dimensions: how to choose the center

$\dim W = 4$ now I know ☺
$\dim W \geq 4$???

Devil: the Moh-Hauser jumping phenomenon.

(Note: In front of the Spanish-Japanese joint effort there stands the Chinese-Austrian devil ☺)

: $\mathrm{inv}_{\mathrm{MON}, ♠, p}(\mathrm{Ape})$

sometimes strictly increases! ☹
(even when there is a unique choice for the center).

How to kill the devil:
Want to show $\mathrm{inv}_{\mathrm{MON}, ♠, p}(\mathrm{Ape})$ eventually decreases, after jumping, to the value lower than the one before jumping.

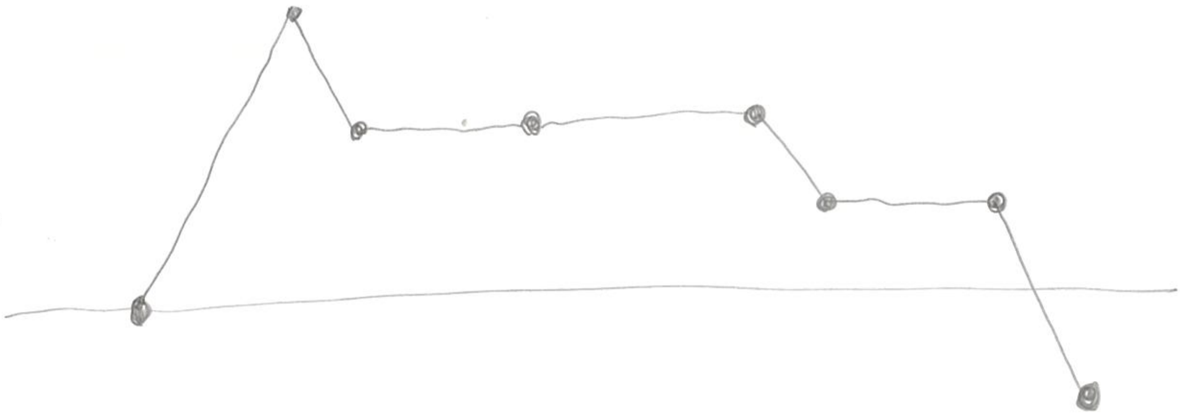
Eventual decrease :

in $\dim W \leq 3$.

$\dim W \geq 4$

still open!.

Eventual decrease
we show in dimension $= 3$



what we expect in dimension > 3